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Best Proximity Point Theorems for the Generalized Fuzzy Interpolative Proximal Contractions

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Abstract: The idea of best proximity points of the fuzzy mappings in fuzzy metric space was intorduced by Vetro and Salimi. We introduce a new type of proximal contractive condition that ensures the existence of best proximity points of fuzzy mappings in the fuzzy complete metric spaces. We establish certain best proximity point theorems for such proximal contractions. We improve and generalize the fuzzy proximal contractions by introducing (Ψ, Φ)-fuzzy proximal contractions and (Ψ, Φ)-fuzzy proximal interpolative contractions. The obtained results improve and generalize many best proximity point theorems published earlier. Moreover, we provide many nontrivial examples to validate our best proximity point theorem.

Keywords: best proximty point; interpolative fuzzy contractions; fuzzy metric space

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1. Introduction

The (fixed point) equation $\ell(x) = x$ is identical to $T(x) = 0$, where $T(x) = \ell(x) - x$. As a result, the concrete solution of such equations takes into account "fixed point theory". Any approximative solution is also worth examining and can be determined using the best proximity point theory in circumstances where such a problem cannot be solved. The best proximity roughly translates to the smallest value of $d(x, \ell(x))$ if $\ell(x)$ is not equal to x. Best proximity theorems, interestingly, are a natural development of fixed point theorems. When the mapping in question is a self-mapping, a best proximity point becomes a fixed point. The existence of a best proximity point can be determined by analyzing different types of proximal contractions [1–5].

The concept of fuzzy sets was given by Zadeh [6]. This idea was successful in altering several mathematical structures within itself. Schweizer and Sklar [7] defined the notion of continuous t-norms. Karamosil and Michlek [8] introduced the notion of fuzzy metric space by using the concept of fuzzy sets, continuous t-norm, and metric space. Gregory and Sapena [9] proved various fixed point results in the context of fuzzy metric spaces.

The triangular inequality that a fuzzy metric space satisfies provides a certain control on how the distances between two points of a triplet are related. However, sometimes, it is not strong enough to complete the proofs of certain results in the field of fixed-point theory. In such a case, an additional assumption is often useful: the non-Archimedean property. This condition establishes that the same real parameter can relate the fuzzy distances between any three points of the underlying space. Such a hypothesis is very useful in practice because the main examples of fuzzy metric spaces that are handled in applications usually satisfy such a constraint. Fuzzy metrics have been demonstrated to be a very consistent notion, leading to significant improvements in many fields.

Pakanazar [10] proved the best proximity point theorems in a fuzzy metric space and Vetro and Salimi [11] considered the problem of finding a best proximity point that achieves the minimum distance between two nonempty sets in a non-Archimedean fuzzy metric space. Recently, Hierro et al. [12] presented the Proinov type fixed point results in a fuzzy metric space. The most important advantage of the cited family of contractions is that it involves very general auxiliary functions that were inspired on Proinov's attractive paper. The obtained results demonstrated that there is a wide field of research that must be explored to better understand the topological, analytical, and algebraic structure of fuzzy metric spaces.

We introduce(Ψ, Φ)-non-Archimedean fuzzy proximal contraction, (Ψ, Φ)-fuzzy interpolative Reich-Rus-Ciric type and (Ψ, Φ)-fuzzy interpolative Hardy Rogers type of the first kind in a non-Archimedean fuzzy metric space. The aim of this paper, is to generalize the non-Archimedean fuzzy proximal contraction in a non-Archimedean fuzzy complete metric space. These results help researchers to better understand the best proximity theory in the setting of (Ψ, Φ)-non-Archimedean fuzzy proximal contraction. In the following, we present the contribution of various mathematicians towards fuzzy proximal contraction in chronological order (see Table 1).

Table 1. Contributions of several authors towards generalized interpolative proximal contraction.

Authors	Year	Contributions
Calogero Vetro and Peyman Salimi [11]	2013	Best proximity points in fuzzy metric space
Erdal Karapinar [13]	2018	Interpolative contraction
Ishak Alton and Aysenur Tasdemir [14]	2020	Interpolative proximal contraction
Petko D. Proinov [15]	2020	Generalized contraction mappings
Khalil et al.	This paper	Generalized fuzzy interpolative proximal contraction

Recently, many nonlinear fuzzy models have appeared in the literature [16] and to show the existence of solutions to such mathematical models, we need generalized fuzzy contractive conditions. In this regard, Hierro et al. [12], Vetro and Salimi [11] have presented some generalized Lipschitz conditions to obtain best proximity point theorems. In this paper, we generalize the results in [11,12,17] and suggest various generalized Lipschitz conditions in the fuzzy metric space that can be used to demonstrate the existence of fuzzy models of nonlinear systems.

2. Preliminaries

This section states some prerequisites.

Definition 1 ([7])**.** *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *satisfying the following conditions:*

(1) $*$ *is commutative and associative;*
(2) $*$ *is continuous;*
(3) $a * 1 = a$ *for all* $a \in [0,1]$*;*
(4) $a * b \leq c * d$*, whenever,* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$*,*

is called continuous t-norm.

Definition 2 ([11])**.** *Let* X *be a non-empty set and* $*$ *be a continuous t-norm. A mapping* $F : X \times X \times [0, +\infty)$ *satisfying the following conditions:*

(i) $F(x, y, t) > 0$;
(ii) $F(x, y, t) = 1 \iff x = y$;
(iii) $F(x, y, t) = F(y, x, t)$;
(iv) $F(x, z, t + s) \geq F(x, y, t) * F(y, t, s)$;
(v) $F(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous,*

for all $x, y, z \in X$ *and* $t, s > 0$ *is called fuzzy metric and the triplet* $(X, F, *)$ *represents fuzzy metric space.*

If we replace axiom (iv) by (iv)'. $F(x,z,\max\{t,s\}) \geq F(x,y,t) * F(y,t,s)$, then $(X,F,*)$ is known as non-Archimedean fuzzy metric space. Since (iv) implies (iv)', each non-Archimedean fuzzy metric space is a fuzzy metric space.

Let A and B be two non-empty subsets of a non-Archimedean fuzzy metric space $(X,F,*)$. We define the sets $A_0(t)$ and $B_0(t)$ as follows:

$$\begin{aligned} A_0(t) &= \{u \in A : F(u,v,t) = F(A,B,t) \text{ for some } v \in B \text{ and } t > 0\}, \\ B_0(t) &= \{v \in B : F(u,v,t) = F(A,B,t) \text{ for some } u \in A \text{ and } t > 0\}, \end{aligned}$$

where $F(A,B,t) = \sup\{F(x,y,t) : u \in A \wedge v \in B \text{ and } t > 0\}$,

For any $(X,F,*)$ non-Archimedean fuzzy metric space and A, B non-empty subsets of X, we say that B is approximately compact with respect to A, if every sequence $\{v_n\}$ in B satisfying the following condition

$$F(u,v_n,t) \to F(x,B,t),$$

for some $u \in A$ and $t > 0$ has a convergent subsequence.

It is evident that every set is approximately compact with respect to itself. If A intersects, then $A \cap B$ is contained in both A_0 and B_0. Further, it can be observed that if A is compact and B is approximately compact with respect to A, then the sets A_0 and B_0 are non-empty.

Definition 3 ([9])**.** *Let $(X,F,*)$ be a non-Archimedean fuzzy metric space and A, B be non-empty subsets of X. An element u in A is called a best proximity point of the mapping $T : A \to B$, if it satisfies the equation:*

$$F(u,Tu,t) = F(A,B,t).$$

A best proximity point of the mapping T is not only an approximate solution of the equation $T(u) = u$ but also an optimal solution of the minimization problem:

$$\min\{F(u,T(u),t) : u \in A\}.$$

3. Main Results

In this section, we define (Ψ,Φ)-proximal contraction and show that it generalizes proximal contraction. We prove the existence of best proximity point of (Ψ,Φ)-proximal contraction in a complete non-Archimedean fuzzy metric space followed by supporting examples.

(Ψ,Φ)-Proximal Contraction of First Kind

Let $(X,F,*)$ be a non-Archimedean fuzzy metric space and A, B be subsets of X. A mapping $T : A \to B$ satisfying

$$\begin{array}{l} F(u_1,Tv_1,t) = F(A,B,t), \\ F(u_2,Tv_2,t) = F(A,B,t) \end{array} \Rightarrow \Psi(F(u_1,u_2,t)) \geq \Phi(F(v_1,v_2,t)), \tag{1}$$

for all distinct $u_1,u_2,v_1,v_2 \in A, t > 0$, with $u_1 \neq u_2$ is called a (Ψ,Φ)-proximal contraction of the first kind, where $\Psi,\Phi : (0,1] \to \mathbb{R}$ are two functions such that $\Phi(q) > \Psi(q)$ for all $q \in (0,1)$.

The following example shows that (Ψ,Φ)-proximal contraction generalizes proximal contraction.

Example 1. *Let $X = \mathbb{R}^2$ and $F : X \times X \times (0,\infty) \to [0,1]$ be the non-Archimedean fuzzy metric given by $F(u,v,t) = e^{-\frac{d((u_1,v_1),(u_2,v_2))}{t}}$ and*

$$d((u_1,v_1),(u_2,v_2)) = \sqrt{(u_1-u_2)^2 + (v_1-v_2)^2}\,,$$

for all $u_1, v_1, u_2, v_2 \in X$ *and* $t > 0$. *Let* A, B *be the subsets of* X *defined by*

$$A = \{(0,u);\, u \in \mathbb{R}\},\ B = \{(1,u); u \in \mathbb{R}\},\ \text{then } F(A,B,t) = e^{-\frac{1}{t}}.$$

Define the functions $\Psi, \Phi : (0,1] \to \mathbb{R}$ *by*

$$\Psi(q) = \sqrt{q} \text{ and } \Phi(q) = q^2 \text{ for } q \in (0,1).$$

Define the mapping $T : A \to B$ *by* $T((0,u)) = (1,2u)$ *for all* $(0,\gamma) \in A$. *We show that* T *is a* (Ψ,Φ)*-proximal contraction of the first kind. For* $u_1 = (0,2)$, $v_1 = (0,1)$ *and* $u_2 = (0,4)$, $v_2 = (0,2)$, $t = 1$, *we have*

$$F(u_1, Tv_1, t) = F((0,2), T(0,1), 1) = F(A,B,t), \tag{2}$$

$$F(u_2, Tv_2, t) = F((0,4), T(0,2), 1) = F(A,B,t). \tag{3}$$

This implies that

$$\Psi(F(u_1,u_2,t)) \geq \Phi(F(v_1,v_2,t))$$

$$\begin{aligned} F(u_1,u_2,t) &= F((0,2),(0,4),t) = 0.1353 \\ F(v_1,v_2,t) &= F((0,1),(0,2),t) = 0.3679. \end{aligned}$$

$$\begin{aligned} \Psi(0.1353) &\geq \Phi(0.3679) \\ 0.3673 &> 0.1354 \end{aligned}$$

This shows that T *is a* (Ψ,Φ)*-proximal contraction. However, the following calculation shows that it is not a proximal contraction. Indeed*

$$\begin{aligned} F(u_1, Tv_1, t) &= F((0,2), T(0,1), 1) = F(A,B,t), \\ F(u_2, Tv_2, t) &= F((0,4), T(0,2), 1) = F(A,B,t), \end{aligned}$$

does not imply to

$$F(u_1,u_2,t) \geq F(v_1,v_2,t).$$

This shows that T *is not a proximal contraction.*

The following lemmas will be applied to obtain the proof of main results.

Lemma 1 ([12])**.** *Let* $(X,F,*)$ *be a non-Archimedean fuzzy metric space and* $\{u_n\} \subset X$ *be a sequence verifying* $\lim_{n\to\infty} F(u_n,u_{n+1},t) > 1-\epsilon$ *for all* $t > 0$. *if the sequence* $\{u_n\}$ *is not Cauchy, then there are subsequences* $\{u_{n_k}\}, \{u_{m_k}\}$ *and* $\varepsilon \in (0,1)$ *such that*

$$\lim_{k\to\infty} F\big(u_{n_{k+1}}, u_{m_{k+1}}, t\big) = \varepsilon - . \tag{4}$$

$$\lim_{k\to\infty} F\big(u_{n_k}, u_{m_k} t\big) = \lim_{k\to\infty} F\big(u_{n_{k+1}}, u_{m_k}, t\big) = \lim_{k\to\infty} F\big(u_{n_k}, u_{m_{k+1}}, t\big) = \varepsilon. \tag{5}$$

Lemma 2 ([15])**.** *Let* $\Psi : (0,1] \to \mathbb{R}$. *Then, the following conditions are equivalent:*

(i) $\inf_{q>\varepsilon} \Psi(q) > -\infty$ *for every* $\varepsilon \in (0,1)$.
(ii) $\lim_{q\to\varepsilon-} \inf \Psi(q) > -\infty$ *for any* $\varepsilon \in (0,1)$.
(iii) $\lim_{n\to\infty} \Psi(q_n) = -\infty$ *implies that* $\lim_{n\to\infty} q_n = 1$.

Lemma 3 ([12])**.** *Let* $\{u_n\}$ *be a sequence in* $(X,F,*)$ *such that* $\lim_{n\to\infty} F(u_n,u_{n+1},t) > 1-\varepsilon$ *for all* $t > 0$ *and* $\varepsilon \in (0,1)$ *and* $T : A \to B$ *be a mapping satisfying (1). If the functions*

$\Psi, \Phi : (0,1] \to \mathbb{R}$ *are such that (1)* $\liminf_{q \to \varepsilon-} \Phi(q) > \Psi(\varepsilon-)$ *for any* $\varepsilon \in (0,1)$. *Then,* $\{u_n\}$ *is Cauchy.*

Proof. We suppose that the sequence $\{u_n\}$ is not Cauchy, by Lemma 1, there exist two subsequences $\{u_{n_k}\}, \{u_{m_k}\}$ and $\varepsilon \in (0,1)$ such that the Equations (4) and (5) hold. By (4), we obtain that $F(u_{n_{k+1}}, u_{m_{k+1}}, t) < 1 - \varepsilon$. For $u_{n_k}, u_{m_k}, u_{m_{k+1}}, u_{n_{k+1}} \in A$, we have

$$\begin{aligned} F(u_{n_{k+1}}, Tu_{n_k}, t) &= F(A,B,t), \\ F(u_{m_{k+1}}, Tu_{n_k}, t) &= F(A,B,t) \text{ for all } k \geq 1. \end{aligned}$$

Thus, by (1) we have

$$\Psi\big(F(u_{n_{k+1}}, u_{m_{k+1}}, t)\big) \geq \Phi\big(F(u_{n_k}, u_{m_k}, t)\big) \text{ for any } k \geq 1.$$

Substituting $g_k = F(u_{n_{k+1}}, u_{m_{k+1}}, t)$ and $h_k = F(u_{n_k}, u_{m_k}, t)$ in the above inequality, we have

$$\Psi(g_k) \geq \Phi(h_k) \text{ for any } k \geq 1. \tag{6}$$

By (4) and (5), we have $\lim_{k\to\infty} g_k = \varepsilon-$ and $\lim_{k\to\infty} h_k = \varepsilon$.
By (6), we infer

$$\Psi(\varepsilon-) = \lim_{k\to\infty} \Psi(g_k) \geq \lim \inf_{k\to\infty} \Phi(h_k) \geq \lim \inf_{l\to\varepsilon} \Phi(l). \tag{7}$$

This contradicts the assumption (i). Consequently, $\{u_n\}$ is a Cauchy sequence in A. □

Now, we present our main results on (Ψ, Φ)-proximal contraction.

Theorem 1. *Let* $(X, F, *)$ *be a complete non-Archimedean fuzzy metric space and* A, B *be non-empty, closed subsets of* X *such that* B *is approximately compact with respect to* A. *Let* $T\colon A \to B$ *be an* (Ψ, Φ)*-proximal contraction of the first kind, satisfying*

(i) Ψ *is a non-decreasing function and* $\liminf_{q\to\varepsilon-} \Phi(q) > \Psi(\varepsilon-)$ *for any* $\varepsilon \in (0,1)$.
(ii) A_0 *is non-empty subset of* A *such that* $T(A_0) \subseteq B_0$.

Then T *admits a best proximity point.*

Proof. Let u_0 be an arbitrary point in A_0. Since $T(u_0) \in T(A_0) \subseteq B_0$, there exists $u_1 \in A_0$ satisfying

$$F(u_1, T(u_0), t) = F(A,B,t).$$

As, $T(u_1) \in T(A_0) \subseteq B_0$, there exists $u_2 \in A_0$ such that

$$F(u_2, T(u_1), t) = F(A,B,t).$$

This process of existence of points in A_0 ends up to a sequence $\{u_n\} \subseteq A_0$ such that

$$F(u_{n+1}, T(u_n), t) = F(A,B,t), \tag{8}$$

for all $n \in \mathbb{N}$. Observe that, if $u_n = u_{n+1}$ for some $n \in \mathbb{N}$, then the point u_n is a best proximity point of the mapping T. On the other hand, if $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, then by (8), we have

$$F(u_n, T(u_{n-1}), t) = F(A,B,t),$$

and

$$F(u_{n+1}, T(u_n), t) = F(A,B,t),$$

for all $n \geq 1$. Thus, by (1)

$$\Psi(F(u_n, u_{n+1}, t)) \geq \Phi(F(u_{n-1}, u_n, t)),$$

for all distinct $u_{n-1}, u_n, u_{n+1} \in A$. Substituting $F(u_n, u_{n+1}, t) = E_n$, we have

$$\Psi(E_n) \geq \Phi(E_{n-1}) > \Psi(E_{n-1}). \tag{9}$$

Since Ψ is non-decreasing, by (9), we have $E_n > E_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{E_n\}$ is strictly non-decreasing. Thus, it converges to some element $E \geq 1$. We show that $E = 1$. Assume, on the contrary, that $E < 1$, so that by (9), we obtain

$$\Psi(\varepsilon-) = \lim_{n\to\infty} \Psi(E_n) \geq \lim_{n\to\infty} \Phi(E_{n-1}) \geq \lim_{t\to E-} \inf \Phi(t).$$

This is a contradiction to assumption (i), thus, $E = 1$ and $\lim_{n\to\infty} F_M(u_n, u_{n+1}, t) = 1$. Presently, keeping in mind the assumption (i) and Lemma 3, we conclude that the sequence $\{u_n\}$ is Cauchy. Since $(X, F, *)$ is a complete non-Archimedean fuzzy metric space and A is closed subset of X, there exists $u \in A$, such that $\lim_{n\to\infty} F(u_n, u, t) = 1$. Moreover,

$$\begin{aligned} F(A,B,t) &= F(u_{n+1}, T(u_n), t) \geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \\ &\geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(u_{n+1}, Tu_n, t) \\ &= F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t). \end{aligned}$$

This implies that

$$F(A,B,t) \geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$F(A,B,t) \geq 1 * \lim_{n\to\infty} F(u, T(u_n), t) \geq 1 * 1 * F(A, B, t).$$

That is,

$$\lim_{n\to\infty} F(u, T(u_n), t) = F(A, B, t).$$

Thus, $F(u, T(u_n), t) \to F(u, B, t)$ as $n \to \infty$. Since B is approximately compact with respect to A, there exists a subsequence $\{T(u_{n_k})\}$ of $\{T(u_n)\}$ such that $(Tu_{n_k}) \to \eta \in S$ as $k \to \infty$. By taking $k \to \infty$ in the following equation:

$$F(u_{n_{k+1}}, T(u_{n_k}), t) = F(A, B, t), \tag{10}$$

we have,

$$F(u, \eta, t) = F(A, B, t).$$

Since $u \in A_0$, $T(u) \in T(A_0) \subseteq B_0$, there exists $\xi \in A_0$ such that

$$F(\xi, Tu, t) = F(A, B, t). \tag{11}$$

Now, keeping in mind the Equations (10), (11), by (1) we have

$$\Psi(F(u_{n_{k+1}}, \xi, t)) \geq \Phi(F(u_{n_k}, u, t)) > \Psi\big(F(u_{n_k}, u, t)\big).$$

Since, Ψ a is non-decreasing function, we have

$$F(u_{n_{k+1}}, \xi, t) > F\big(u_{n_k}, u, t\big).$$

As $k \to \infty$, we have $F(u, \xi, t) = 1$ or $u = \xi$. Finally, by (11) we have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping T. $\square$

Theorem 2. *Let $(X, F, *)$ be a complete non-Archimedean fuzzy metric space and A, B be non-empty, closed subsets of X such that B is approximately compact with respect to A. Let $T\colon A \to B$ be an (Ψ, Φ)-proximal contraction of the first kind, satisfying*

(i) Ψ *is non-decreasing, if* $\{\Psi(q_n)\}$ *and* $\{\Phi(q_n)\}$ *are convergent sequences such that* $\lim_{n\to\infty} \Psi(q_n) = \lim_{n\to\infty} \Phi(q_n)$, *then* $\lim_{n\to\infty} q_n = 1$.

(ii) A_0 *is a non-empty subset of A such that* $T(A_0) \subseteq B_0$.

Then, T admits a best proximity point.

Proof. Proceeding as in the proof of Theorem 1, we have

$$\Psi(E_n) \geq \Phi(E_{n-1}) > \Psi(E_{n-1}). \tag{12}$$

By (12), we infer that the sequence $\{\Psi(E_n)\}$ is strictly non-decreasing. We have two cases here; either it is bounded above or not. If $\{\Psi(E_n)\}$ is not bounded above, then

$$\inf_{E_n > \varepsilon} \Psi(E_n) > -\infty \text{ for every } \varepsilon \in (0,1),\ n \in \mathbb{N}.$$

It follows from Lemma 2 that $E_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\Psi(E_n)\}$ is bounded above, then, it is a convergent sequence. By (12), the sequence $\{\Phi(E_n)\}$ also converges, moreover, both have the same limit. By assumption (i), we have $\lim_{n\to\infty} E_n = 1$ or $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$, for any sequence $\{u_n\}$ in A. Presently, the arguments given in the proof of Theorem 1 leads to have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping T. □

4. Best Proximity Points of Interpolative Proximal Contractions in Non-Archimedean Fuzzy Metric Spaces

The interpolative contraction principles consist of products of distances having exponents satisfying some conditions. The term "interpolative contraction" was introduced by the renowned mathematician Erdal Karapinar in his paper [13] published in 2018. Recently, many classical and advanced contractions have been revisited via interpolation (see [18–22]); among them are the following interpolative contraction:

$$\begin{aligned}
d(Tx, Ty) &\leq K(d(x, Tx))^{\alpha}(d(y, Ty))^{1-\alpha},\\
d(Tx, Ty) &\leq K(d(x, Tx))^{\alpha}(d(x, Ty))^{1-\alpha},\\
d(Tx, Ty) &\leq K(d(x, r))^{\beta}(d(x, Tx))^{\alpha}(d(y, Ty))^{1-\alpha-\beta},\ \alpha + \beta < 1\\
d(Tx, Ty) &\leq K(d(x, r))^{\alpha}(d(x, Tx))^{\beta}(d(y, Ty))^{\gamma}\\
&\quad \left(\frac{1}{2}(d(x, Ty) + d(y, Tx))\right)^{1-\beta-\alpha-\gamma},\ \alpha + \beta + \gamma < 1
\end{aligned}$$

for all $x, y \in \mathcal{A}$, $\nu \in (0,1]$ and $K \in [0,1)$, known as the interpolative Kannan type contraction, interpolative Chatterjea type contraction, interpolative Ćirić-Reich-Rus type contraction and interpolative Hardy Rogers type contraction, respectively.

Altun et al. [14] (2020), revisited all the interpolative contractions introduced in [19] and defined interpolative proximal contractions. They presented best proximity theorems on such contractions. In this section, we establish some best proximity point theorems for (Ψ, Φ) fuzzy interpolative proximal contractions, thereby extending Proinov type fixed point results in a fuzzy metric space [12] to the case of non-self mappings. The (Ψ, Φ)-proximal interpolative contractions generalize interpolative proximal contractions introduced in [14].

4.1. (Ψ,Φ)-Interpolative Reich-Rus-Ciric Type Proximal Contraction of the First Kind

Let $(X, F, *)$ be a non-Archimedean fuzzy metric space and A, B be subsets of X. A mapping $T : A \to B$ satisfying

$$\begin{array}{l} F(u_1, Tv_1, t) = F(A, B, t), \\ F(u_2, Tv_2, t) = F(A, B, t) \end{array} \quad \Rightarrow \Psi(F(u_2, u_1, t)) \geq \Phi\left(\begin{array}{l} (F(v_1, v_2, t))^{\alpha}(F(v_1, u_1, t))^{\beta} \\ (F(v_2, u_2, t))^{1-\alpha-\beta} \end{array} \right), \tag{13}$$

for all distinct $u_1, u_2, v_1, v_2 \in A, t > 0$, with $u_1 \neq u_2$, and $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$ is called a (Ψ,Φ)-interpolative Reich-Rus-Ciric type proximal contraction of the first kind, where $\Psi, \Phi : (0,1] \to \mathbb{R}$ are two functions such that $\Phi(q) > \Psi(q)$ for all $q \in (0,1)$.

Example 2. *Let $X = \mathbb{R}$ and $F : X \times X \times (0,\infty) \to [0,1]$ be a non-Archimedean fuzzy metric given by*

$$F(u, v, t) = \frac{t}{t + d(u, v)},$$

where $d(u, v) =\mid u - v \mid$ for all $u, v \in X$ and $t > 0$. Let A, B are subsets of X and defined as

$$A = \{1,2,3,4,5\},\ B = \{1,2,3,4,5,6,7\},\ \textit{then } F(A, B, t) = 1.$$

Define the functions $\Psi, \Phi : (0,1] \to \mathbb{R}$ by

$$\Psi(q) = \sqrt{q} \textit{ and } \Phi(q) = q \textit{ for all } q \in (0,1).$$

Define the mapping $T : A \to B$ by $T(u) = u + 1$. We show that T is a (Ψ,Φ)-non-Archimedean fuzzy interpolative Reich-Rus-Ćirić type proximal contraction of the first kind. For $u_1 = 4, u_2 = 2$, $v_1 = 3, v_2 = 1$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ and $t = 1$. We have,

$$\begin{aligned} F(u_1, Tv_1, t) &= F(4, T3, 1) = F(A, B, t), \\ F(u_1, Tv_1, t) &= F(2, T1, 1) = F(A, B, t). \end{aligned}$$

This implies to

$$\begin{aligned} \Psi(F(u_2, u_1, t)) &\geq \Phi\Big((F(v_1, v_2, t))^{\alpha}(F(v_1, u_1, t))^{\beta}(F(v_2, u_2, t))^{1-\alpha-\beta}\Big), \\ \Psi(0.3333) &\geq \Phi\Big((0.3333)^{\frac{1}{2}}(0.5)^{\frac{1}{3}}(0.3333)^{1-\frac{1}{2}-\frac{1}{3}}\Big), \\ \Psi(0.3333) &\geq \Phi(0.4079), \\ 0.5773 &\geq 0.4079. \end{aligned}$$

Hence, T is a (Ψ,Φ)-interpolative Rich-Rus Ciric type contraction of the first kind. However, the following calculation shows that it is not a interpolative Rich-Rus Ciric type contraction of the first kind. Indeed,

$$\begin{aligned} F(u_1, Tv_1, t) &= F(4, T3, t) = F(A, B, t) \textit{ and} \\ F(u_1, Tv_1, t) &= F(2, T1, t) = F(A, B, t), \end{aligned}$$

does not imply to,

$$F(u_2, u_1, t) \geq \Big((F(v_1, v_2, t))^{\alpha}(F(v_1, u_1, t))^{\beta}(F(v_2, u_2, t))^{1-\alpha-\beta}\Big).$$

Note that the results will not change if the number of elements in the sets A and B is increased or decreased or if non-integer numbers are taken.

Theorem 3. *Let $(X, F, *)$ be a complete non-Archimedean nfuzzy metric space and A, B be non-empty, closed subsets of X such that B is approximately compact with respect to A. Let $T\colon A \to B$ be an (Ψ,Φ)-interpolative Reich-Rus-Ciric type proximal contraction of the first kind satisfying*

(i) Ψ *is non-decreasing function and* $\liminf_{q\to\varepsilon-}\Phi(q) > \Psi(\varepsilon-)$ *for any* $\varepsilon \in (0,1)$.
(ii) A_0 *is non-empty subset of* A *such that* $T(A_0) \subseteq B_0$.

Then T *has a best proximity point.*

Proof. Let u_0 be an arbitrary point in A_0. Since $T(u_0) \in T(A_0) \subseteq B_0$, there exists $u_1 \in A_0$ satisfying

$$F(u_1, T(u_0), t) = F(A, B, t).$$

Also $T(u_1) \in T(A_0) \subseteq B_0$ implies that there exist $u_2 \in A_0$ such that

$$F(u_2, T(u_1), t) = F(A, B, t).$$

This process of existence of points in A_0 ends up to a sequence $\{u_n\} \subseteq A_0$ satisfying

$$F(u_{n+1}, T(u_n), t) = F(A, B, t), \tag{14}$$

for all $n \in \mathbb{N}$. If $u_n = u_{n+1}$ for some $n \in \mathbb{N}$, then the point u_n is a best proximity point of the mapping T. On the other hand, if $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. By (14), we have

$$F(u_n, T(u_{n-1}), t) = F(A, B, t),$$

and

$$F(u_{n+1}, T(u_n), t) = F(A, B, t),$$

for all $n \geq 1$. Thus, by (13)

$$\Psi(F(u_n, u_{n+1}, t)) \geq \Phi\Big((F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{1-\alpha-\beta}\Big), \tag{15}$$

for all distinct $u_{n-1}, u_n, u_{n+1} \in A$. Since, $\Phi(q) > \Psi(q)$ for all $q \in (0,1)$, by (15), we have

$$\Psi(F(u_n, u_{n+1}, t)) > \Psi\Big((F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{1-\alpha-\beta}\Big).$$

Since Ψ is a non decreasing function, we have

$$F(u_n, u_{n+1}, t) > (F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{1-\alpha-\beta}.$$

This implies that

$$(F(u_n, u_{n+1}, t))^{\alpha+\beta} > (F_M(u_{n-1}, u_n, t))^{\alpha+\beta}.$$

Letting $F(u_n, u_{n+1}, t) = E_n$, we have

$$\Psi((E_n)) \geq \Phi\Big((E_{n-1})^{\alpha+\beta}(E_n)^{1-\alpha-\beta}\Big) > \Psi\Big((E_{n-1})^{\alpha+\beta}(E_n)^{1-\alpha-\beta}\Big).$$

Since Ψ is non-decreasing, by (15), we have $E_n > E_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{E_n\}$ is positive and strictly non-decreasing. Thus, it converges to some element $E \geq 1$. We show that $E = 1$. Assume, on the contrary that $E < 1$, so that by (15), we obtain the following:

$$\Psi(\varepsilon-) = \lim_{n\to\infty} \Psi(E_n) \geq \lim_{n\to\infty} \Phi\Big((E_{n-1})^{\alpha+\beta}(E_n)^{1-\alpha-\beta}\Big) \geq \lim_{t\to E-} \inf \Phi(t).$$

This contradicts assumption (i), hence, $E = 1$ and $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$. Presently, keeping in mind the assumption (i) and Lemma 3, we conclude that $\{u_n\}$ is Cauchy. Since

$(X, F, *)$ is a complete non-Archimedean fuzzy metric space and A is closed subset of X. Then there exists $u \in A$, such that $\lim_{n\to\infty} F(u_n, u, t) = 1$.. Moreover,

$$\begin{aligned} F(A,B,t) &= F(u_{n+1}, T(u_n), t \\ &\geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \\ &\geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(u_{n+1}, Tu_n, t) \\ &= F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t). \end{aligned}$$

This implies

$$F(A,B,t) \geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t).$$

Taking $n \to \infty$ in the above inequality, we obtain

$$F(A,B,t) \geq 1 * \lim_{n\to\infty} F(u, T(u_n), t) \geq 1 * 1 * F(A,B,t).$$

That is,

$$\lim_{n\to\infty} F(u^*, T(u_n), t) = F(A,B,t).$$

Therefore, $F(u, T(u_n), t) \to F(u, B, t)$ as $n \to \infty$. Since B is approximately compact with respect to A, there exists a subsequence $\{T(u_{n_k})\}$ of $\{T(u_n)\}$ such that $(Tu_{n_k}) \to \eta \in B$ as $k \to \infty$. Therefore, by taking $k \to \infty$ in the following equation,

$$F(u_{n_{k+1}}, T(u_{n_k}), t) = F(A,B,t), \tag{16}$$

we have,

$$F(u, \eta, t) = F(A,B,t).$$

Since $u \in A_0$, $T(u) \in T(A_0) \subseteq B_0$ so that there exists $\xi \in A_0$ such that

$$F(\xi, Tu, t) = F(A,B,t). \tag{17}$$

Now, keeping in mind the Equations (16), (17), by (13) we have

$$\begin{aligned} \Psi(F(u_{n_{k+1}}, \xi, t)) &\geq \Phi\Big((F(u_{n_k}, u, t))^{\alpha} (F(u_{n_k}, u_{n_{k+1}}, t))^{\beta} (F(u, \xi, t))^{1-\alpha-\beta} \Big), \\ &> \Psi\Big((F(u_{n_k}, u, t))^{\alpha} (F(u_{n_k}, u_{n_{k+1}}, t))^{\beta} (F(u, \xi, t))^{1-\alpha-\beta} \Big). \end{aligned}$$

Since, Ψ is a non-decreasing function, we have

$$F(u_{n_{k+1}}, \xi, t) > (F(u_{n_k}, u, t))^{\alpha} (F(u_{n_k}, u_{n_{k+1}}, t))^{\beta} (F(u, \xi, t))^{1-\alpha-\beta}.$$

Thus, as $k \to \infty$, we have $F(u, \xi, t) = 1$ or $u = \xi$. Finally, by (17) we have

$$F(u, T(u), t) = F(A,B,t).$$

This shows that the point u is a best proximity point of the mapping, T. □

Theorem 4. *Let $(X, F, *)$ be a complete non-Archimedean fuzzy metric space and A, B be non-empty, closed subsets of X such that A is approximately compact with respect to B. Let $T\colon A \to B$ be an (Ψ, Φ)-interpolative Rus-Reich-Ciric type proximal contraction of the first kind, satisfying*

(i) *Ψ is non-decreasing, $\{\Psi(q_n)\}$ and $\{\Phi(q_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \Psi(q_n) = \lim_{n\to\infty} \Phi(q_n)$, then $\lim_{n\to\infty} t_n = 1$.*

(ii) *A_0 is non-empty subset of A such that $T(A_0) \subseteq B_0$.*

Then, T has a best proximity point.

Proof. Proceeding as in the proof of Theorem 3, we have

$$\Psi((E_n)) \geq \Phi\Big((E_{n-1})^{\alpha+\beta}(E_n)^{1-\alpha-\beta}\Big) > \Psi\Big((E_{n-1})^{\alpha+\beta}(E_n)^{1-\alpha-\beta}\Big). \tag{18}$$

By (18), we infer that $\{\Psi(E_n)\}$ is strictly non-decreasing sequence. We have two cases here; either the sequence $\{\Psi(E_n)\}$ is bounded above or not. If $\{\Psi(E_n)\}$ is not bounded above, then

$$\inf_{E_n > \varepsilon} \Psi(E_n) > -\infty \text{ for every } \varepsilon \in (0,1),\ n \in \mathbb{N}.$$

It follows from Lemma 2 that $E_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\Psi(E_n)\}$ is bounded above, then, it is the convergent sequence. By (18), the sequence $\{\Phi(E_n)\}$ also converges; moreover, both have the same limit. By assumption (i), we have $\lim_{n\to\infty} E_n = 1$ or $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$, for any sequence $\{u_n\}$ in A. By the steps conducted in the proof of theorem 3, we have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping T. □

4.2. (Ψ, Φ)-Interpolative Kannan Type Proximal Contraction of the First Kind

Let $(X, F, *)$ be a non-Archimedean fuzzy metric space and A, B be non-empty subsets of X. A mapping $T : A \to B$ satisfies

$$\begin{array}{l} F(u_1, Tv_1, t) = F(A, B, t) \\ F(u_2, Tv_2, t) = F(A, B, t) \end{array} \quad \Rightarrow \Psi(F(u_1, u_2, t)) \geq \Phi\Big((F(v_1, u_1, t))^{\alpha}(F(v_2, u_2, t))^{1-\alpha}\Big), \tag{19}$$

for all distinct $u_1, u_2, v_1, v_2 \in A$, $t > 0$, with $u_1 \neq u_2$, is called (Ψ, Φ)-interpolative Kannan type proximal contraction of the first kind, where $\Psi, \Phi : (0,1] \to \mathbb{R}$ are two functions such that $\Phi(q) > \Psi(q)$ for all $q \in (0,1)$ and $\alpha \in (0,1)$.

Example 3. *Let $X = \mathbb{R}$ and $F : X \times X \times (0,\infty) \to [0,1]$ be the non-Archimedean fuzzy metric given by*

$$F(u, v, t) = e^{-\frac{d(u,v)}{t}},$$

where $d(u,v) = \mid u - v \mid$ for all $u, v \in X$ and $t > 0$. Let A, B be the subsets of X defined by

$$A = \{1,2,3,4,5\},\ B = \{1,2,3,4,5,6,7\},\ \text{then } F(A,B,t) = 1.$$

Define the functions $\Psi, \Phi : (0,1] \to \mathbb{R}$ by

$$\Psi(q) = \sqrt{q} \text{ and } \Phi(q) = q \text{ for all } q \in (0,1).$$

Define the mapping $T : A \to B$ by $T(u) = u + 1$ for all $u \in A$. We show that T is a (Ψ, Φ)-non-Archimedean fuzzy interpolative Kannan type fuzzy proximal contraction of the first kind. For $u_1 = 3$, $u_2 = 5$, $v_1 = 2$, $v_2 = 4$, $\alpha = \frac{1}{2}$ and $t = 1$, we have

$$\begin{aligned} F(u_1, Tv_1, t) &= F(3, T2, 1) = F(A, B, t), \\ F(u_2, Tv_2, t) &= F(5, T4, 1) = F(A, B, t). \end{aligned}$$

This implies that

$$\begin{aligned}\Psi(F(u_1,u_2,t)) &\geq \Phi\Big((F(v_1,u_1,t))^{\alpha}(F(v_2,u_2,t))^{1-\alpha}\Big),\\ \Psi(F(3,5,1)) &\geq \Phi\Big((F(2,3,1))^{\frac{1}{2}}(F(4,5,1))^{1-\frac{1}{2}}\Big),\\ \Psi(0.1353) &\geq \Phi\Big((0.3678)^{\frac{1}{2}}(0.3678)^{\frac{1}{2}}\Big),\\ \Psi(0.1353) &\geq \Phi(0.3678),\\ 0.3678 &\geq 0.3678.\end{aligned}$$

This shows that T is a (Ψ,Φ)-interpolative Kannan type fuzzy proximal contraction of the first kind. However, the following calculations demonstrate that it is not a interpolative Kannan type proximal contraction of the first kind. Indeed,

$$\begin{aligned}F(u_1,Tv_1,t) &= F(3,T2,1)=F(A,B,t)\\ F(u_2,Tv_2,t) &= F(5,T4,1)=F(A,B,t),\end{aligned}$$

does not imply to

$$F(u_1,u_2,t)\geq\Big((F(v_1,u_1,t))^{\alpha}(F(v_2,u_2,t))^{1-\alpha}\Big).$$

Hence, T is not an interpolative Kannan type proximal contraction of the first kind.

Theorem 5. *Let $(X,F,*)$ be a complete non-Archimedean fuzzy metric space and A,B be non-empty, closed subsets of X such that B is approximately compact with respect to A. Let $T\colon A\to B$ be an (Ψ,Φ)-interpolative Kannan type proximal contraction of the first kind, satisfying*

(i) *Ψ is non-decreasing function and $\liminf_{q\to\varepsilon-}\Phi(q)>\Psi(\varepsilon-)$ for any $\varepsilon\in(0,1)$.*
(ii) *A_0 is non-empty subset of A such that $T(A_0)\subseteq B_0$.*

Then T admits a best proximity point.

Proof. Let u_0 be an arbitrary point in A_0. Since $T(u_0)\in T(A_0)\subseteq B_0$, there exists $u_1\in A_0$ such that,

$$F(u_1,T(u_0),t)=F(A,B,t).$$

Also for $T(u_1)\in T(A_0)\subseteq B_0$, there exists $u_2\in A_0$ such that

$$F(u_2,T(u_1),t)=F(A,B,t).$$

This process of existence of points in A_0 ends up to a sequence $\{u_n\}\subseteq A_0$ satisfying

$$F(u_{n+1},T(u_n),t)=F(A,B,t),\ \text{for all } n\in\mathbb{N}. \tag{20}$$

Observe that, for some $n\in\mathbb{N}$, if $u_n=u_{n+1}$ then u_n is a best proximity point of the mapping T. On the other hand, if $u_n\neq u_{n+1}$ for all $n\in\mathbb{N}$. By (20), we have

$$F(u_n,T(u_{n-1}),t)=F(A,B,t),$$

and

$$F(u_{n+1},T(u_n),t)=F(A,B,t),$$

for all $n\geq 1$. Thus, by (19)

$$\Psi(F(u_n,u_{n+1},t))\geq\Phi\Big((F(u_{n-1},u_n,t))^{\alpha}(F(u_n,u_{n+1},t))^{1-\alpha}\Big), \tag{21}$$

for all distinct $u_{n-1},u_n,u_{n+1}\in A$. Since, $\Phi(q)>\Psi(q)$ for all $q\in(0,1)$, by (21), we have

$$\Psi(F(u_n,u_{n+1},t))>\Psi\Big((F(u_{n-1},u_n,t))^{\alpha}(F(u_n,u_{n+1},t))^{1-\alpha}\Big).$$

Since, Ψ is a nondecreasing function, we have

$$F(u_n, u_{n+1}, t) > \left((F(u_{n-1}, u_n, t))^{\alpha}(F(u_n, u_{n+1}, t))^{1-\alpha}\right).$$

This implies that

$$(F(u_n, u_{n+1}, t))^{\alpha} > (F(u_{n-1}, u_n, t))^{\alpha}.$$

Letting $F(u_n, u_{n+1}, t) = E_n$, we have

$$\Psi((E_n)) \geq \Phi\left((E_{n-1})^{\alpha}(E_n)^{1-\alpha}\right) > \Psi\left((E_{n-1})^{\alpha}(E_n)^{1-\alpha}\right).$$

Since Ψ is non-decreasing, by (21), we have $E_n > E_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{E_n\}$ is strictly non-decreasing. Thus, it converges to some element $E \geq 1$. We show that $E = 1$. Assume on contrary that $E < 1$, by (21) we obtain the following inequality:

$$\Psi(\varepsilon-) = \lim_{n\to\infty} \Psi(E_n) \geq \lim_{n\to\infty} \Phi\left((E_{n-1})^{\alpha}(E_n)^{1-\alpha}\right) \geq \lim_{t\to E-} \inf \Phi(t).$$

This contradicts assumption (i), hence $E = 1$ and $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$. Now keeping in mind the assumption (i) and Lemma 3, we conclude that the sequence $\{u_n\}$ is Cauchy. $(X, F, *)$ is a complete non-Archimedean fuzzy metric space and A is closed subset of X. Then, there exists $u \in A$, such that $\lim_{n\to\infty} F(u_n, u, t) = 1$. Moreover,

$$\begin{aligned} F(A, B, t) &= F(u_{n+1}, T(u_n), t) \geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \\ &\geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(u_{n+1}, Tu_n, t) \\ &= F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t). \end{aligned}$$

This implies

$$F(A, B, t) \geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t).$$

As $n \to \infty$ in the above inequality, we obtain

$$F(A, B, t) \geq 1 * \lim_{n\to\infty} F(u, T(u_n), t) \geq 1 * 1 * F(A, B, t).$$

That is,

$$\lim_{n\to\infty} F(u, T(u_n), t) = F(A, B, t).$$

Therefore, $F(u, T(u_n), t) \to F(u, B, t)$ as $n \to \infty$. Since B is approximately compact with respect to A, there exists a subsequence $\{T(u_{n_k})\}$ of $\{T(u_n)\}$ such that $(Tu_{n_k}) \to \eta \in B$ as $k \to \infty$. By taking $k \to \infty$ in the following equation:

$$F(u_{n_{k+1}}, T(u_{n_k}), t) = F(A, B, t). \tag{22}$$

we have,

$$F(u, \eta, t) = F(A, B, t).$$

Since $u^* \in A_0$, $T(u^*) \in T(A_0) \subseteq B_0$, there exists $\xi \in A_0$, such that

$$F(\xi, Tu, t) = F(A, B, t). \tag{23}$$

Now, keeping in mind the Equations (22) and (22), and utilizing (19), we have

$$\Psi(F(u_{n_{k+1}}, \xi, t)) \geq \Phi\left((F(u_{n_k}, u_{n_{k+1}}, t))^{\alpha}(F(u, \xi, t))^{1-\alpha}\right) > \Psi\left((F(u_{n_k}, u_{n_{k+1}}, t))^{\alpha}(F(u, \xi, t))^{1-\alpha}\right).$$

Since, Ψ is a non-decreasing function, we have

$$F(u_{n_{k+1}}, \xi, t) > \left(F(u_{n_k}, u_{n_{k+1}}, t)\right)^{\alpha} \left(F(u, \xi, t)\right)^{1-\alpha}.$$

Thus, as $k \to \infty$, we have $F(u, \xi, t) = 1$ or $u = \xi$ and by (23) we have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping T. □

Theorem 6. *Let $(X, F, *)$ be a complete non-Archimedean fuzzy metric space and A, B be non-empty, closed subsets of X such that B is approximately compact with respect to A. Let $T\colon A \to B$ be an (Ψ, Φ)-interpolative Kannan type proximal contraction of the first kind, satisfying*

(i) Ψ is non-decreasing and $\{\Psi(q_n)\}$ and $\{\Phi(q_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \Psi(q_n) = \lim_{n\to\infty} \Phi(q_n)$, then $\lim_{n\to\infty} q_n = 1$.

(ii) A_0 is non-empty subset of A such that $T(A_0) \subseteq B_0$.

Then, T has a best proximity point.

Proof. Proceeding as in the proof of Theorem 5, we have

$$\Psi((E_n)) \geq \Phi\Big((E_{n-1})^{\alpha}(E_n)^{1-\alpha}\Big) > \Psi\Big((E_{n-1})^{\alpha}(E_n)^{1-\alpha}\Big). \tag{24}$$

By (24), we infer that $\{\Psi(E_n)\}$ is a strictly non-decreasing sequence. We have two cases here; either the sequence $\{\Psi(E_n)\}$ is bounded above or not. If $\{\Psi(E_n)\}$ is not bounded above, then

$$\inf_{E_n > \varepsilon} \Psi(E_n) > -\infty \text{ for every } \varepsilon \in (0,1); n \in \mathbb{N}.$$

It follows from Lemma 2 that $E_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\Psi(E_n)\}$ is bounded above, then, it is a convergent sequence. By (24), the sequence $\{\Phi(E_n)\}$ also converges, moreover, both have the same limit. By assumption (i), we have $\lim_{n\to\infty} E_n = 1$ or $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$, for any sequence $\{u_n\}$ in A. Presently, following the proof of Theorem 5, we have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping T. □

4.3. (Ψ, Φ)-Interpolative Hardy Rogers Type Proximal Contraction of the First Kind

Let $(X, F, *)$ be a non-Archimedean fuzzy metric space and A, B be a non-empty subsets of X. A mapping $T : A \to B$ is said to be a (Ψ, Φ)-interpolative Hardy Rogers type proximal contraction of the first kind, if there exists real numbers $\alpha, \beta, \gamma, \delta \in (0,1)$, such that $\alpha + \beta + \gamma + \delta < 1$ and satisfying the inequality

$$\begin{aligned} & F(u_1, Tv_1, t) = F(A, B, t), \\ & F(u_2, Tv_2) = F(A, B, t), \\ \Rightarrow \quad & \Psi(F(u_1, u_2, t)) \geq \Phi\left(\begin{array}{c} (F(v_1, v_2, t))^{\alpha}(F(v_1, u_1, t))^{\beta}(F(v_2, u_2, t))^{\gamma} \\ (F(v_1, u_2, t))^{\delta}(F(v_2, u_1, t))^{1-\alpha-\beta-\gamma-\delta} \end{array} \right), \end{aligned} \tag{25}$$

for all $u_1, u_{2,} v_1, v_2 \in A$, $t > 0$, $u_i \neq v_i; i \in \{1,2\}$ with $F(u_1, u_2, t) > 0$, where $\Psi, \Phi : (0,1] \to \mathbb{R}$ are two functions such that $\Phi(q) > \Psi(q)$ for $q \in (0,1)$.

Example 4. *Let $X = \mathbb{R}^2$ and $F : X \times X \times (0, \infty) \to [0,1]$ be the non-Archimedean fuzzy metric given by*

$$F(u, v, t) = \frac{t}{t + d(u, v)}$$

where $d(u,v) = \mid u_1 - v_1 \mid + \mid u_2 - v_2 \mid$ *for all* $u_1, v_1, u_2, v_2 \in X$ *with* $t > 0$. *Let* A, B *be the subset of* X *defined by*

$$A = \{(0,u), u \in \mathbb{R}\},\ B = \{(1,u), u \in \mathbb{R}\},\ \textit{then}\ F(A,B,t) = \frac{t}{t+1}.$$

Define the functions $\Psi, \Phi : (0,1] \to \mathbb{R}$ *by*

$$\Psi(q) = \sqrt{q}\ \textit{and}\ \Phi(q) = q^2\ \textit{for all}\ q \in (0,1).$$

Define the mapping $T : A \to B$ *by*

$$T(q) = \begin{cases} (1,q) & \textit{if } q \in [-1,1] \\ (1,q^2) & \textit{otherwise.} \end{cases}$$

We show that T *is an* (Ψ,Φ)*-interpolative Hardy Rogers type proximal contraction of the first kind. Letting* $u = (0,4)$, $v = (0,2)$, $x = (0,9)$, $y = (0,3)$, $\alpha = 0.01$, $\beta = 0.02$, $\gamma = 0.03$, $\delta = 0.04$ *and* $t = 1$ *we have*

$$\begin{aligned} F(u,Tv,t) &= F((0,4),T(0,2),1) = F(A,B,t), \\ F(x,Ty,t) &= F((0,9),T(0,3),1) = F(A,B,t). \end{aligned}$$

This implies that

$$\begin{aligned} \Psi(F(u,x,t)) &\geq \Phi\left(\begin{array}{l} (F(v,y,t))^{\alpha}(F(v,u,t))^{\beta}(F(y,x,t))^{\gamma} \\ (F(v,x,t))^{\delta}(F(y,u,t))^{1-\alpha-\beta-\gamma-\delta} \end{array} \right), \\ \Psi(0.1667) &\geq \Phi\left(\begin{array}{l} (0.5)^{0.01}(0.3333)^{0.02}(0.1429)^{0.03} \\ (0.125)^{0.04}(0.5)^{0.9} \end{array} \right), \\ \Psi(0.1667) &\geq \Phi(0.4519), \\ 0.4082 &\geq 0.2042. \end{aligned}$$

This shows that T *is an* (Ψ,Φ)*-interpolative Hardy Rogers type proximal contraction of the first kind. However, the following calculations show that it is not a interpolative Hardy Rogers type proximal contraction of the first kind. Indeed, for* $\alpha = 0.01$, $\beta = 0.02$, $\gamma = 0.03$, $\delta = 0.04$ *and* $t = 1$, *we have*

$$\begin{aligned} F(u,Tv,t) &= F((0,4),T(0,2),1) = F(A,B,t), \\ F(x,Ty,t) &= F((0,9),T(0,3),1) = F(A,B,t). \end{aligned}$$

This implies that

$$\begin{aligned} F(u,x,t) &\geq \begin{array}{l} (F(v,y,t))^{\alpha}(F(v,u,t))^{\beta}(F(y,x,t))^{\gamma} \\ (F(v,x,t))^{\delta}(F(y,u,t))^{1-\alpha-\beta-\gamma-\delta} \end{array} \\ 0.1667 &\geq 0.4519. \end{aligned}$$

This is a contradiction. Hence, T *is not interpolative Hardy Rogers type proximal contraction of the first kind.*

Theorem 7. *Let* $(X, F, *)$ *be a complete non-Archimedean fuzzy metric space and* A, B *be non-empty, closed subsets of* X *such that* B *is approximately compact with respect to* A. *Let* $T\colon A \to B$ *be an* (Ψ,Φ)*-interpolative Hardy Rogers type proximal contraction of the first kind satisfying:*

(i) Ψ *is non-decreasing and* $\liminf_{q \to \varepsilon-} \Phi(q) > \Psi(\varepsilon-)$ *for any* $\varepsilon \in (0,1)$.
(ii) A_0 *is non-empty subset of* A *such that* $T(A_0) \subseteq B_0$.

Then T *admits a best proximity point.*

Proof. Let u_0 be an arbitrary point in A_0. Since $T(u_0) \in T(A_0) \subseteq B_0$, there exists $u_1 \in A_0$ such that

$$F(u_1, T(u_0), t) = F(A, B, t).$$

Again for $T(u_1) \in T(A_0) \subseteq B_0$, there exists $u_2 \in A_0$ such that

$$F(u_2, T(u_1), t) = F(A, B, t).$$

This process of existence of points in A_0 implies to have a sequence $\{u_n\} \subseteq A_0$ such that

$$F(u_{n+1}, T(u_n), t) = F(A, B, t), \tag{26}$$

for all $n \in \mathbb{N}$. Observe that, for some $n \in \mathbb{N}$ such that $u_n = u_{n+1}$, from (26), we infer that the point u_n is a best proximity point of the mapping T. On the other hand, if $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then, by (26), we have

$$F(u_n, T(u_{n-1}), t) = F(A, B, t),$$

and

$$F(u_{n+1}, T(u_n), t) = F(A, B, t),$$

for all $n \geq 1$. By (25), we get

$$\begin{aligned}
\Psi(F(u_n, u_{n+1}, t)) &\geq \Phi\left(\begin{array}{l} (F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{\gamma} \\ (F(u_{n-1}, u_{n+1}, t))^{\delta}(F(u_n, u_n, t))^{1-\alpha-\beta-\gamma-\delta} \end{array}\right), \\
\Psi(F(u_n, u_{n+1}, t)) &\geq \Phi\left(\begin{array}{l} (F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{\gamma} \\ (F(u_{n-1}, u_{n+1}, t))^{\delta}(1)^{1-\alpha-\beta-\gamma-\delta} \end{array}\right), \\
\Psi(F(u_n, u_{n+1}, t)) &\geq \Phi\left(\begin{array}{l} (F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{\gamma} \\ (F(u_{n-1}, u_{n+1}, t))^{\delta} \end{array}\right), \\
\Psi(F(u_n, u_{n+1}, t)) &\geq \Phi\left(\begin{array}{l} (F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{\gamma} \\ (F(u_{n-1}, u_n, t))^{\delta}(F(u_n, u_{n+1}, t))^{\delta} \end{array}\right),
\end{aligned} \tag{27}$$

for all distinct $u_{n-1}, u_n, u_{n+1} \in A$. Since, $\Phi(q) > \Psi(q)$ for all $q \in (0,1)$, by (27), we have

$$\Psi(F(u_n, u_{n+1}, t)) > \Psi\left(\begin{array}{l} (F(u_{n-1}, u_n, t))^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{\gamma} \\ (F(u_{n-1}, u_n, t))^{\delta}(F(u_n, u_{n+1}, t))^{\delta} \end{array}\right).$$

Since, Ψ is non decreasing function, we have

$$F(u_n, u_{n+1}, t) > \begin{array}{l} F(u_{n-1}, u_n, t)^{\alpha}(F(u_{n-1}, u_n, t))^{\beta}(F(u_n, u_{n+1}, t))^{\gamma} \\ (F(u_{n-1}, u_n, t))^{\delta}(F(u_n, u_{n+1}, t))^{\delta}. \end{array}$$

This implies that

$$F(u_n, u_{n+1}, t) > F(u_{n-1}, u_n, t)^{\alpha+\beta+\delta} F(u_n, u_{n+1}, t)^{\gamma+\delta}.$$

Let $F(u_n, u_{n+1}, t) = E_n$, so that

$$\Psi((E_n)) \geq \Phi\Big((E_{n-1})^{\alpha+\beta+\delta}(E_n)^{\gamma+\delta}\Big) > \Psi\Big((E_{n-1})^{\alpha+\beta+\delta}(E_n)^{\gamma+\delta}\Big).$$

Suppose that $E_{n-1} > E_n$ for some $n \geq 1$. Since Ψ is non-decreasing, we have $(E_n)^{\alpha+\beta+\delta} < (E_n)^{\alpha+\beta+\delta}$. Consequently, we have $E_n > E_{n-1}$ for all $n \in \mathbb{N}$. This implies $E_n > E_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{E_n\}$ is strictly non-decreasing.

Thus, it converges to some element $E \geq 1$. Assuming on the contrary $E < 1$, we obtain the following:

$$\Psi(\varepsilon-) = \lim_{n\to\infty} \Psi(E_n) \geq \lim_{n\to\infty} \Phi\Big((E_{n-1})^{\alpha+\beta+\delta}(E_n)^{\gamma+\delta}\Big) \geq \lim_{t\to E-} \inf \Phi(t).$$

This contradicts the assumption (i), hence, $E = 1$ and $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$. Now keeping in mind the assumption (i) and Lemma 3, we conclude that $\{u_n\}$ is a cauchy sequence. Since $(X, F, *)$ is a complete non-Archimedean fuzzy metric space and A is closed subset of X. Then, there exists $u \in A$, such that $\lim_{n\to\infty} F(u_n, u, t) = 1$. Moreover,

$$\begin{aligned} F(A,B,t) &= F(u_{n+1}, T(u_n), t) \\ &\geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \\ &\geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(u_{n+1}, Tu_n, t) \\ &= F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t). \end{aligned}$$

This implies that

$$F(A,B,t) \geq F(u_{n+1}, u, t) * F(u, T(u_n), t) \geq F(u_{n+1}, u, t) * F(u, u_{n+1}, t) * F(A, B, t).$$

As $n \to \infty$ in the above inequality, we obtain

$$F(A,B,t) \geq 1 * \lim_{n\to\infty} F(u, T(u_n), t) \geq 1 * 1 * F(A, B, t).$$

That is,

$$\lim_{n\to\infty} F(u, T(u_n), t) = F(A, B, t).$$

Therefore, $F(u, T(u_n), t) \to F(u, B, t)$ as $n \to \infty$. Since B is approximately compact with respect to A, there exists a subsequence $\{T(u_{n_k})\}$ of $\{T(u_n)\}$ such that $(Tu_{n_k}) \to \eta \in B$ as $k \to \infty$. Therefore, applying $k \to \infty$ in the following equation,

$$F(u_{n_{k+1}}, T(u_{n_k}), t) = F(A, B, t), \tag{28}$$

we have,

$$F(u, \eta, t) = F(A, B, t).$$

Since, $u \in A_0$, $T(u) \in T(A_0) \subseteq B_0$, there exists $\xi \in A_0$ such that

$$F(\xi, Tu, t) = F(A, B, t). \tag{29}$$

Now, keeping in mind the Equations (28) and (29), by (25), we have

$$\begin{aligned} \Psi(F(u_{n_{k+1}}, \xi, t)) &\geq \Phi\left(\begin{array}{c} (F(u_{n_k}, u, t))^{\alpha} (F(u_{n_k}, u_{n_{k+1}}, t))^{\beta} (F(u, \xi, t))^{\gamma} \\ (F(u_{n_k}, \xi, t))^{\delta} (F(u, u_{n_{k+1}}, t))^{1-\alpha-\beta-\gamma-\delta} \end{array} \right), \\ &> \Psi\left(\begin{array}{c} (F(u_{n_k}, u, t))^{\alpha} (F(u_{n_k}, u_{n_{k+1}}, t))^{\beta} (F(u, \xi, t))^{\gamma} \\ (F(u_{n_k}, \xi, t))^{\delta} (F(u, u_{n_{k+1}}, t))^{1-\alpha-\beta-\gamma-\delta} \end{array} \right). \end{aligned}$$

Since, Ψ is non-decreasing function, we have

$$F(u_{n_{k+1}}, \xi, t) > \begin{array}{l} (F(u_{n_k}, u, t))^{\alpha} (F(u_{n_k}, u_{n_{k+1}}, t))^{\beta} (F(u, \xi, t))^{\gamma} \\ (F(u_{n_k}, \xi, t))^{\delta} (F(u, u_{n_{k+1}}, t))^{1-\alpha-\beta-\gamma-\delta}. \end{array}$$

Thus, as $k \to \infty$, we have $F(u, \xi, t) = 1$ or $u = \xi$. Finally, by (29) we have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping, T. □

Theorem 8. *Let $(X, F, *)$ be a complete non-Archimedean fuzzy metric space and A, B be non-empty, closed subsets of X such that B is approximately compact with respect to A. Let $T\colon A \to B$ be an (Ψ, Φ)-interpolative Hardy Rorgers type proximal contraction of the first kind, satisfying:*

(i) Ψ *is non-decreasing, the sequences* $\{\Psi(q_n)\}$ *and* $\{\Phi(q_n)\}$ *are convergent such that* $\lim_{n\to\infty} \Psi(q_n) = \lim_{n\to\infty} \Phi(q_n)$, *then* $\lim_{n\to\infty}(q_n) = 1$.

(ii) *A_0 is a non-empty subset of A such that $T(A_0) \subseteq B_0$.*

Then T admits a best proximity point.

Proof. Proceeding as in the proof of Theorem 7, we have

$$\Psi((E_n)) \geq \Phi\Big((E_{n-1})^{\alpha+\beta+\delta}(E_n)^{\gamma+\delta}\Big) > \Psi\Big((E_{n-1})^{\alpha+\beta+\delta}(E_n)^{\gamma+\delta}\Big). \tag{30}$$

By (30), we infer that $\{\Psi(E_n)\}$ is strictly non-decreasing sequence. We have two cases here; either the sequence $\{\Psi(E_n)\}$ is bounded above or not. If $\{\Psi(E_n)\}$ is not bounded above, then

$$\inf_{E_n > \varepsilon} \Psi(E_n) > -\infty \text{ for every } \varepsilon \in (0,1), n \in \mathbb{N}.$$

It follows from Lemma 2 that $E_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\Psi(E_n)\}$ is bounded above, then, it is a convergent sequence. By (30), the sequence $\{\Phi(E_n)\}$ also converges, moreover, both have the same limit. By assumption (i), we have $\lim_{n\to\infty} E_n = 1$ or $\lim_{n\to\infty} F(u_n, u_{n+1}, t) = 1$ for any sequence $\{u_n\}$ in A. Presently, using the arguments given in the proof of theorem 7, we have

$$F(u, T(u), t) = F(A, B, t).$$

This shows that the point u is a best proximity point of the mapping T. □

5. Conclusions

We have produced several new types of contractive condition that ensures the existence of best proximity points in non-Archimedean complete fuzzy metric spaces. The examples show that the new contractive conditions generalize the corresponding contractions given in earlier works. According to the nature (linear and nonlinear) of contractions (13), (19) and (25), these can be used to demonstrate the existence of solutions to fuzzy models of linear and nonlinear dynamic systems. The study carried out in this paper generalizes the valuable research work presented in [12,13,15,23,24]. This work can be extended by using the ideas given in [23,24].

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