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On the Absorbing Problems for Wiener, Ornstein–Uhlenbeck, and Feller Diffusion Processes: Similarities and Differences

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Abstract: For the Wiener, Ornstein–Uhlenbeck, and Feller processes, we study the transition probability density functions with an absorbing boundary in the zero state. Particular attention is paid to the proportional cases and to the time-homogeneous cases, by obtaining the first-passage time densities through the zero state. A detailed study of the asymptotic average of local time in the presence of an absorbing boundary is carried out for the time-homogeneous cases. Some relationships between the transition probability density functions in the presence of an absorbing boundary in the zero state and between the first-passage time densities through zero for Wiener, Ornstein–Uhlenbeck, and Feller processes are proven. Moreover, some asymptotic results between the first-passage time densities through zero state are derived. Various numerical computations are performed to illustrate the role played by parameters.

Keywords: Wiener process; Ornstein–Uhlenbeck process; Feller process; asymptotic average of the local time; first-passage time and its moments

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1. Introduction and Background

Diffusion models are widely used to describe dynamical systems in economics, finance, biology, genetics, physics, engineering, neuroscience, queueing, and other fields (cf. Bailey [1], Ricciardi [2], Gardiner [3], Stirzaker [4], Janssen et al. [5], Pavliotis [6]). In various applications, it is useful to consider diffusion processes with linear infinitesimal drift and linear infinitesimal variance. This class incorporates Wiener, Ornstein–Uhlenbeck, and Feller diffusion processes. In population dynamics, these processes can be used to describe the growth of a population and the zero state represents the absorbing extinction threshold. With this aim, we study the absorbing problem for linear diffusion processes.

In the remaining part of this section, we shall briefly review some background results on the absorbing problems that will be used in the next sections for Wiener, Ornstein–Uhlenbeck and Feller diffusion processes.

Let $\{\mathcal{Z}(t), t \geq t_0\}$ be a time-inhomogeneous diffusion (TNH-D) process with state-space $\mathcal{D} = (r_1, r_2)$, which satisfies the stochastic differential equation

$$d\mathcal{Z}(t) = \zeta_1[\mathcal{Z}(t), t]\, dt + \sqrt{\zeta_2[\mathcal{Z}(t), t]}\, dW(t), \qquad \mathcal{Z}(t_0) = x_0,$$

with $\zeta_1(x,t)$ and $\zeta_2(x,t)$ denoting, respectively, the infinitesimal drift and the infinitesimal variance of $\mathcal{Z}(t)$ and where $W(t)$ is a standard Brownian motion. Often, $\mathcal{D} = (-\infty, +\infty)$, with $\pm\infty$ unattainable endpoints, but in some cases $\mathcal{Z}(t)$ is confined to the state space $\mathcal{D} = (0, +\infty)$ and in the zero state is imposed an absorbing condition.

When the endpoints $\pm\infty$ of $\mathcal{D}$ are unattainable boundaries, the transition probability density function (PDF) $f_{\mathcal{Z}}(x,t|x_0,t_0)$ is the solution of the backward Kolmogorov equation (cf. Dynkin [7])

$$\frac{\partial f_{\mathcal{Z}}(x,t|x_0,t_0)}{\partial t_0} + \zeta_1(x_0,t_0)\,\frac{\partial f_{\mathcal{Z}}(x,t|x_0,t_0)}{\partial x_0} + \frac{1}{2}\,\zeta_2(x_0,t_0)\,\frac{\partial^2 f_{\mathcal{Z}}(x,t|x_0,t_0)}{\partial x_0^2} = 0, \qquad (1)$$

with the initial delta condition $\lim_{t_0\uparrow t} f_{\mathcal{Z}}(x,t|x_0,t_0) = \delta(x-x_0)$. In the backward Kolmogovov Equation (1), the forward variables x and t are constant and enter only through the initial and boundary conditions.

We remark that the PDF $f_{\mathcal{Z}}(x,t|x_0,t_0)$ is also solution of a forward Kolmogorov equation, also known as the Fokker–Planck equation (cf. Dynkin [7]), in which the backward variables x_0 and t_0 are essentially constant. In this paper, we choose to use the Kolmogorov backward equation because we will address absorption problems. Indeed, if one is interested to the first-passage time distribution through a fixed state S as a function of the initial position x_0, then the backward Kolmogorov equation provides the most appropriate method (cf. Cox and Miller [8]).

For a diffusion process $\mathcal{Z}(t)$, the first-passage time (FPT) problem can be reduced to estimate the density of the random variable

$$T_{\mathcal{Z}}(S|x_0,t_0) = \begin{cases} \inf_{t\geq t_0}\{t : \mathcal{Z}(t) \geq S\}, & \mathcal{Z}(t_0) = x_0 < S, \\ \inf_{t\geq t_0}\{t : \mathcal{Z}(t) \leq S\}, & \mathcal{Z}(t_0) = x_0 > S, \end{cases}$$

which describes the FPT of $\mathcal{Z}(t)$ through the state S starting from $\mathcal{Z}(t_0) = x_0 \neq S$.

The FPT problem plays an important role in various biological applications. For instance, in the context of population dynamics the FPT problem is suitable to model population's extinction or persistence (see Bailey [1], Ricciardi [2], Allen [9,10]).

Let $g_{\mathcal{Z}}(S,t|x_0,t_0) = dP\{T_{\mathcal{Z}}(S|x_0,t_0) \leq t\}/dt$ be the FPT density, being $P\{T_{\mathcal{Z}}(S|x_0,t_0) \leq t\}$ the distribution function of the random variable $T_{\mathcal{Z}}(S|x_0,t_0)$. If the endpoints of $\mathcal{D}$ are unattainable boundaries, the densities $f_{\mathcal{Z}}(x,t|x_0,t_0)$ and $g_{\mathcal{Z}}(S,t|x_0,t_0)$ are related by the following renewal equation (cf. Blake and Lindsey [11]):

$$f_{\mathcal{Z}}(x,t|x_0,t_0) = \int_{t_0}^{t} g_{\mathcal{Z}}(S,\tau|x_0,t_0)\, f_{\mathcal{Z}}(x,t|S,\tau)\, d\tau, \qquad (x_0 < S \leq x) \text{ or } (x \leq S < x_0). \quad (2)$$

Equation (2) indicates that any sample path that reaches $x \geq S$ $[x \leq S]$, after starting from $x_0 < S$ $[x_0 > S]$ at time t_0, must necessarily cross S for the first time at some intermediate instant $\tau \in (t_0,t)$.

For diffusion processes, closed form expressions for FPT densities through constant boundaries are not available, except in some special cases (see Ricciardi et al. [12], Ding and Rangarajan [13], Molini et al. [14], Giorno and Nobile [15], Masoliver [16]). In particular, closed form expressions are available in the following cases: (i) the Wiener process through an arbitrary constant boundary; (ii) the Ornstein–Uhlenbeck process through the boundary in which the drift vanishes; and (iii) the Feller process through the zero state. In the literature many efforts have been devoted to determining the asymptotic behavior of FPT density and its moments for large boundaries or large times and to search efficient numerical and simulation methods to estimate the FPT densities (cf. Ricciardi et al. [12], Linetsky [17]). Furthermore, the FPT problems play a relevant role also in the context of fractional processes (see, for instance, Guo et al. [18], Wiese [19], Abundo [20], Leonenko and Pirozzi [21]).

For a TNH-D process $\mathcal{Z}(t)$ confined to interval $(0,+\infty)$, with 0 absorbing boundary and $+\infty$ unattainable boundary, we denote with

$$a_{\mathcal{Z}}(x,t|x_0,t_0) = \frac{\partial}{\partial x} P\{\mathcal{Z}(t) \leq x;\ \mathcal{Z}(\theta) > 0, \forall\theta < t|\mathcal{Z}(t_0) = x_0\}, \quad x > 0, x_0 > 0$$

the PDF of $\mathcal{Z}(t)$ with an absorbing condition in the zero state. The PDF $a_{\mathcal{Z}}(x,t|x_0,t_0)$ satisfies the Kolmogorov Equation (1) with the initial condition $\lim_{t_0 \uparrow t} a_{\mathcal{Z}}(x,t|x_0,t_0) = \delta(x-x_0)$ and the absorbing condition $\lim_{x_0 \downarrow 0} a_{\mathcal{Z}}(x,t|x_0,t_0) = 0$.

The densities $f_{\mathcal{Z}}(x,t|x_0,t_0)$, $g_{\mathcal{Z}}(0,t|x_0,t_0)$, and $a_{\mathcal{Z}}(x,t|x_0,t_0)$ are related by the following integral equations (cf. Siegert [22]):

$$a_{\mathcal{Z}}(x,t|x_0,t_0) = f_{\mathcal{Z}}(x,t|x_0,t_0) - \int_{t_0}^{t} g_{\mathcal{Z}}(0,\theta|x_0,t_0)\, f_{\mathcal{Z}}(x,t|0,\theta)\, d\theta, \quad x_0 > 0, x > 0, \tag{3}$$

$$\int_0^{+\infty} a_{\mathcal{Z}}(x,t|x_0,t_0)\, dx + \int_{t_0}^{t} g_{\mathcal{Z}}(0,\theta|x_0,t_0)\, d\theta = 1, \qquad x_0 > 0. \tag{4}$$

In the context of population dynamics, the first integral in (4) gives the survival probability, i.e., the probability that the trajectories of the process $\mathcal{Z}(t)$ are not absorbed in the zero state in (t_0,t). Moreover, from (4) one obtains the FPT density

$$g_{\mathcal{Z}}(0,t|x_0,t_0) = -\frac{\partial}{\partial t}\int_0^{+\infty} a_{\mathcal{Z}}(x,t|x_0,t_0)\, dx, \qquad x_0 > 0, \tag{5}$$

and the ultimate FPT probability of $\mathcal{Z}(t)$ through the zero-state

$$P_{\mathcal{Z}}(0|x_0,t_0) = \int_{t_0}^{+\infty} g_{\mathcal{Z}}(0,\tau|x_0,t_0)\, d\tau = 1 - \lim_{t\to+\infty}\int_0^{+\infty} a_{\mathcal{Z}}(x,t|x_0,t_0)\, dx, \quad x_0 > 0. \tag{6}$$

In population dynamics, $g_{\mathcal{Z}}(0,t|x_0,t_0)$ in Equation (5) represents the density of the time required to reach the zero state for the first time (extinction density); instead, $P_{\mathcal{Z}}(0|x_0,t_0)$ in Equation (6) provides the probability that the population will become extinct sooner or later.

For a TNH-D process $\mathcal{Z}(t)$, the *local time* $\mathcal{L}(t,x|t_0)$ at an interior state $x \in \mathcal{D}$ is a random variable defined as (cf. Karlin and Taylor [23], Aït-Sahalia and Park [24]):

$$\mathcal{L}(t,x|t_0) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_{t_0}^{t} 1\{|\mathcal{Z}(\theta)-x| \le \varepsilon\}\, d\theta, \qquad t > t_0, \tag{7}$$

where, for $\varepsilon > 0$, we have set

$$1\{|\mathcal{Z}(\theta)-x| \le \varepsilon\} = \begin{cases} 1, & |\mathcal{Z}(\theta)-x| \le \varepsilon, \\ 0, & \text{otherwise}. \end{cases}$$

The asymptotic average of the local time in the presence of an absorbing boundary in the zero state, for $x > 0$ and $x_0 > 0$ is:

$$L_{\mathcal{Z}}(x|x_0,t_0) = \lim_{t\to+\infty} E[\mathcal{L}(t,x|t_0)|\mathcal{Z}(t_0) = x_0] = \int_{t_0}^{+\infty} a_{\mathcal{Z}}(x,\theta|x_0,t_0)\, d\theta. \tag{8}$$

For a time-homogeneous diffusion (TH-D) process $\mathcal{Z}(t)$ one has $\zeta_1(x,t) = \zeta_1(x)$ and $\zeta_2(x,t) = \zeta_2(x)$ for all t. In this case, the classification of the endpoints of the state space $\mathcal{D}$, due to Feller [25,26], is based on integrability properties of the functions

$$h_{\mathcal{Z}}(x) = \exp\Big\{-2\int^{x}\frac{\zeta_1(u)}{\zeta_2(u)}\, du\Big\}, \qquad s_{\mathcal{Z}}(x) = \frac{2}{\zeta_2(u)\, h_{\mathcal{Z}}(u)}, \qquad x \in \mathcal{D}, \tag{9}$$

called scale function and speed density, respectively. Such functions allow us to determine the FPT moments for TH-D processes thanks to the Siegert formula (cf. Siegert [22]). Specifically, if $\mathcal{Z}(t)$ is a TH-D process with state space $\mathcal{D} = (r_1,r_2)$, for $n = 1,2,\ldots$ it results in

- for $x_0 < S$, if $P_{\mathcal{Z}}(S|x_0) = \int_0^{+\infty} g_{\mathcal{Z}}(S,t|x_0)\, dt = 1$ and if $\int_{r_1}^{z} s_{\mathcal{Z}}(u)\, du$ converges one has:

$$t_n^{(\mathcal{Z})}(S|x_0) = \int_0^{+\infty} t^n \, g_{\mathcal{Z}}(S,t|x_0)\, dt = n \int_{x_0}^{S} dz\, h_{\mathcal{Z}}(z) \int_{r_1}^{z} s_{\mathcal{Z}}(u)\, t_{n-1}(S|u)\, du, \quad x_0 < S, \tag{10}$$

- for $x_0 > S$, if $P_{\mathcal{Z}}(S|x_0) = 1$, and if $\int_z^{r_2} s_{\mathcal{Z}}(u)\, du$ converges one has

$$t_n^{(\mathcal{Z})}(S|x_0) = \int_0^{+\infty} t^n \, g_{\mathcal{Z}}(S,t|x_0)\, dt = n \int_{S}^{x_0} dz\, h_{\mathcal{Z}}(z) \int_{z}^{r_2} s_{\mathcal{Z}}(u)\, t_{n-1}(S|u)\, du, \quad x_0 > S, \tag{11}$$

with $t_0^{(\mathcal{Z})}(S|x_0) = P_{\mathcal{Z}}(S|x_0)$.

In the sequel, for the FPT of TH-D process $\mathcal{Z}(t)$ we denote by

$$\mathrm{Var}^{(\mathcal{Z})}(S|x_0) = t_2^{(\mathcal{Z})}(S|x_0) - [t_1^{(\mathcal{Z})}(S|x_0)]^2, \qquad \mathrm{Cv}^{(\mathcal{Z})}(S|x_0) = \frac{\sqrt{\mathrm{Var}^{(\mathcal{Z})}(S|x_0)}}{t_1^{(\mathcal{Z})}(S|x_0)},$$

$$\Sigma^{(\mathcal{Z})}(S|x_0) = \frac{t_3^{(\mathcal{Z})}(S|x_0) - 3\, t_1^{(\mathcal{Z})}(S|x_0)\, t_2^{(\mathcal{Z})}(S|x_0) + 2\, [t_1^{(\mathcal{Z})}(S|x_0)]^3}{[\mathrm{Var}^{(\mathcal{Z})}(S|x_0)]^{3/2}}.$$

the variance, the coefficient of variation, and the skewness, respectively.

For a TH-D process in $(0, +\infty)$, with 0 absorbing boundary, if $x_0 > 0$ and $x > 0$ the asymptotic average of the local time is (cf. Giorno and Nobile [27]):

$$L_{\mathcal{Z}}(x|x_0) = \begin{cases} s_{\mathcal{Z}}(x) \int_0^{x_0 \wedge x} h_{\mathcal{Z}}(z) dz, & +\infty \text{ unattainable, nonattracting,} \\ \\ s_{\mathcal{Z}}(x) P_{\mathcal{Z}}(0|x \vee x_0) \int_0^{x_0 \wedge x} h_{\mathcal{Z}}(z) dz, & +\infty \text{ unattainable, attracting,} \end{cases} \tag{12}$$

where $x_0 \wedge x = \min(x_0, x)$ and $x_0 \vee x = \max(x_0, x)$.

For a TH-D process $\mathcal{Z}(t)$, in the sequel we denote by

$$\varphi_\lambda^{(\mathcal{Z})}(x|x_0) = \int_0^{+\infty} e^{-\lambda\, t} \varphi_{\mathcal{Z}}(x,t|x_0)\, dt$$

the Laplace transform (LT) of the function $\varphi_{\mathcal{Z}}(x,t|x_0)$.

Plan of the Paper

In Section 2, we consider the time-inhomogeneous Wiener (TNH-W) process $X(t)$, with infinitesimal drift and infinitesimal variance $A_1(t) = \beta(t)$ and $A_2(t) = \sigma^2(t)$, respectively. For $\beta(t) = \gamma\, \sigma^2(t)$, with $\gamma \in \mathbb{R}$, we determine the PDF $a_X(x,t|x_0,t_0)$ and the FPT density $g_X(0,t|x_0,t_0)$. Furthermore, for the time-homogeneous Wiener (TH-W) process, the FPT moments through a boundary $S \in \mathbb{R}$ and the asymptotic average of the local time are studied.

In Section 3, we take into account the time-inhomogeneous Ornstein–Uhlenbeck (TNH-OU) process $Y(t)$, with infinitesimal drift and infinitesimal variance $B_1(x,t) = \alpha(t)\, x + \beta(t)$ and $B_2(t) = \sigma^2(t)$, respectively. For $\beta(t) = \gamma\, \sigma^2(t)\, e^{-A(t|0)}$, with $\gamma \in \mathbb{R}$ and $A(t|0) = \int_0^t \alpha(u)\, du$, we determine $a_Y(x,t|x_0,t_0)$ and $g_Y(0,t|x_0,t_0)$. Moreover, for the TH-OU process, the FPT mean through a constant boundary and the asymptotic average of the local time are evaluated.

In Section 4, we consider the time-inhomogeneous Feller (TNH-F) process $Z(t)$ with infinitesimal drift and infinitesimal variance $C_1(x,t) = \alpha(t)\, x + \beta(t)$ and $C_2(x,t) = 2\, r(t)\, x$, respectively, with an absorbing boundary in the zero-state. For $\beta(t) = \xi\, r(t)$, with $0 \leq \xi < 1$, we obtain $a_Z(x,t|x_0,t_0)$ and $g_Z(0,t|x_0,t_0)$. Furthermore, for the TH-F process, the FPT mean through a constant boundary and the asymptotic average of the local time are examined.

We remark that time-inhomogeneous Wiener, Ornstein–Uhlenbeck and Feller diffusion processes are used in biological systems to model the growth of a population. In such a context, $\alpha(t)$ represents the growth intensity function and $\beta(t)$ denotes the immigration/emigration intensity function. The functions $\sigma^2(t)$ (in Wiener and Ornstein–Uhlenbeck processes) and $r(t)$ (in the Feller process) are the noise intensity functions and take into account the environmental fluctuations.

In Sections 2–4, by using Siegert Formulas (10) and (11), extensive computation are performed with MATHEMATICA to obtain the mean, the variance, the coefficient of variation, and the skewness of FPT for the TH-W, TH-OU, and TH-F processes for various choices of parameters. For these processes, some considerations on the asymptotic average of the local time in the presence of an absorbing boundary in the zero state are also made.

In Section 5, for $\beta(t) = r(t)/2$, some relationships between the PDF in the presence of an absorbing boundary in the zero state and between the FPT densities through zero for Wiener, Ornstein–Uhlenbeck and Feller processes are proved. Moreover, for $\beta(t) = \xi\, r(t)$ $(0 < \xi < 1)$ some asymptotic results for large times between the FPT densities are provided.

2. Wiener-Type Diffusion Process

Let $\{X(t), t \geq t_0\}$, $t_0 \geq 0$, be a TNH-W process, having infinitesimal drift and infinitesimal variance

$$A_1(t) = \beta(t), \qquad A_2(t) = \sigma^2(t), \tag{13}$$

with the state space $\mathbb{R}$, where $\beta(t) \in \mathbb{R}$ and $\sigma(t) > 0$ are continuous functions.

The Wiener process arises as the mathematical limit of other stochastic processes, such as random walks (see Knight [28]). This process has been originally used in physics to model the motion of particles suspended in a fluid and it is still used as a mathematical model for various random phenomena in applied mathematics, economics, quantitative finance, evolutionary biology, and physics.

For $t \geq t_0$, the PDF of $X(t)$ is normal,

$$f_X(x,t|x_0,t_0) = \frac{1}{\sqrt{2\pi V_X(t|t_0)}}\, \exp\Big\{-\frac{\big[x - M_X(t|x_0,t_0)\big]^2}{2\, V_X(t|t_0)}\Big\}, \qquad x, x_0 \in \mathbb{R}, \tag{14}$$

with

$$M_X(t|x_0,t_0) = x_0 + \int_{t_0}^{t} \beta(u)\, du, \qquad V_X(t|t_0) = \int_{t_0}^{t} \sigma^2(u)\, du.$$

We now consider the TNH-W process $X(t)$, having infinitesimal moments given in (13), restricted to the state space $(0, +\infty)$ with 0 absorbing boundary; we denote by $a_X(x,t|x_0,t_0)$ its PDF. For the Wiener process $X(t)$ in the presence of an absorbing boundary in the zero state, we analyze two cases: the proportional case with $\beta(t) = \gamma\, \sigma^2(t)$, being $\gamma \in \mathbb{R}$ and $\sigma(t) > 0$, and the time-homogeneous case.

2.1. Proportional Case for the Wiener Process

Proposition 1. *Let $\beta(t) = \gamma\, \sigma^2(t)$, with $\gamma \in \mathbb{R}$ and $\sigma(t) > 0$ in (13). For the TNH-W process $X(t)$ one has*

$$a_X(x,t|x_0,t_0) = f_X(x,t|x_0,t_0) - e^{2\gamma x} f_X(-x,t|x_0,t_0), \qquad x > 0, x_0 > 0, \tag{15}$$

with $f_X(x,t|x_0,t_0)$ given in (14).

Proof. If $\beta(t) = \gamma\, \sigma^2(t)$, from (14) the following symmetry relation holds,

$$f_X(x,t|0,t_0) = e^{2\gamma x}\, f_X(-x,t|0,t_0), \qquad x \in \mathbb{R},$$

so that from (3) one has

$$a_X(x,t|x_0,t_0) = f_X(x,t|x_0,t_0) - e^{2\gamma x} \int_{t_0}^{t} g_X(0,\tau|x_0,t_0)\, f_X(-x,t|0,\tau)\, d\tau, \qquad x_0 > 0, x > 0. \tag{16}$$

Hence, by virtue of the renewal Equation (2), Equation (15) follows from (16). □

From (15), for $\beta(t) = \gamma\,\sigma^2(t)$, with $\gamma \in \mathbb{R}$ and $\sigma(t) > 0$, one explicitly obtains

$$a_X(x,t|x_0,t_0) = \frac{1}{\sqrt{2\pi V_X(t|t_0)}}\,\Bigg[\exp\Big\{-\frac{\big[x-x_0-\gamma\,V_X(t|t_0)\big]^2}{2\,V_X(t|t_0)}\Big\} - e^{2\gamma\,x}\exp\Big\{-\frac{\big[x+x_0+\gamma\,V_X(t|t_0)\big]^2}{2\,V_X(t|t_0)}\Big\}\Bigg], \qquad x_0>0, x>0. \tag{17}$$

We note that Equation (17) for $t_0 = 0$ is in agreement with Equation (25) in Molini et al. [14].

Proposition 2. *Under the assumptions of Proposition 1, for the TNH-W process $X(t)$ one has*

$$g_X(0,t|x_0,t_0) = \frac{x_0\,\sigma^2(t)}{\sqrt{2\pi\,\big[V_X(t|t_0)\big]^3}}\,\exp\Big\{-\frac{\big[x_0+\gamma\,V_X(t|t_0)\big]^2}{2V_X(t|t_0)}\Big\}, \qquad x_0>0. \tag{18}$$

Moreover, if $\lim_{t\to+\infty} V_X(t|t_0) = +\infty$, the ultimate FPT probability of $X(t)$ through zero is

$$P_X(0|x_0,t_0) = \int_{t_0}^{+\infty} g_X(0,t|x_0,t_0)\,dt = \begin{cases} 1, & \gamma \le 0, \\ e^{-2\gamma\,x_0}, & \gamma > 0, \end{cases} \qquad x_0 > 0. \tag{19}$$

Proof. For $x_0 > 0$, from (17) one obtains

$$\int_0^{+\infty} a_X(x,t|x_0,t_0)\,dx = \frac{1}{2}\Bigg[1+\mathrm{Erf}\Big(\frac{x_0+\gamma\,V_X(t|t_0)}{\sqrt{2\,V_X(t|t_0)}}\Big) - e^{-2\gamma x_0}\,\mathrm{Erfc}\Big(\frac{x_0-\gamma\,V_X(t|t_0)}{\sqrt{2\,V_X(t|t_0)}}\Big)\Bigg], \tag{20}$$

where $\mathrm{Erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-z^2}\,dz$ denotes the error function and $\mathrm{Erfc}(x) = 1-\mathrm{Erf}(x)$ is the complementary error function. Hence, due to (5) and recalling (20), Equation (18) follows. Finally, if $\lim_{t\to+\infty} V_X(t|t_0) = +\infty$, Equation (19) follows, making use of (20) in (6) and by noting that

$$\lim_{t\to+\infty}\mathrm{Erf}\Big(\frac{x_0+\gamma\,V_X(t|t_0)}{\sqrt{2\,V_X(t|t_0)}}\Big) = \begin{cases} -1, & \gamma<0, \\ 0, & \gamma=0, \\ 1, & \gamma>0, \end{cases} \qquad \lim_{t\to+\infty}\mathrm{Erfc}\Big(\frac{x_0-\gamma\,V_X(t|t_0)}{\sqrt{2\,V_X(t|t_0)}}\Big) = \begin{cases} 0, & \gamma<0, \\ 1, & \gamma=0, \\ 2, & \gamma>0, \end{cases}$$

for any x_0. □

Equation (19) shows that if $\beta(t) = \gamma\,\sigma^2(t)$, with $\gamma \in \mathbb{R}$ and $\sigma(t) > 0$ in (13), the first-passage for the Wiener process through zero is a sure event for $\gamma > 0$ and $x_0 > 0$.

2.2. Time-Homogeneous Case for the Wiener Process

We consider the TH-W process, obtained from (13) by setting $\beta(t) = \beta$ and $\sigma^2(t) = \sigma^2$, with $\beta \in \mathbb{R}$ and $\sigma > 0$. When $\beta > 0$ ($\beta < 0$) the end point $-\infty$ is a nonattracting (attracting) natural boundary and the end point $+\infty$ is an attracting (nonattracting) natural boundary. Instead, for $\beta = 0$ the end points $-\infty$ and $+\infty$ are nonattracting natural boundaries. The scale function and the speed density, defined in (9) for the TH-W process $X(t)$ are

$$h_X(x) = \exp\Big\{-\frac{2\beta}{\sigma^2}\,x\Big\}, \qquad s_X(x) = \frac{2}{\sigma^2}\,\exp\Big\{\frac{2\beta}{\sigma^2}\,x\Big\}, \tag{21}$$

respectively.

The FPT density of the TH-W process $X(t)$ through the constant boundary S starting from x_0 is

$$g_X(S,t|x_0) = \frac{|S-x_0|}{\sigma\sqrt{2\pi\,t^3}}\,\exp\Big\{-\frac{(S-x_0-\beta\,t)^2}{2\sigma^2\,t}\Big\}, \qquad S \ne x_0 \tag{22}$$

and the ultimate FPT probability is

$$P_X(S|x_0) = \int_0^{+\infty} g_X(S,t|x_0)\,dt = \begin{cases} 1, & \beta = 0 \text{ or } \beta(S-x_0) > 0, \\ \exp\left\{\frac{2\beta(S-x_0)}{\sigma^2}\right\}, & \beta(S-x_0) < 0. \end{cases} \tag{23}$$

For $\beta(S-x_0) > 0$, the FPT moments of the TH-W process $X(t)$ are finite and from (22) one has

$$t_n^{(X)}(S|x_0) = \frac{2\,|S-x_0|}{\sigma\sqrt{2\pi}} \left(\frac{S-x_0}{\beta}\right)^{n-1/2} \exp\left\{\frac{\beta(S-x_0)}{\sigma^2}\right\} K_{n-1/2}\left[\frac{\beta(S-x_0)}{\sigma^2}\right], \quad n = 1,2,\ldots$$

where $K_\nu(z)$ denotes the modified Bessel function of the third kind, which can be expressed in terms of the modified Bessel function of first kind $I_\nu(z)$ (see Abramowitz and Stegun [29], p. 375, n. 9.6.2),

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\,\pi)}, \qquad I_\nu(z) = \sum_{k=0}^{+\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{24}$$

where $\Gamma(\nu) = \int_0^{+\infty} y^{\nu-1}\,e^{-y}\,dy$, with $\mathrm{Re}\,\nu > 0$, is the Euler gamma function.

In particular, for $\beta(S-x_0) > 0$ the first three FPT moments of the TH-W process $X(t)$ are

$$t_1^{(X)}(S|x_0) = \frac{S-x_0}{\beta}, \qquad t_2^{(X)}(S|x_0) = \left(\frac{S-x_0}{\beta}\right)^2 \left\{1 + \frac{\sigma^2}{\beta\,(S-x_0)}\right\},$$
$$t_3^{(X)}(S|x_0) = \left(\frac{S-x_0}{\beta}\right)^3 \left\{1 + \frac{3\,\sigma^2}{\beta\,(S-x_0)} + \frac{3\,\sigma^4}{\beta^2\,(S-x_0)^2}\right\}.$$

In Tables 1 and 2, the mean $t_1^{(X)}(S|x_0)$, the variance $\mathrm{Var}^{(X)}(S|x_0)$, the coefficient of variation $\mathrm{Cv}^{(X)}(S|x_0)$, and the skewness $\Sigma^{(X)}(S|x_0)$ of the FPT are listed for $x_0 = 4$, $\sigma = 1$ and some choices of β and S.

Table 1. For the Wiener process, with $A_1(x) = \beta$ and $A_2(x) = 1$, the mean, the variance, the coefficient of variation, and the skewness of FPT are listed for $x_0 = 4$, $\beta = 0.1, 0.2$ and for increasing values the boundary $S > x_0$.

	S	$t_1^{(X)}(S\|x_0)$	$\mathrm{Var}^{(X)}(S\|x_0)$	$\mathrm{Cv}^{(X)}(S\|x_0)$	$\Sigma^{(X)}(S\|x_0)$
$\beta = 0.1$	100	960	96,000	0.322749	0.968246
	500	4960	496,000	0.141990	0.425971
	1000	9960	996,000	0.100201	0.300602
	1500	14,960	1,496,000	0.0817587	0.245276
	2000	19,960	1,996,000	0.0707815	0.212344
	2500	24,960	2,496,000	0.0632962	0.189889
	3000	29,960	2,996,000	0.0577736	0.173321
$\beta = 0.2$	100	480	12,000	0.228218	0.684653
	500	2480	62,000	0.100402	0.301207
	1000	4980	124,500	0.0708525	0.212558
	1500	7480	187,000	0.0578122	0.173436
	2000	9980	249,500	0.0500501	0.150150
	2500	12,480	312,000	0.0447572	0.134272
	3000	14,980	374,500	0.0408521	0.122556

Table 2. As in Table 1, with $x_0 = 4$, $\sigma = 1$, $\beta = -0.1, -0.2$ and for decreasing values the boundary $S \in [0, x_0)$.

	S	$t_1^{(X)}(S\|x_0)$	$\mathrm{Var}^{(X)}(S\|x_0)$	$\mathrm{Cv}^{(X)}(S\|x_0)$	$\Sigma^{(X)}(S\|x_0)$
$\beta = -0.1$	3.5	5	500	4.47214	13.4164
	3.0	10	1000	3.16228	9.48683
	2.5	15	1500	2.58199	7.74597
	2.0	20	2000	2.23607	6.7082
	1.5	25	2500	2.0	6.0
	1.0	30	3000	1.82574	5.47723
	0.5	35	3500	1.69031	5.07093
	0.0	40	4000	1.58114	4.74342
$\beta = -0.2$	3.5	2.5	62.5	3.16228	9.48683
	3.0	5	125	2.23607	6.7082
	2.5	7.5	187.5	1.82574	5.47723
	2.0	10	250	1.58114	4.74342
	1.5	12.5	312.5	1.41421	4.24264
	1.0	15	375	1.29099	3.87298
	0.5	17.5	437.5	1.19523	3.58569
	0.0	20	500	1.11803	3.3541

As shown in Tables 1 and 2, for the TH-W process $X(t)$ the coefficient of variation and the skewness of the FPT decrease when S moves away from x_0.

Moreover, by setting $\beta(t) = \beta$, $\sigma^2(t) = \sigma^2$ and $\gamma = \beta/\sigma^2$ in (17), for the TH-W process $X(t)$ one has

$$a_X(x,t|x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \left[\exp\left\{ -\frac{(x - x_0 - \beta t)^2}{2\sigma^2 t} \right\} - \exp\left\{ \frac{2\beta x}{\sigma^2} \right\} \exp\left\{ -\frac{(x + x_0 + \beta t)^2}{2\sigma^2 t} \right\} \right] \tag{25}$$

with $x_0 > 0$ and $x > 0$.

Proposition 3. *For the TH-W process $X(t)$, the asymptotic average of the local time is*

$$\begin{aligned} L_X(x|x_0) &= \int_0^{+\infty} a_X(x,t|x_0)\, dt \\ &= \begin{cases} \frac{1}{|\beta|} \exp\left\{ \frac{\beta(x-x_0)}{\sigma^2} \right\} \left[\exp\left\{ -\frac{|\beta(x-x_0)|}{\sigma^2} \right\} - \exp\left\{ -\frac{|\beta(x+x_0)|}{\sigma^2} \right\} \right], & \beta \neq 0, \\ \frac{|x+x_0|}{\sigma^2} - \frac{|x-x_0|}{\sigma^2} = \frac{2\,(x_0 \wedge x)}{\sigma^2}, & \beta = 0, \end{cases} \end{aligned} \tag{26}$$

with $x_0 \wedge x = \min(x_0, x)$ and $x_0 \vee x = \max(x_0, x)$.

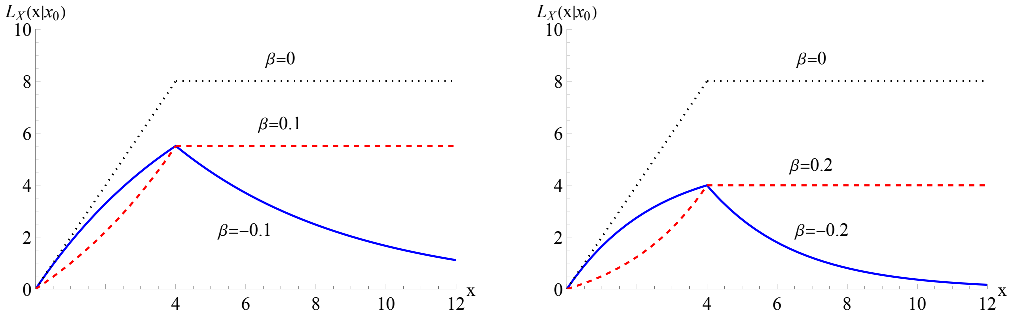
Proof. Because $+\infty$ is a nonattracting boundary for $\beta \leq 0$ and attracting for $\beta > 0$, Equation (26) follows from (12) making use of (21) and (23). □

From (26), for $\beta \in \mathbb{R}$ and $\sigma > 0$ one has $\lim_{x \downarrow 0} L_X(x|x_0) = 0$ and

$$\lim_{x \uparrow +\infty} L_X(x|x_0) = \begin{cases} 0, & \beta < 0, \\ \frac{2\,x_0}{\sigma^2}, & \beta = 0, \\ \frac{1 - e^{-2\beta x_0/\sigma^2}}{\beta}, & \beta > 0. \end{cases}$$

We note that $L_X(x|x_0)$ tends to zero as x increases if $\beta < 0$, and it approaches a positive value when $\beta \geq 0$.

In Figure 1, the asymptotic average of the local time for the TH-W process $X(t)$ is plotted for $x_0 = 4$, $\sigma = 1$ and some choices of β. We note that $L_X(x|x_0)$ tends to zero only if $\beta < 0$, otherwise approaches to a positive value.



(a) **(b)**

Figure 1. $L_X(x|x_0)$, given in (26), with $x_0 = 4$, $\sigma = 1$ and some choices of β. In (**a**) $\beta = -0.1, 0, 0.1$ and in (**b**) $\beta = -0.2, 0, 0.2$.

3. Ornstein–Uhlenbeck-Type Diffusion Process

Let $\{Y(t), t \geq t_0\}$, $t_0 \geq 0$, be a TNH-OU process, having infinitesimal drift and infinitesimal variance

$$B_1(x,t) = \alpha(t)\, x + \beta(t), \qquad B_2(t) = \sigma^2(t), \qquad x \in \mathbb{R}, \tag{27}$$

with state space $\mathbb{R}$, where $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$, $\sigma(t) > 0$ are continuous functions. Note that when $\alpha(t) = 0$ for all t, the process $Y(t)$ identifies with the TNH-W process $X(t)$ having infinitesimal moments (13).

Although the Ornstein–Uhlenbeck process has been originally used in physics to model the velocity of a Brownian particle (see Uhlenbeck and Ornstein [30]), it finds many applications in several scientific fields. In particular, the Ornstein–Uhlenbeck process is frequently proposed as a stochastic model for the single neuronal activity (see Ricciardi and Sacerdote [31], Lánský and Ditlevsen [32]). A wide field of applications of the Ornstein–Uhlenbeck process lies also in mathematical finance to model the evolution of the interest rate of financial markets (cf. Vasicek [33], Hull and White [34]).

The PDF of $Y(t)$ is normal,

$$f_Y(x,t|x_0,t_0) = \frac{1}{\sqrt{2\pi V_Y(t|t_0)}}\, \exp\Big\{-\frac{[x - M_Y(t|x_0,t_0)]^2}{2\, V_Y(t|t_0)}\Big\}, \qquad x, x_0 \in \mathbb{R}, \tag{28}$$

with

$$M_Y(t|x_0,t_0) = x_0\, e^{A(t|t_0)} + \int_{t_0}^{t} \beta(\theta)\, e^{A(t|\theta)}\, d\theta, \qquad V_Y(t|t_0) = \int_{t_0}^{t} \sigma^2(\theta)\, e^{2A(t|\theta)}\, d\theta, \tag{29}$$

being

$$A(t|t_0) = \int_{t_0}^{t} \alpha(\theta)\, d\theta. \tag{30}$$

We now consider the TNH-OU process $Y(t)$, having infinitesimal moments given in (27), restricted to the state space $(0, +\infty)$, with 0 absorbing boundary, and denote by $a_Y(x,t|x_0,t_0)$ its PDF. For the TNH-OU process $Y(t)$ with 0 absorbing boundary, we take into account two cases: the proportional case in which $\beta(t) = \gamma\, \sigma^2(t)\, e^{-A(t|0)}$, with $\gamma \in \mathbb{R}$, $\alpha(t) \in \mathbb{R}$ and $\sigma(t) > 0$, and the time-homogeneous case.

3.1. Proportional Case for the Ornstein-Uhlenbeck Process

Proposition 4. *Let $\beta(t) = \gamma\,\sigma^2(t)\,e^{-A(t|0)}$, with $\gamma \in \mathbb{R}$, $\alpha(t) \in \mathbb{R}$, $\sigma(t) > 0$ in (27) and $A(t|0)$ defined in (30). For the TNH-OU process $Y(t)$ one has*

$$a_Y(x,t|x_0,t_0) = f_Y(x,t|x_0,t_0) - \exp\Big\{2\gamma\,x\,e^{-A(t|0)}\Big\}\,f_Y(-x,t|x_0,t_0), \qquad x>0, x_0>0, \tag{31}$$

with $f_Y(x,t|x_0,t_0)$ given in (28).

Proof. By choosing $\beta(t) = \gamma\,\sigma^2(t)\,e^{-A(t|0)}$, from (28) the following symmetry relation holds,

$$f_Y(x,t|0,t_0) = \exp\Big\{2\gamma\,x\,e^{-A(t|0)}\Big\}\,f_Y(-x,t|0,t_0), \qquad x \in \mathbb{R},$$

so that from (3) one obtains

$$a_Y(x,t|x_0,t_0) = f_Y(x,t|x_0,t_0) - \exp\Big\{2\gamma\,x\,e^{-A(t|0)}\Big\}\int_{t_0}^{t} g_Y(0,\tau|x_0,t_0)\,f_Y(-x,t|0,\tau)\,d\tau \tag{32}$$

for $x_0 > 0$ and $x > 0$. Hence, by virtue of the renewal Equation (2), Equation (31) follows from (32). □

From (31), if $\beta(t) = \gamma\,\sigma^2(t)\,e^{-A(t|0)}$, for $x_0 > 0$ and $x > 0$ one obtains

$$\begin{aligned} a_Y(x,t|x_0,t_0) = \frac{1}{\sqrt{2\pi V_Y(t|t_0)}}\Bigg[&\exp\Bigg\{-\frac{\big[x - x_0\,e^{A(t|t_0)} - \gamma\,e^{-A(t|0)}\,V_Y(t|t_0)\big]^2}{2\,V_Y(t|t_0)}\Bigg\} \\ &- \exp\Big\{2\gamma\,x\,e^{-A(t|0)}\Big\}\,\exp\Bigg\{-\frac{\big[x + x_0\,e^{A(t|t_0)} + \gamma\,e^{-A(t|0)}\,V_Y(t|t_0)\big]^2}{2\,V_Y(t|t_0)}\Bigg\}\Bigg]. \end{aligned} \tag{33}$$

Proposition 5. *Under the assumptions of Proposition 4, for the TNH-OU process $Y(t)$ one has*

$$g_Y(0,t|x_0,t_0) = \frac{x_0\,\sigma^2(t)\,e^{A(t|t_0)}}{\sqrt{2\pi\,[V_Y(t|t_0)]^3}}\,\exp\Bigg\{-\frac{\big[x_0\,e^{A(t|t_0)} + \gamma\,e^{-A(t|0)}V_Y(t|t_0)\big]^2}{2\,V_Y(t|t_0)}\Bigg\}, \qquad x_0 > 0. \tag{34}$$

Furthermore, if $\lim_{t\to+\infty}[e^{-2A(t|t_0)}V_Y(t|t_0)] = +\infty$, the ultimate FPT probability for $x_0 > 0$ is

$$P_Y(0|x_0,t_0) = \int_{t_0}^{+\infty} g_Y(0,t|x_0,t_0)\,dt = \begin{cases} 1, & \gamma \le 0, \\ \exp\Big\{-2\gamma\,x_0\,e^{-A(t_0|0)}\Big\}, & \gamma > 0. \end{cases} \tag{35}$$

Proof. Recalling (33), one obtains

$$\begin{aligned} \int_0^{+\infty} a_Y(x,t|x_0,t_0)\,dx = \frac{1}{2}\Bigg[&1 + \mathrm{Erf}\Bigg(\frac{x_0\,e^{A(t|t_0)} + \gamma\,e^{-A(t|0)}\,V_Y(t|t_0)}{\sqrt{2\,V_Y(t|t_0)}}\Bigg) \\ &- \exp\Big\{-2\gamma\,x_0\,e^{-A(t_0|0)}\Big\}\,\mathrm{Erfc}\Bigg(\frac{x_0\,e^{A(t|t_0)} - \gamma\,e^{-A(t|0)}\,V_Y(t|t_0)}{\sqrt{2\,V_Y(t|t_0)}}\Bigg)\Bigg], \qquad x_0 > 0. \end{aligned} \tag{36}$$

By virtue of (5) and recalling (36), Equation (34) follows. Moreover, under the assumption $\lim_{t\to+\infty}[e^{-2A(t|t_0)}V_Y(t|t_0)] = +\infty$, Equation (35) follows, making use of (36) in (6) by noting that

$$\lim_{t\to+\infty} \mathrm{Erf}\left(\frac{x_0\, e^{A(t|t_0)} + \gamma\, e^{-A(t|0)}\, V_Y(t|t_0)}{\sqrt{2\, V_Y(t|t_0)}}\right) = \begin{cases} -1, & \gamma < 0, \\ 0, & \gamma = 0, \\ 1, & \gamma > 0, \end{cases}$$

$$\lim_{t\to+\infty} \mathrm{Erfc}\left(\frac{x_0\, e^{A(t|t_0)} - \gamma\, e^{-A(t|0)}\, V_Y(t|t_0)}{\sqrt{2\, V_Y(t|t_0)}}\right)\Bigg] = \begin{cases} 0, & \gamma < 0, \\ 1, & \gamma = 0, \\ 2, & \gamma > 0, \end{cases}$$

for any x_0. $\square$

3.2. Time-Homogeneous Case for the Ornstein–Uhlenbeck Process

We consider the TH-OU process $Y(t)$, by setting in (27) $\alpha(t) = \alpha$, $\beta(t) = \beta$, $\sigma^2(t) = \sigma^2$, with $\alpha \neq 0$, $\beta \in \mathbb{R}$ and $\sigma > 0$. The end points $-\infty$ and $+\infty$ are nonattracting natural boundaries for $\alpha < 0$ and attracting natural boundaries for $\alpha > 0$. The scale function and the speed density, defined in (9), for the TH-OU process $Y(t)$ are

$$h_Y(x) = \exp\left\{-\frac{\alpha}{\sigma^2}\left(x^2 + \frac{2\beta}{\alpha}\, x\right)\right\}, \qquad s_Y(x) = \frac{2}{\sigma^2}\, \exp\left\{\frac{\alpha}{\sigma^2}\left(x^2 + \frac{2\beta}{\alpha}\, x\right)\right\}, \tag{37}$$

respectively. The LT of $f_Y(x,t|x_0)$ is

$$f_\lambda^{(Y)}(x|x_0) = \begin{cases} \dfrac{2^{\frac{\lambda}{|\alpha|}-1}}{\sigma\pi\sqrt{|\alpha|}}\Gamma\left(\frac{\lambda}{2|\alpha|}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2|\alpha|}\right)\exp\left\{-\frac{|\alpha|}{2\sigma^2}\left[\left(x+\frac{\beta}{\alpha}\right)^2 - \left(x_0+\frac{\beta}{\alpha}\right)^2\right]\right\} & \\ \quad\times D_{-\frac{\lambda}{|\alpha|}}\left(-\frac{\sqrt{2|\alpha|}}{\sigma}\left[x_0\wedge x+\frac{\beta}{\alpha}\right]\right)D_{-\frac{\lambda}{|\alpha|}}\left(\frac{\sqrt{2|\alpha|}}{\sigma}\left[x_0\vee x+\frac{\beta}{\alpha}\right]\right), & \alpha < 0, \\ & \\ \dfrac{2^{\frac{\lambda}{\alpha}}}{\sigma\pi\sqrt{\alpha}}\Gamma\left(1+\frac{\lambda}{2\alpha}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2\alpha}\right)\exp\left\{-\frac{\alpha}{2\sigma^2}\left[\left(x_0+\frac{\beta}{\alpha}\right)^2 - \left(x+\frac{\beta}{\alpha}\right)^2\right]\right\} & \\ \quad\times D_{-\frac{\lambda}{\alpha}-1}\left(-\frac{\sqrt{2\alpha}}{\sigma}\left[x_0\wedge x+\frac{\beta}{\alpha}\right]\right)D_{-\frac{\lambda}{\alpha}-1}\left(\frac{\sqrt{2\alpha}}{\sigma}\left[x_0\vee x+\frac{\beta}{\alpha}\right]\right), & \alpha > 0, \end{cases} \tag{38}$$

where $D_\nu(z)$ is the parabolic cylinder function defined as (cf. Gradshteyn and Ryzhik [35], p. 1028, no. 9.240). We have

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4}\left\{\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\Phi\left(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\right) - \frac{z\sqrt{2\,\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\Phi\left(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\right)\right\} \tag{39}$$

in terms of Kummer's confluent hypergeometric function

$$\Phi(a,c;x) = 1 + \sum_{n=1}^{+\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!},$$

with $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n = 1,2,\ldots$ In the following, we will make use of the relations (cf. Gradshteyn and Ryzhik [35], p. 1030, no. 9.251 and no. 9.254).

$$D_0(x) = e^{-x^2/4}, \quad D_1(x) = x\, e^{-x^2/4}, \quad D_{-1}(x) = \sqrt{\frac{\pi}{2}}\, e^{x^2/4}\,\mathrm{Erfc}\left(\frac{x}{\sqrt{2}}\right). \tag{40}$$

For the TH-OU process, taking the Laplace transform in (2) and recalling (38), for $x_0 \neq S$ one has

$$g_\lambda^{(Y)}(S|x_0) = \begin{cases} \exp\Big\{\frac{|\alpha|}{2\sigma^2}\Big[\Big(x_0+\frac{\beta}{\alpha}\Big)^2-\Big(S+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} \dfrac{D_{-\frac{\lambda}{|\alpha|}}\Big(\mathrm{sign}(x_0-S)\frac{\sqrt{2|\alpha|}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)}{D_{-\frac{\lambda}{|\alpha|}}\Big(\mathrm{sign}(x_0-S)\frac{\sqrt{2|\alpha|}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)}, & \alpha<0, \\ \exp\Big\{-\frac{\alpha}{2\sigma^2}\Big[\Big(x_0+\frac{\beta}{\alpha}\Big)^2-\Big(S+\frac{\beta}{\alpha}\Big)^2\Big]\Big\} \dfrac{D_{-\frac{\lambda}{\alpha}-1}\Big(\mathrm{sign}(x_0-S)\frac{\sqrt{2\alpha}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)}{D_{-\frac{\lambda}{\alpha}-1}\Big(\mathrm{sign}(x_0-S)\frac{\sqrt{2\alpha}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)}, & \alpha>0, \end{cases} \tag{41}$$

where $\mathrm{sign}(z)$ denotes the sign function that returns -1 if $z<0$, $+1$ if $z>0$ and 0 otherwise. Moreover, by setting $\lambda=0$ in (41) and recalling (40), for $x_0 \neq S$ one has

$$P_Y(S|x_0) = \int_0^{+\infty} g_Y(S,t|x_0)\,dt = \begin{cases} 1, & \alpha<0, \\ \dfrac{\mathrm{Erfc}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big)}{\mathrm{Erfc}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(S+\frac{\beta}{\alpha}\Big)\Big)}, & \alpha>0, \end{cases} \tag{42}$$

so that the first passage through the state S is a sure event for $\alpha<0$.

The inverse LT of $g_\lambda^{(Y)}(S|x_0)$ can be obtained in closed form only if $S=-\beta/\alpha$.

Proposition 6. *For the TH-OU process, the FPT density through the boundary $S=-\beta/\alpha$ is*

$$g_Y\Big(-\frac{\beta}{\alpha},t\Big|x_0\Big) = \frac{2\,e^{\alpha t}|x_0+\beta/\alpha|}{\sigma\sqrt{\pi}}\Big[\frac{\alpha}{e^{2\alpha t}-1}\Big]^{3/2}\exp\Big\{-\frac{\alpha\,e^{2\alpha t}\,(x_0+\beta/\alpha)^2}{\sigma^2(e^{2\alpha t}-1)}\Big\}, \quad x_0\neq-\beta/\alpha, \tag{43}$$

and the ultimate FPT probability is

$$P_Y\Big(-\frac{\beta}{\alpha}\Big|x_0\Big) = \int_0^{+\infty} g_Y\Big(-\frac{\beta}{\alpha},t\Big|x_0\Big)\,dt = \begin{cases} 1, & \alpha<0, \\ \mathrm{Erfc}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(x_0+\frac{\beta}{\alpha}\Big)\Big), & \alpha>0. \end{cases} \tag{44}$$

Proof. Because

$$D_\nu(0) = \frac{2^{\nu/2}\sqrt{\pi}}{\Gamma\Big(\frac{1-\nu}{2}\Big)},$$

from (41) for $\alpha\neq0$ and $x_0\neq-\beta/\alpha$ one has

$$g_\lambda^{(Y)}\Big(-\frac{\beta}{\alpha}\Big|x_0\Big) = \begin{cases} \exp\Big\{\frac{|\alpha|}{2\sigma^2}\Big(x_0+\frac{\beta}{\alpha}\Big)^2\Big\}\frac{2^{\frac{\lambda}{2|\alpha|}}}{\sqrt{\pi}}\Gamma\big(\tfrac12+\tfrac{\lambda}{2|\alpha|}\big)\,D_{-\frac{\lambda}{|\alpha|}}\Big(\frac{\sqrt{2|\alpha|}}{\sigma}\Big|x_0+\frac{\beta}{\alpha}\Big|\Big), & \alpha<0, \\ \exp\Big\{-\frac{\alpha}{2\sigma^2}\Big(x_0+\frac{\beta}{\alpha}\Big)^2\Big\}\frac{2^{\frac{\lambda}{2\alpha}+\frac12}}{\sqrt{\pi}}\Gamma\big(1+\tfrac{\lambda}{2\alpha}\big)\,D_{-\frac{\lambda}{\alpha}-1}\Big(\frac{\sqrt{2\alpha}}{\sigma}\Big|x_0+\frac{\beta}{\alpha}\Big|\Big), & \alpha>0. \end{cases} \tag{45}$$

Equation (43) follows by taking the inverse LT of (45) and making use of the following result (cf. Erdèlyi et al. [36], p. 290, no. 9):

$$\int_0^{+\infty} e^{-pt}\Big[\frac{e^{t/2}}{(e^t-1)^{\nu+1/2}}\exp\Big\{-\frac{\gamma^2}{4(e^t-1)}\Big\}D_{2\nu}\Big(\frac{\gamma}{\sqrt{1-e^{-t}}}\Big)\Big]\,dt = 2^{p+\nu}\Gamma(p+\nu)\,D_{-2p}(\gamma), \qquad \mathrm{Re}\,p>0.$$

Moreover, by setting $\lambda=0$ in (45) and recalling (40), one obtains (44). □

When $\alpha < 0$, the FPT moments through S starting from x_0 can be evaluated by making use of Siegert Formulas (10) and (11) with $r_1 = -\infty$ and $r_2 = +\infty$. In particular, for $n = 1$ and $\alpha < 0$ one has

$$t_1^{(Y)}(S|x_0) = \frac{1}{|\alpha|}\left\{\frac{\pi}{2}\left[\mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0 \vee S + \frac{\beta}{\alpha}\Big)\Big) - \mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0 \wedge S + \frac{\beta}{\alpha}\Big)\Big)\right] + \psi_1\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(S + \frac{\beta}{\alpha}\Big)\Big) - \psi_1\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0 + \frac{\beta}{\alpha}\Big)\Big)\right\}, \qquad x_0 \neq S,,$$

where

$$\mathrm{Erfi}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{u^2}\,du = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{z^{2k+1}}{(2k+1)\,k!}, \qquad \psi_1(z) = \sum_{k=0}^{+\infty}\frac{2^k z^{2k+2}}{(k+1)\,(2k+1)!!}.$$

Furthermore, for $\alpha < 0$ from (10) and (11) one obtains (cf. Ricciardi et al. [12])

$$\lim_{S\to+\infty}\frac{t_n^{(Y)}(S|x_0)}{[t_1^{(Y)}(S|x_0)]^n} = n! \quad (x_0 < S)$$

for $n = 1, 2, \ldots$ so that for $\alpha < 0$ the FPT density of the Ornstein–Uhlenbeck process exhibits an exponential asymptotic behavior as the boundary moves away from the starting point.

In Tables 3 and 4, the mean $t_1^{(Y)}(S|x_0)$, the variance $\mathrm{Var}^{(Y)}(S|x_0)$, the coefficient of variation $\mathrm{Cv}^{(Y)}(S|x_0)$, and the skewness $\Sigma^{(Y)}(S|x_0)$ of the FPT, obtained by using (10) and (11), are listed for $x_0 = 4$, $\alpha = -0.02$, $\sigma = 1$ and some choices of β and S.

Table 3. For the TH-OU process, with $B_1(x) = -0.02\,x + \beta$ and $B_2(x) = 1$, the mean, the variance, the coefficient of variation, and the skewness of FPT are listed for $x_0 = 4$, $\beta = -0.1, 0, 0.1$ and for increasing values the boundary $S > x_0$.

	S	$t_1^{(Y)}(S\|x_0)$	$\mathrm{Var}^{(Y)}(S\|x_0)$	$\mathrm{Cv}^{(Y)}(S\|x_0)$	$\Sigma^{(Y)}(S\|x_0)$
$\beta = -0.1$	5	1.491996×10^2	1.265053×10^5	2.383893	3.759894
	10	3.97436×10^3	1.842427×10^7	1.080010	2.020738
	15	1.005474×10^5	1.017251×10^{10}	1.003097	1.999075
	20	7.036678×10^6	4.951899×10^{13}	1.000042	1.986165
	25	1.413375×10^9	1.997625×10^{18}	0.9999992	1.970391
$\beta = 0$	5	3.077237×10^1	5.966963×10^3	2.510243	4.446831
	10	4.475225×10^2	2.545499×10^5	1.127383	2.098001
	15	4.272683×10^3	1.855231×10^7	1.008088	2.000674
	20	1.008457×10^5	1.017263×10^{10}	1.000136	1.993045
	25	7.036975×10^6	4.951901×10^{13}	1.000000	1.980451
$\beta = 0.1$	5	1.161050×10^1	7.316673×10^2	2.329732	4.818854
	10	1.162808×10^2	1.533674×10^4	1.065021	2.181767
	15	5.330310×10^2	2.639197×10^5	0.9637923	2.002620
	20	4.358192×10^3	1.856168×10^7	0.9885583	1.997674
	25	1.009312×10^5	1.017264×10^{10}	0.9992894	1.998622

From Table 3, we note that for the TH-OU process $Y(t)$ the coefficient of variation approaches the value 1 and the skewness approaches the value 2 for large boundaries. Hence, when $\alpha < 0$ the FPT density of the TH-OU process exhibits an exponential behavior for large boundaries S, such that $S > x_0$.

Table 4. As in Table 3 with $x_0 = 4$, $\alpha = 0.02$, $\sigma = 1$, $\beta = -0.1, 0, 0.1$ and for decreasing values the boundary $S \in [0, x_0)$.

	S	$t_1^{(Y)}(S\|x_0)$	$\mathbf{Var}^{(Y)}(S\|x_0)$	$\mathbf{Cv}^{(Y)}(S\|x_0)$	$\Sigma^{(Y)}(S\|x_0)$
$\beta = -0.1$	3.5	2.321819	38.35014	2.667198	6.633648
	3.0	4.740922	81.06132	1.899083	4.708105
	2.5	7.264863	128.7704	1.561998	3.861617
	2.0	9.902021	182.2269	1.363272	3.362382
	1.5	12.66172	242.3166	1.229416	3.026467
	1.0	15.55435	310.0904	1.132119	2.782932
	0.5	18.59156	386.8000	1.057858	2.597858
	0.0	21.7864	473.9432	0.9992582	2.452724
$\beta = 0$	3.5	3.763743	130.8265	3.03898	6.851246
	3.0	7.756523	281.9104	2.164654	4.855690
	2.5	12.00269	457.1913	1.781437	3.978597
	2.0	16.53010	661.5233	1.555955	3.462116
	1.5	21.37079	900.9213	1.404503	3.115650
	1.0	26.56169	1182.882	1.294836	2.865643
	0.5	32.14558	1516.804	1.211556	2.676884
	0.0	38.17219	1914.549	1.146268	2.530113
$\beta = 0.1$	3.5	7.745813	697.7332	3.410183	6.676430
	3.0	16.23615	1550.603	2.425310	4.719137
	2.5	25.58427	2600.685	1.993291	3.859455
	2.0	35.92567	3903.424	1.739074	3.354977
	1.5	47.42327	5532.504	1.568445	3.018737
	1.0	60.27406	7586.697	1.445093	2.778458
	0.5	74.71751	10199.60	1.351666	2.599416
	0.0	91.04643	13553.66	1.278690	2.462534

From Table 4, we note that for the TH-OU process $Y(t)$ the coefficient of variation and the skewness decreases as S decreases.

Moreover, taking the Laplace transform in (3) one has

$$a_\lambda^{(Y)}(x|x_0) = f_\lambda^{(Y)}(x|x_0) - g_\lambda^{(Y)}(0|x_0)\, f_\lambda^{(Y)}(x|0), \qquad x_0 > 0, x > 0, \tag{46}$$

so that, recalling (38) and (41), one can obtain the LT of $a_Y(x,t|x_0)$ for the TH-OU process in $(0, +\infty)$ with 0 absorbing boundary.

Proposition 7. *Let $Y(t)$ be a TH-OU process.*

- *For $\alpha < 0$, one has*

$$L_Y(x|x_0) = \int_0^{+\infty} a_Y(x,t|x_0)\, dt = \frac{1}{\sigma}\sqrt{\frac{\pi}{|\alpha|}} \exp\Big\{-\frac{|\alpha|}{\sigma^2}\Big(x + \frac{\beta}{\alpha}\Big)^2\Big\}$$
$$\times \Big[\mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\Big(x_0 \wedge x + \frac{\beta}{\alpha}\Big)\Big) - \mathrm{Erfi}\Big(\frac{\sqrt{|\alpha|}}{\sigma}\,\frac{\beta}{\alpha}\Big)\Big], \qquad x_0 > 0, x > 0. \tag{47}$$

- *For $\alpha > 0$, it results in*

$$L_Y(x|x_0) = \int_0^{+\infty} a_Y(x,t|x_0)\, dt = \frac{1}{\sigma}\sqrt{\frac{\pi}{\alpha}} \exp\Big\{\frac{\alpha}{\sigma^2}\Big(x + \frac{\beta}{\alpha}\Big)^2\Big\} \frac{\mathrm{Erfc}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(x_0 \vee x + \frac{\beta}{\alpha}\Big)\Big)}{\mathrm{Erfc}\Big(\frac{\sqrt{\alpha}}{\sigma}\,\frac{\beta}{\alpha}\Big)}$$
$$\times \Big[\mathrm{Erf}\Big(\frac{\sqrt{\alpha}}{\sigma}\Big(x_0 \wedge x + \frac{\beta}{\alpha}\Big)\Big) - \mathrm{Erf}\Big(\frac{\sqrt{\alpha}}{\sigma}\,\frac{\beta}{\alpha}\Big)\Big], \quad x_0 > 0, x > 0. \tag{48}$$

Proof. Because $+\infty$ is a nonattracting boundary for $\alpha < 0$ and attracting for $\alpha > 0$, Equations (47) and (48) follow from (12) making use of (37) and (42). □

From (47) and (48), for $\alpha \neq 0$, $\beta \in \mathbb{R}$ and $\sigma > 0$ one obtains $\lim_{x \downarrow 0} L_Y(x|x_0) = 0$ and $\lim_{x \uparrow +\infty} L_Y(x|x_0) = 0$.

In Figure 2, the asymptotic average of the local time for the TH-OU process $Y(t)$ is plotted for $x_0 = 4$, $\sigma = 1$ and some choices of α and β.

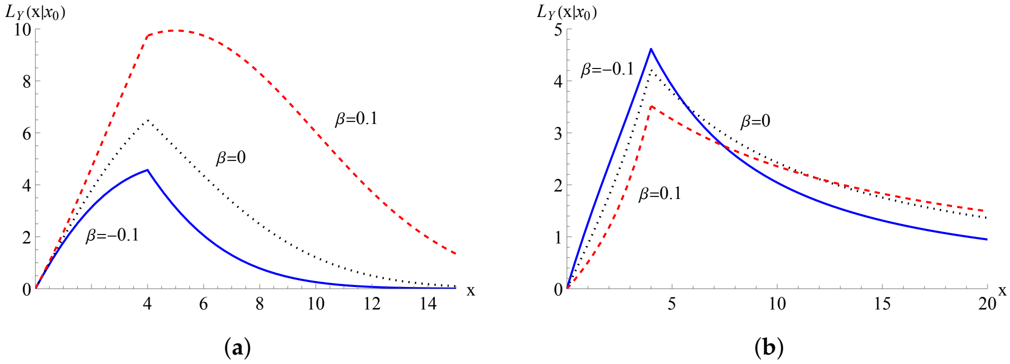


Figure 2. $L_Y(x|x_0)$, given in Proposition 7, with $x_0 = 4$, $\sigma = 1$, and some choices of β. In (**a**) $\alpha = -0.02$ and in (**b**) $\alpha = 0.02$.

4. Feller-Type Diffusion Process

Let $\{Z(t), t \geq t_0\}$, $t_0 \geq 0$ be a TNH-F process having infinitesimal drift and infinitesimal variance

$$C_1(x,t) = \alpha(t)\, x + \beta(t), \qquad C_2(x,t) = 2\, r(t)\, x, \tag{49}$$

with state space $(0, +\infty)$, where $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$, $r(t) > 0$ continuous functions.

We point out that the processes (27) and (49) have identical infinitesimal drifts; instead, the infinitesimal variances are different in terms of the involved noise intensity functions.

The TNH-F process is used to describe the growth of a population (cf. Feller [37], Giorno and Nobile [38]) and the number of customers in queueing models (cf. Di Crescenzo and Nobile [39]). This process is also applied in mathematical finance to study stochastic volatility and interest rates (see Tian and Zhang [40], Cox et al. [41], Di Nardo and D'Onofrio [42]) and in neurobiology to model the input–output behavior of single neurons (see Ditlevsen and Lánský [43], D'Onofrio et al. [44]).

We consider the TNH-F process $Z(t)$, having infinitesimal moments (49), with an absorbing condition placed in the zero state and we denote with $a_Z(x,t|x_0,t_0)$ its PDF. We assume that $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$, $r(t) > 0$, $\beta(t) \leq \xi\, r(t)$, with $0 \leq \xi < 1$. For the TNH-F process $Z(t)$ with an absorbing boundary in zero, we consider two cases: the proportional case in which $\beta(t) = \xi\, r(t)$, with $0 \leq \xi < 1$ and $r(t) > 0$, and the time-homogeneous case.

4.1. Proportional Case for the Feller Process

We assume that $\alpha(t) \in \mathbb{R}$, $r(t) > 0$ and $\beta(t) = \xi\, r(t)$, with $0 \leq \xi < 1$, in (49). As proven in Giorno and Nobile [45] one has

$$a_Z(x,t|x_0,t_0) = \begin{cases} \dfrac{e^{-A(t|t_0)}}{\Gamma(2-\xi)} \left[\dfrac{1}{R(t|t_0)}\right]^{2-\xi} x_0^{1-\xi} \exp\Big\{-\dfrac{x_0}{R(t|t_0)}\Big\}, & x = 0, \\[2ex] \dfrac{e^{-A(t|t_0)}}{R(t|t_0)} \left(\dfrac{x_0}{x}\right)^{(1-\xi)/2} \exp\Big\{-\dfrac{x_0 + x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\} & \\ \qquad \times \exp\Big\{\dfrac{1-\xi}{2}\, A(t|t_0)\Big\}\, I_{1-\xi}\left[\dfrac{2\sqrt{x\, x_0\, e^{-A(t|t_0)}}}{R(t|t_0)}\right], & x > 0, \end{cases} \tag{50}$$

with $A(t|t_0)$ given in (30), $I_\nu(z)$ defined in (24) and

$$R(t|t_0) = \int_{t_0}^{t} r(\theta)\, e^{-A(\theta|t_0)}\, d\theta. \tag{51}$$

Proposition 8. *Let $\alpha(t) \in \mathbb{R}$, $r(t) > 0$ and $\beta(t) = \xi\, r(t)$, with $0 \leq \xi < 1$, in (49). For the TNH-F process $Z(t)$ one has*

$$g_Z(0,t|x_0,t_0) = \frac{1}{\Gamma(1-\xi)}\,\frac{r(t)\,e^{-A(t|t_0)}}{R(t|t_0)}\left[\frac{x_0}{R(t|t_0)}\right]^{1-\xi}\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}, \qquad x_0 > 0, \tag{52}$$

with $R(t|t_0)$ given in (51). Moreover, it results in

$$P_Z(0|x_0,t_0) = \int_{t_0}^{+\infty} g_Z(0,t|x_0,t_0)\,dt = \begin{cases} 1, & \lim_{t\to+\infty} R(t|t_0) = +\infty, \\ \frac{\Gamma\left(1-\xi,\frac{x_0}{c}\right)}{\Gamma\left(1-\xi\right)}, & \lim_{t\to+\infty} R(t|t_0) = c. \end{cases} \tag{53}$$

Proof. From (50), one has (cf. Erdèlyi et al. [36], p. 197, no. 19)

$$\int_0^{+\infty} a_Z(x,t|x_0,t_0)\,dx = \frac{1}{\Gamma(1-\xi)}\,\gamma\Big(1-\xi,\frac{x_0}{R(t|t_0)}\Big), \qquad 0 \leq \xi < 1, \tag{54}$$

where $\Gamma(\nu)$ is the Euler gamma function and $\gamma(\nu,z) = \int_0^z y^{\nu-1}\,e^{-y}\,dy$, with $\nu > 0$, is the incomplete gamma function. Hence, due to (5) and recalling (54), Equation (52) follows. Finally, Equation (53) is obtained, making use of (54) in (6). □

We point out that the general TNH-F process with an absorbing boundary in zero is considered in Giorno and Nobile [45], Masoliver and Perelló [46], Masoliver [47] and Lavigne and Roques [48].

4.2. Time-Homogeneous Case for the Feller Process

Let $Z(t)$ be the TH-F process, obtained by setting $\alpha(t) = \alpha$, $\beta(t) = \beta$ and $r(t) = r$ in (49). From (9), the scale function and the speed density of the TH-F process $Z(t)$ are

$$h_Z(x) = x^{-\beta/r}\exp\Big\{-\frac{\alpha x}{r}\Big\}, \qquad s_Z(x) = \frac{x^{\beta/r-1}}{r}\,\exp\Big\{\frac{\alpha x}{r}\Big\}, \tag{55}$$

respectively. As proven by Feller, the state 0 is an exit boundary for $\beta \leq 0$, regular for $0 < \beta < r$ and entrance for $\beta \geq 0$. Furthermore, the end point $+\infty$ is a nonattracting natural boundary for $\alpha \leq 0$ and an attracting natural boundary for $\alpha > 0$. In the sequel, we assume that $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $r > 0$, with $\beta < r$, and an absorbing condition is set in the zero-state.

As proven in Giorno and Nobile [45], for a TH-F process $Z(t)$ having $\beta \in \mathbb{R}$, $r > 0$, with $\beta < r$, one has

- If $\alpha = 0$ one has

$$a_Z(x,t|x_0) = \begin{cases} \frac{1}{\Gamma(2-\beta/r)}\left(\frac{1}{r\,t}\right)^{2-\beta/r} x_0^{1-\beta/r}\,\exp\Big\{-\frac{x_0}{r\,t}\Big\}, & x = 0, \\ \frac{1}{r\,t}\left(\frac{x_0}{x}\right)^{(1-\beta/r)/2}\exp\Big\{-\frac{x+x_0}{r\,t}\Big\}\,I_{1-\beta/r}\left[\frac{2\sqrt{x\,x_0}}{r\,t}\right], & x > 0. \end{cases} \tag{56}$$

- If $\alpha \neq 0$ one obtains

$$a_Z(x,t|x_0) = \begin{cases} \frac{e^{-\alpha t}}{\Gamma(2-\beta/r)}\left[\frac{\alpha\,e^{\alpha t}}{r(e^{\alpha t}-1)}\right]^{2-\beta/r} x_0^{1-\beta/r}\,\exp\Big\{-\frac{\alpha\,x_0\,e^{\alpha t}}{r(e^{\alpha t}-1)}\Big\}, & x = 0, \\ \frac{\alpha\,e^{\alpha(1-\beta/r)\,t/2}}{r(e^{\alpha t}-1)}\left(\frac{x_0}{x}\right)^{(1-\beta/r)/2}\exp\Big\{-\frac{\alpha(x+x_0\,e^{\alpha t})}{r(e^{\alpha t}-1)}\Big\}\,I_{1-\beta/r}\left[\frac{2\alpha\sqrt{x\,x_0\,e^{\alpha t}}}{r(e^{\alpha t}-1)}\right], & x > 0. \end{cases} \tag{57}$$

Proposition 9. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $r > 0$, with $\beta < r$. For the TH-F process $Z(t)$, with $x_0 > 0$, one has*

$$g_Z(0,t|x_0) = \begin{cases} \frac{1}{t\,\Gamma(1-\beta/r)} \left(\frac{x_0}{r\,t}\right)^{1-\beta/r} \exp\Big\{-\frac{x_0}{r\,t}\Big\}, & \alpha = 0, \\ \frac{1}{\Gamma(1-\beta/r)}\,\frac{\alpha}{e^{\alpha t}-1}\left[\frac{\alpha\,x_0\,e^{\alpha t}}{r(e^{\alpha t}-1)}\right]^{1-\beta/r} \exp\Big\{-\frac{\alpha\,x_0\,e^{\alpha t}}{r\,(e^{\alpha t}-1)}\Big\}, & \alpha \neq 0 \end{cases} \tag{58}$$

and

$$P_Z(0|x_0) = \int_0^{+\infty} g_Z(0,t|x_0)\,dt = \begin{cases} 1, & \alpha \leq 0, \\ \frac{\Gamma\left(1-\frac{\beta}{r},\frac{\alpha\,x_0}{r}\right)}{\Gamma\left(1-\beta/r\right)}, & \alpha > 0. \end{cases} \tag{59}$$

Proof. From (56) and (57), one obtains (cf. Erdèlyi et al. [36], p. 197, no. 19)

$$\int_0^{+\infty} a_Z(x,t|x_0)\,dx = \begin{cases} \frac{1}{\Gamma(1-\beta/r)}\gamma\left(1-\frac{\beta}{r},\frac{x_0}{r\,t}\right), & \alpha = 0, \\ \frac{1}{\Gamma(1-\beta/r)}\gamma\left(1-\frac{\beta}{r},\frac{\alpha\,x_0\,e^{\alpha t}}{r\,(e^{\alpha t}-1)}\right), & \alpha \neq 0. \end{cases} \tag{60}$$

Making use of (60) in (5), Equation (58) follows. Finally, by virtue of (6) and (60), we obtain the FPT probability (59). □

By applying the Siegert Formula (11) with $r_2 = +\infty$ and recalling (55), for $\alpha = 0$ and $\beta < r$ one has that the FPT mean $t_1^{(Z)}(0|x_0)$ diverges, whereas for $\alpha < 0$ and $\beta < r$ one obtains

$$t_1^{(Z)}(S|x_0) = \frac{1}{|\alpha|}\int_{|\alpha|S/r}^{|\alpha|x_0/r} x^{-\beta/r} e^x\,\Gamma\left(\frac{\beta}{r},x\right)dx, \qquad x_0 > S \geq 0.$$

In Table 5, the mean $t_1^{(Z)}(S|x_0)$, the variance $\mathrm{Var}^{(Z)}(S|x_0)$, the coefficient of variation $\mathrm{Cv}^{(Z)}(S|x_0)$, and the skewness $\Sigma^{(Z)}(S|x_0)$ of the FPT, obtained by using the Siegert Formula (11), are listed for $x_0 = 4$ and some choices of S, with $\alpha = -0.02$, $\beta = -0.1, 0, 0.1$ and $r = 0.5$.

As shown in Table 5, for the TH-F process $Z(t)$ the mean and the variance of the FPT increases as S decreases; instead, the coefficient of variation and the skewness decrease as S decreases.

Proposition 10. *Let $Z(t)$ be a TH-F process having $\beta \in \mathbb{R}$, $r > 0$, with $\beta < r$.*

- *If $\alpha \leq 0$, for $x_0 > 0$ and $x > 0$ one has*

$$\begin{aligned} L_Z(x|x_0) &= \int_0^{+\infty} a_Z(x,t|x_0)\,dt \\ &= \begin{cases} \frac{1}{r}\left(\frac{r}{|\alpha|x}\right)^{1-\beta/r} e^{-|\alpha|\,x/r}\int_0^{|\alpha|(x_0\wedge x)/r} y^{-\beta/r}e^y\,dy, & \alpha < 0 \\ \frac{1}{r}\,\frac{1}{1-\beta/r}\left(\frac{x_0\wedge x}{x}\right)^{1-\beta/r}, & \alpha = 0. \end{cases} \end{aligned} \tag{61}$$

- *If $\alpha > 0$, for $x_0 > 0$ and $x > 0$ one obtains*

$$\begin{aligned} L_Z(x|x_0) &= \int_0^{+\infty} a_Z(x,t|x_0)\,dt \\ &= \frac{1}{r}\left(\frac{r}{\alpha x}\right)^{1-\beta/r} e^{\alpha\,x/r}\gamma\left(1-\frac{\beta}{r},\frac{\alpha(x_0\wedge x)}{r}\right)\frac{\Gamma(1-\frac{\beta}{r},\frac{\alpha\,(x_0\vee x)}{r})}{\Gamma(1-\frac{\beta}{r})}. \end{aligned} \tag{62}$$

Proof. Because $+\infty$ is a nonattracting boundary for $\alpha \leq 0$ and attracting for $\alpha > 0$, Equations (61) and (62) follow from (12), making use of (55) and (59). $\square$

Table 5. For the TH-F process, with $C_1(x) = -0.02\,x + \beta$ and $C_2(x) = x$, the mean, the variance, the coefficient of variation, and the skewness of FPT are listed for $x_0 = 4$, $\beta = -0.1, 0, 0.1$ and for decreasing values the boundary $S \in [0, x_0)$.

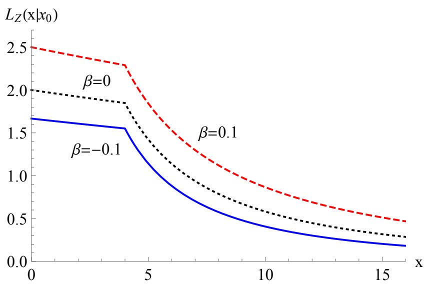
	S	$t_1^{(Z)}(S\|x_0)$	$\mathrm{Var}^{(Z)}(S\|x_0)$	$\mathrm{Cv}^{(Z)}(S\|x_0)$	$\Sigma^{(Z)}(S\|x_0)$
$\beta = -0.1$	3.5	1.392620	43.55136	4.738799	13.55745
	3.0	2.859008	89.01922	3.300096	9.501619
	2.5	4.412074	136.5521	2.648539	7.684201
	2.0	6.069704	186.2977	2.248724	6.585614
	1.5	7.858374	238.3733	1.964699	5.823477
	1.0	9.821571	292.7690	1.742134	5.249299
	0.5	12.04587	348.9465	1.550747	4.792382
	0.0	14.86611	401.3413	1.347596	4.429951
$\beta = 0$	3.5	1.702121	69.83433	4.909578	13.0398
	3.0	3.512494	144.3878	3.420973	9.119134
	2.5	5.452850	224.4206	2.747311	7.356073
	2.0	7.554206	310.9409	2.334265	6.285290
	1.5	9.864220	405.3553	2.041057	5.536382
	1.0	12.46511	509.782	1.811324	4.964136
	0.5	15.53300	627.8781	1.613178	4.494585
	0.0	19.91651	768.9171	1.392280	4.068225
$\beta = 0.1$	3.5	2.126455	115.1039	5.045321	12.41484
	3.0	4.415487	240.9926	3.515793	8.659792
	2.5	6.904328	380.0698	2.823646	6.964637
	2.0	9.647790	535.8021	2.399244	5.929435
	1.5	12.73337	713.5752	2.097861	5.199456
	1.0	16.31983	922.7997	1.861394	4.631900
	0.5	20.77405	1184.008	1.656366	4.155107
	0.0	28.39302	1607.070	1.411906	3.602509

From (61) and (62), for $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $r > 0$, with $\beta < r$, one has $\lim_{x \uparrow +\infty} L_Z(x|x_0) = 0$ and

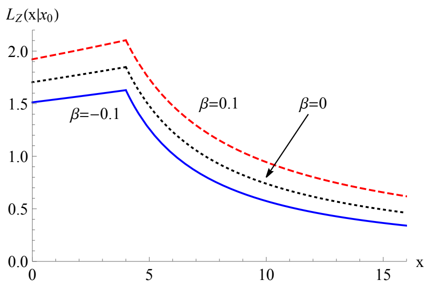
$$\lim_{x \downarrow 0} L_Z(x|x_0) = \begin{cases} \frac{1}{r}\,\frac{1}{1-\beta/r} & \alpha \leq 0, \\ \frac{1}{r}\,\frac{1}{1-\beta/r}\,\frac{\Gamma\left(1-\frac{\beta}{r}, \frac{\alpha\, x_0}{r}\right)}{\Gamma\left(1-\frac{\beta}{r}\right)}, & \alpha > 0. \end{cases}$$

Therefore, for the TH-F process the asymptotic average of local time tend to zero as x increases, whereas it is positive for $x \downarrow 0$.

In Figure 3, the asymptotic average of the local time for the TH-F process $Z(t)$ is plotted for $x_0 = 4$, $r = 0.5$ and some choices of α and β.

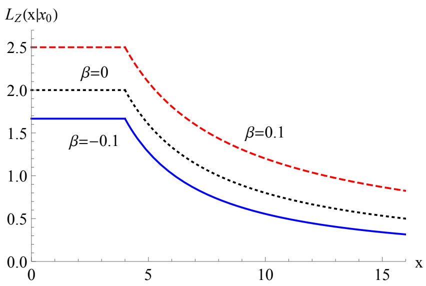


(a)



(b)

Figure 3. *Cont.*



(**c**)

Figure 3. $L_Z(x|x_0)$, given in Proposition 12, with $x_0 = 4$, $r = 0.5$, and some choices of β. In (**a**) $\alpha = -0.02$, in (**b**) $\alpha = 0.02$ and in (**c**) $\alpha = 0$.

5. Relationships and Asymptotic Results

In this section, for $\beta(t) = r(t)/2$ some relationships between the PDF in the presence of an absorbing boundary in the zero state and between the FPT densities through zero for Wiener, Ornstein–Uhlenbeck and Feller processes are proven; moreover, for $\beta(t) = \xi\, r(t)$ $(0 < \xi < 1)$ some asymptotic results for large times between the FPT densities are provided.

5.1. Relations between the Transition Densities with an Absorbing Boundary in the Zero State

We consider the TNH-F process (49) with $\beta(t) = r(t)/2$ in the presence of an absorbing boundary in the zero state, and we show that its PDF can be related to the PDF of the Wiener and of the Ornstein–Uhlenbeck processes with an absorbing boundary in the zero state.

Proposition 11. *Let $Z(t)$ be a TNH-F process with $C_1(t) = r(t)/2$ and $C_2(x,t) = 2\,r(t)\,x$, where $r(t) > 0$, and let $X(t)$ be a TNH-W process with $A_1 = 0$ and $A_2(t) = r(t)/2$. One has*

$$a_Z(x,t|x_0,t_0) = \frac{1}{2\sqrt{x}}\, a_X(\sqrt{x}, t|\sqrt{x_0}, t_0), \qquad x_0 > 0, x > 0, \tag{63}$$

$$g_Z(0,t|x_0,t_0) = g_X(0,t|\sqrt{x_0},t_0), \qquad x_0 > 0. \tag{64}$$

Proof. For the TNH-F process $Z(t)$, by setting $\alpha(t) = 0$ and $\beta(t) = r(t)/2$ in (50) and in Proposition 9, recalling that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sinh(x)}{\sqrt{x}}, \qquad \gamma\Big(\frac{1}{2}, x\Big) = \sqrt{\pi}\,\mathrm{Erf}(\sqrt{x}), \tag{65}$$

one has

$$a_Z(x,t|x_0,t_0) = \begin{cases} 2\sqrt{\frac{x_0}{\pi\,[\widetilde{R}(t|t_0)]^3}}\,\exp\Big\{-\frac{x_0}{\widetilde{R}(t|t_0)}\Big\}, & x = 0, \\ \frac{1}{2\sqrt{\pi\,\widetilde{R}(t|t_0)x}}\left[\exp\Big\{-\frac{(\sqrt{x}-\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Big\} - \exp\Big\{-\frac{(\sqrt{x}+\sqrt{x_0})^2}{\widetilde{R}(t|t_0)}\Big\}\right], & x > 0, \end{cases} \tag{66}$$

and

$$g_Z(0,t|x_0,t_0) = r(t)\sqrt{\frac{x_0}{\pi\,[\widetilde{R}(t|t_0)]^3}}\,\exp\Big\{-\frac{x_0}{\widetilde{R}(t|t_0)}\Big\}, \qquad x_0 > 0, \tag{67}$$

where $\widetilde{R}(t|t_0) = \int_{t_0}^t r(\theta)\,d\theta$. Furthermore, for the TNH-W process $X(t)$ with $\beta(t) = 0$ and $\sigma^2(t) = r(t)/2$, one has $V_X(t|t_0) = \widetilde{R}(t|t_0)/2$. Then, (63) and (64) follow by comparing (66) and (67) with (17) and (18), respectively. □

Under the assumptions of the Proposition 13, one has $L_Z(x|x_0,t_0) = L_X(\sqrt{x}\,|\sqrt{x_0},t_0)/(2\sqrt{x})$ for $x > 0, x_0 > 0$ and, if $\lim_{t\to+\infty}\widetilde{R}(t|t_0) = +\infty$, one obtains $t_n^{(Z)}(0|x_0,t_0) = t_n^{(X)}(0|\sqrt{x_0},t_0)$ for $n = 1,2,\ldots$ with $x_0 > 0$.

Proposition 12. *Let $Z(t)$ be a TNH-F process with $C_1(x,t) = \alpha(t)\,x + r(t)/2$ and $C_2(x,t) = 2\,r(t)\,x$, where $\alpha(t)$ is not always zero and $r(t) > 0$, and let $Y(t)$ be a TNH-OU process with $B_1(x,t) = \alpha(t)\,x/2$ and $B_2(t) = r(t)/2$. One has*

$$a_Z(x,t|x_0,t_0) = \frac{1}{2\sqrt{x}}\,a_Y(\sqrt{x},t|\sqrt{x_0},t_0), \qquad x_0 > 0, x > 0, \tag{68}$$

$$g_Z(0,t|x_0,t_0) = g_Y(0,t|\sqrt{x_0},t_0), \qquad x_0 > 0. \tag{69}$$

Proof. For the TNH-F process $Z(t)$, by setting $\beta(t) = r(t)/2$ in (50) and in Proposition 9, recalling (65), one obtains

$$a_Z(x,t|x_0,t_0) = \begin{cases} 2\,e^{-A(t|t_0)}\sqrt{\frac{x_0}{\pi\,[R(t|t_0)]^3}}\,\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}, & x = 0,\\ \frac{e^{-A(t|t_0)/2}}{2\sqrt{\pi\,R(t|t_0)x}}\Bigg[\exp\Bigg\{-\frac{\big(\sqrt{x\,e^{-A(t|t_0)}}-\sqrt{x_0}\big)^2}{R(t|t_0)}\Bigg\} \\ \qquad -\exp\Bigg\{-\frac{\big(\sqrt{x\,e^{-A(t|t_0)}}+\sqrt{x_0}\big)^2}{R(t|t_0)}\Bigg\}\Bigg], & x > 0, \end{cases} \tag{70}$$

and

$$g_Z(0,t|x_0,t_0) = r(t)\,e^{-A(t|t_0)}\sqrt{\frac{x_0}{\pi\,[R(t|t_0)]^3}}\,\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}, \qquad x_0 > 0, \tag{71}$$

with $A(t|t_0)$ and $R(t|t_0)$ given in (30) and (51), respectively. Moreover, in the TNH-OU process $Y(t)$ we set $\beta(t) = 0$, $\sigma^2(t) = r(t)/2$ and we change $\alpha(t)$ into $\alpha(t)/2$, so that, by virtue of (29) and (51), one has $V_Y(t|t_0) = R(t|t_0)e^{A(t|t_0)}/2$. Then, (68) and (69) follow by comparing (70) and (71) with (33) and (34), respectively. □

Under the assumptions of Proposition 14, one has $L_Z(x|x_0,t_0) = L_Y(\sqrt{x}\,|\sqrt{x_0},t_0)/(2\sqrt{x})$ for $x > 0, x_0 > 0$ and, if $\lim_{t\to+\infty} R(t|t_0) = +\infty$, one obtains $t_n^{(Z)}(0|x_0,t_0) = t_n^{(Y)}(0|\sqrt{x_0},t_0)$ for $n = 1,2,\ldots$ with $x_0 > 0$.

5.2. Asymptotic Behaviors between the FPT Densities

In this section, for $\beta(t) = \xi\,r(t)$ $(0 < \xi < 1)$ some asymptotic results for large times between the FPT densities of TNH-W, TNH-OU and TNH-F processes are shown.

Proposition 13. *Let $Z(t)$ be a TNH-F process with $C_1(t) = \xi\,r(t)$ and $C_2(x,t) = 2\,r(t)\,x$, where $r(t) > 0$, $0 < \xi < 1$, and let $X(t)$ be a TNH-W process with $A_1 = 0$ and $A_2(t) = \xi\,r(t)$. If $\lim_{t\to+\infty}\widetilde{R}(t|t_0) = +\infty$, and one has*

$$\lim_{t\to+\infty}\frac{g_Z(0,t|x_0,t_0)\,[\widetilde{R}(t|t_0)]^{1/2-\xi}}{g_X(0,t|x_0^{1-\xi},t_0)} = \frac{\sqrt{2\,\pi\,\xi}}{\Gamma(1-\xi)}, \qquad x_0 > 0. \tag{72}$$

Proof. Recalling (18) and (67) and noting that $V_X(t|t_0) = \xi\,\widetilde{R}(t|t_0)$, one has

$$\frac{g_Z(0,t|x_0,t_0)\,[\widetilde{R}(t|t_0)]^{1/2-\xi}}{g_X(0,t|x_0^{1-\xi},t_0)} = \frac{\sqrt{2\,\pi\,\xi}}{\Gamma(1-\xi)}\,\exp\Big\{-\frac{x_0}{\widetilde{R}(t|t_0)} + \frac{x_0^{2(1-\xi)}}{2\,\xi\,\widetilde{R}(t|t_0)}\Big\},$$

from which, under the assumption $\lim_{t\to+\infty}\widetilde{R}(t|t_0) = +\infty$, Equation (72) follows. □

Proposition 14. *Let $Z(t)$ be a TNH-F process having $C_1(x,t) = \alpha(t)\,x + \xi\,r(t)$ and $C_2(x,t) = 2\,r(t)\,x$, with $\alpha(t)$ not always zero, $r(t) > 0$, $0 < \xi < 1$, and let $Y(t)$ be a TNH-OU process with $B_1(x,t) = \alpha(t)\,x/2$ and $B_2(t) = \xi\,r(t)$. If $\lim_{t\to+\infty} R(t|t_0) = +\infty$, and one has*

$$\lim_{t\to+\infty}\frac{g_Z(0,t|x_0,t_0)\,[R(t|t_0)]^{1/2-\xi}}{g_Y(0,t|x_0^{1-\xi},t_0)} = \frac{\sqrt{2\,\pi\,\xi}}{\Gamma(1-\xi)}, \qquad x_0 > 0. \tag{73}$$

Proof. Making use of (34) and (52) and noting that $V_Y(t|t_0) = \xi\, R(t|t_0)\, e^{A(t|t_0)}$, one obtains

$$\frac{g_Z(0,t|x_0,t_0)\,[R(t|t_0)]^{1/2-\xi}}{g_Y(0,t|x_0^{1-\xi},t_0)} = \frac{\sqrt{2\,\pi\,\xi}}{\Gamma(1-\xi)}\,\exp\Big\{-\frac{x_0}{R(t|t_0)} + \frac{x_0^{2(1-\xi)}}{2\,\xi\,R(t|t_0)}\Big\},$$

from which, recalling that $\lim_{t\to+\infty} R(t|t_0) = +\infty$, Equation (73) follows. □

6. Conclusions

For the Wiener, Ornstein–Uhlenbeck, and Feller processes, we analyze the transition densities in the presence of an absorbing boundary in the zero state and the FPT problem to the zero state. Particular attention is dedicated to the proportional cases and to the time-homogeneous cases, by achieving the FPT densities through the zero state. Extensive computation are performed with MATHEMATICA to obtain the mean, the variance, the coefficient of variation and the skewness of FPT for TH-W, TH-OU and TH-F processes. Moreover, for these processes, a detailed study of the asymptotic average of local time with an absorbing boundary in the zero-state is carried out.

In Table 6, a summary containing the conditions and the most important equations numbering in Sections 2–4 concerning the absorbing problem for Wiener, Ornstein–Uhlenbeck and Feller diffusion processes is given.

Table 6. Summary containing conditions and the most important equations numbering in Sections 2–4 for Wiener, Ornstein–Uhlenbeck and Feller diffusion processes.

	Conditions	Results—Equations Numbering
Wiener process $A_1(t) = \beta(t)$ $A_2(t) = \sigma^2(t)$ $(\beta(t) \in \mathbb{R}, \sigma(t) > 0)$	$\beta(t) = \gamma\,\sigma^2(t)$ $(\gamma \in \mathbb{R}, \sigma(t) > 0)$	$a_X(x,t|x_0,t_0)$ − (17) $g_X(0,t|x_0,t_0)$ − (18) $P_X(0|x_0,t_0)$ − (19)
	$\beta(t) = \beta, \sigma^2(t) = \sigma^2$ $(\beta \in \mathbb{R}, \sigma > 0)$	$g_X(S,t|x_0)$ − (22) $P_X(S|x_0)$ − (23) $a_X(x,t|x_0)$ − (25) $L_X(x|x_0)$ − (26)
Ornstein–Uhlenbeck process $B_1(x,t) = \alpha(t)\,x + \beta(t)$ $B_2(t) = \sigma^2(t)$ $(\alpha(t) \in \mathbb{R}, \beta(t) \in \mathbb{R}, \sigma(t) > 0)$	$\beta(t) = \gamma\,\sigma^2(t)\,e^{-A(t|0)}$ $(\gamma \in \mathbb{R}, \alpha(t) \in \mathbb{R}, \sigma(t) > 0)$	$a_Y(x,t|x_0,t_0)$ − (33) $g_Y(0,t|x_0,t_0)$ − (34) $P_Y(0|x_0,t_0)$ − (35)
	$\alpha(t) = \alpha, \beta(t) = \beta, \sigma^2(t) = \sigma^2$ $(\alpha \neq 0, \beta \in \mathbb{R}, \sigma > 0)$	$g_\lambda^{(Y)}(S|x_0)$ − (41) $P_Y(S|x_0)$ − (42) $a_\lambda^{(Y)}(x|x_0)$ − (46) $L_Y(x|x_0)$ − (47), (48)
Feller process $C_1(x,t) = \alpha(t)\,x + \beta(t)$ $C_2(x,t) = 2\,r(t)\,x$ $(\alpha(t) \in \mathbb{R}, \beta(t) \in \mathbb{R}, r(t) > 0,$ $(\beta(t) \le \xi\,r(t),\ 0 \le \xi < 1)$	$\beta(t) = \xi\,r(t)$ $(0 \le \xi < 1, r(t) > 0)$	$a_Z(x,t|x_0,t_0)$ − (50) $g_Z(0,t|x_0,t_0)$ − (52) $P_Z(0|x_0,t_0)$ − (53)
	$\alpha(t) = \alpha, \beta(t) = \beta, r(t) = r$ $(\alpha \in \mathbb{R}, \beta \in \mathbb{R}, r > 0, \beta < r)$	$a_Z(x,t|x_0)$ − (56), (57) $g_Z(0,t|x_0)$ − (58) $P_Z(0|x_0)$ − (59) $L_Z(x|x_0)$ − (61), (62)

As shown in Table 6, by setting $\beta(t) = 0$ in TNH-W, TNH-OU and TNH-F processes, the PDF in the presence of an absorbing boundary in the zero state and the FPT density through zero are given in closed form. Moreover, in TH-W, TH-F processes, the previous densities are obtainable, whereas for the TH-OU process only the LT is available.

The knowledge of the PDF in the presence of an absorbing boundary in the zero state is of interest in the context of biological systems because it allows us to evaluate the survival probabilities (20), (36) and (54) for Wiener, Ornstein–Uhlenbeck and Feller processes, respectively. Moreover, such PDF allows one to get information on the FPT density through zero (extinction density) (18), (34), and (52) and on the probability of extinction (19), (35) and (53) of the considered processes. Furthermore, the asymptotic average of the local time for TH-W, TH-OU, and TH-F processes provides information on the average of the sojourn time in the various states before the absorption occurs in the zero state.

The results of Section 5 show that the same FPT density through the zero-state (extinction density) may correspond to different diffusion processes with modified initial states.

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Abbreviations

The following abbreviations are used in this manuscript:

PDF	Transition Probability Density Function
FPT	First Passage Time
TNH-D	Time Inhomogeneous Diffusion
TNH-W	Time Inhomogeneous Wiener
TNH-OU	Time Inhomogeneous Ornstein-Uhlenbeck
TNH-F	Time Inhomogeneous Feller
TH-D	Time Homogeneous Diffusion
TH-W	Time Homogeneous Wiener
TH-OU	Time Homogeneous Ornstein-Uhlenbeck
TH-F	Time Homogeneous Feller

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