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Asymptotic Behavior on a Linear Self-Attracting Diffusion Driven by Fractional Brownian Motion

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Abstract: Let $B^H = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst index $\frac{1}{2} \leq H < 1$. In this paper, we consider the linear self-attracting diffusion: $dX_t^H = dB_t^H + \sigma X_t^H dt - \theta\left(\int_0^t (X_s^H - X_u^H)ds\right)dt + \nu dt$ with $X_0^H = 0$, where $\theta > 0$ and $\sigma, \nu \in \mathbb{R}$ are three parameters. The process is an analogue of the self-attracting diffusion (Cranston and Le Jan, *Math. Ann.* **303** (1995), 87–93). Our main aim is to study the large time behaviors. We show that the solution $(t - \frac{\sigma}{\theta})^H (X_t^H - X_\infty^H)$ converges in distribution to a normal random variable, as t tends to infinity, and obtain two strong laws of large numbers associated with the solution X^H.

Keywords: diffusion process; fractional Brownian motion; rate of convergence; law of large numbers

MSC: 58J65; 60G22; 41A25; 60F15

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1. Introduction

In 1995, Cranston and Le Jan [1] introduced a linear self-attracting diffusion

$$X_t = B_t + \nu t - \theta \int_0^t \int_0^s (X_s - X_u) du ds, \quad t \geq 0 \tag{1}$$

with $\theta > 0$ and $\nu \in \mathbb{R}$, where B is a 1-dimensional standard Brownian motion. It showed that the process X_t converges in L^2 and almost surely as t tends to infinity. This is a special case of path-dependent stochastic differential equations. In 2008, inspired by research on fractional Brownian motion as a polymer model, Yan et al. [2] considered the analogue driven by fractional Brownian motion with $\frac{1}{2} \leq H < 1$, and, moreover, Sun and Yan [3] studied related parameter estimation. In fact, such a path-dependent stochastic differential equation was first developed by Durrett and Rogers [4] and was introduced in 1992 as a model for the shape of a growing polymer (Brownian polymer):

$$X_t = B_t + \int_0^t \int_0^s f(X_s - X_u) du ds \tag{2}$$

where B_t is a standard Brownian motion on $\mathbb{R}^d$ and f Lipschitz continuous (which is called the interacting function). X_t corresponds to the location of the end of the polymer at time t. The model is a continuous analogue of the motion of edge self-interacting random walk (see Pemantle [5]). We may call this solution a Brownian motion interacting with its own passed trajectory, i.e., a *self-interacting motion*. In general, Equation (2) defines a self-interacting diffusion without any assumption on f. We will call it self-repelling (resp. self-attracting) if, for all $x \in \mathbb{R}^d$, $x \cdot f(x) \geq 0$ (resp. ≤ 0); in other words, if it is more likely to stay away from (and, respectively, come back to) the places that it has already visited before. In 2002, Benaïm et al. [6] also introduced a self-interacting diffusion with dependence on the (convolved) empirical measure. A great difference between these diffusions and Brownian polymers is that the drift term is divided by t. It is important to note that, in many cases of f, the interaction potential is attractive enough to compare the diffusion (a bit modified) to an Ornstein–Uhlenbeck process, which gives access to its ergodic behaviour.

On the other hand, in 2015, Benaim et al. [7] studied the self-repelling diffusion of the form

$$X_t = B_t + \int_0^t g(X_s)ds - \int_0^t \int_0^s f(X_s - X_u)duds,$$

where B is a Brownian motion and f is a 2π-periodic function with sufficient regularity. Under a suitable condition of the initial drift profile g, they introduced the Feller property and invariant measure of the transition semigroup. More works can be found in the works by Chakravarti and Sebastian [8], Cherayil and Biswas [9], Cranston and Mountford [10], Gauthier [11], Herrmann and Roynette [12], Herrmann and Scheutzow [13], Mountford and P. Tarrés [14], Sun and Yan [15], Chen and Shen [16] and the references. Motivated by these results, in this paper, we consider the equation

$$X_t^H = B_t^H + \sigma \int_0^t X_s^H ds + \nu t - \theta \int_0^t \int_0^s \left(X_s^H - X_u^H \right) duds, \tag{3}$$

where $\theta > 0, \sigma, \nu \in \mathbb{R}$ are three parameters and B^H is a fBm with Hurst index $\frac{1}{2} \leq H < 1$. Perhaps this process should be called the fractional Ornstein–Uhlenbeck process with linear self-attracting drift. Our main aim is to expand and prove the next statements.

(I) Let $\frac{1}{2} \leq H < 1$ and $\theta > 0$. Define the function

$$h_{\theta,\sigma}(s) = 1 - (\theta s - \sigma) e^{\frac{\theta}{2}s^2 - \sigma s} \int_s^\infty e^{-\frac{\theta}{2}u^2 + \sigma u} du.$$

Then, X_t^H converges to

$$X_\infty^H = \int_0^\infty h_{\theta,\sigma}(s) dB_s^H + \nu \int_0^\infty h_{\theta,\sigma}(s) ds$$

in L^2 and almost surely as t tends to infinity. Moreover, we have

$$\left(t - \frac{\sigma}{\theta} \right)^H \left(X_t^H - X_\infty^H \right) \longrightarrow N(0, \zeta(H,\theta)) \tag{4}$$

in distribution, as t tends to infinity, where $N(0,\sigma)$ denotes a central normal random variable with the variance σ and

$$\zeta(H,\theta) = H\theta^{-2H}\Gamma(2H).$$

(II) Let $\frac{1}{2} \leq H < 1$ and $\theta > 0$. Define the process

$$Y_t^H = \int_0^t \left(r - \frac{\sigma}{\theta} \right) dX_r^H, t \geq 0.$$

Then, we have

$$\frac{1}{T} \int_0^T Y_t^H dt \longrightarrow \frac{\nu}{\theta} \tag{5}$$

and

$$\frac{1}{T^{3-2H}} \int_0^T (Y_t^H)^2 dt \longrightarrow \frac{H\theta^{-2H}}{3-2H} \Gamma(2H) \tag{6}$$

in L^2 and almost surely as T tends to infinity.

It is important to note that the above convergences are not true if $\theta = 0$, i.e., the process (3) is an Ornstein–Uhlenbeck process. This also points out, in general, that the long-time behavior of the process (3) is much more complex than that of the Ornstein–Uhlenbeck process, so many cases cannot be observed in the Ornstein–Uhlenbeck process. When $\theta > 0$, we can basically conclude that the asymptotic behavior of the system is very sensitive to the complexity of the dependent structure of a driving noise, and the driving noise is the main contradiction that leads to the complexity of the asymptotic behavior of such processes. Guo et al. [17] considered the model driven by sub-fBm and $\sigma = 0$. When $\theta < 0$, the asymptotic behavior of the process (3) basically does not depend on the selection of noise (in fact, the results in the study by Sun and Yan [15] support this judgment). We also need to say that such an equation can be written as

$$X_t^H = B_t^H + \int_0^t \int_0^s (\theta r + \sigma) dX_r^H ds + \nu t$$

with $X_0^H = 0$, which is a special case of the equation

$$X_t^H = x + B_t^H + \int_0^t \int_0^s g_1(r) dX_r^H ds + g_2(t)$$

with $X_0^H = x$, where g_1 and g_2 are two Borel measurable functions. We will consider this general equation in a future paper. This paper is organized as follows. In Section 2, we present some preliminaries for fractional Brownian motion and Malliavin calculus. In Section 3, we prove the statement (I). The statement (II) is given in Section 4.

2. Preliminaries

In this section, we briefly recall some basic definitions and results of fractional Brownian motion. For more aspects of this material we refer to Duncan et al. (2000) [18], Hu (2005) [19], Mishura (2008) [20] and the references therein. Throughout this paper, we assume that $\frac{1}{2} \leq H < 1$ is arbitrary but fixed and we let $B^H = \{B_t^H, t \geq 0\}$ be a one-dimensional fBm with Hurst index H defined on $(\Omega, \mathcal{F}^H, P)$ such that $\mathcal{F}^H$ is the sigma-field generated by B^H. A fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geq 0\}$ with Hurst index H is a mean zero Gaussian process such that $B_1^H = 0$ and

$$E\left[B_t^H B_s^H\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $t, s > 0$. For $H = \frac{1}{2}$, B^H coincides with the standard Brownian motion B. B^H is neither a semi-martingale nor a Markov process unless $H = \frac{1}{2}$, so many of the powerful techniques from stochastic analysis are not available when dealing with B^H.

Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}, t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $s, t \in [0,T]$. When $\frac{1}{2} < H < 1$, we have

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \varphi(t) \varphi(s) |t-s|^{2H-2} ds dt < \infty$$

for all $\varphi, \psi \in \mathcal{E}$ with $\alpha_H = H(2H-1)$. The elements of the Hilbert space $\mathcal{H}$ may not be functions but distributions of negative order (see, for instance, Pipiras and Taqqu (2001)). In order to avoid unnecessary trouble, we introduce a subspace of $\mathcal{H}$ as follows:

$$|\mathcal{H}| = \left\{\varphi : [0,T] \to \mathbb{R} \mid \|\varphi\|_{|\mathcal{H}|} < \infty\right\}$$

for all $\frac{1}{2} < H < 1$, where

$$\|\varphi\|^2_{|\mathcal{H}|} = \alpha_H \int_0^T \int_0^T |\varphi(t)||\varphi(s)||t-s|^{2H-2} ds dt.$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}|}$ and that $\mathcal{E}$ is dense in $|\mathcal{H}|$. Moreover, we have

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_{a,b} \int_0^T \int_0^T \varphi(t)\psi(s)(t \wedge s)^a |t-s|^{b-1} ds dt$$

for all $\varphi, \psi \in |\mathcal{H}|$ and we also have

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}$$

for any $T > 0$.

As usual, we define the Wiener integral

$$B^H(\varphi) = \int_0^T \varphi(t) dB_t^H, \quad \varphi \in \mathcal{H} \tag{7}$$

as the limit in probability of a *Riemann sum*, which is a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by B^H, and it can be understood as an extension of the mapping $1_{[0,t]} \mapsto B_t^H$. The Wiener integral $B^H(\varphi)$ is well-defined as a mean zero Gaussian random variable such that

$$E\left| \int_0^T \varphi(t) dB_t^H \right|^2 = \|\varphi\|^2_{\mathcal{H}}$$

for all $\varphi \in \mathcal{H}$. If the Wiener integral $\int_0^T \varphi(t) dB_t^H$ is well-defined for every $T > 0$, then we can define the integral

$$\int_0^\infty \varphi(t) dB_t^H$$

for any φ satisfying

$$\|\varphi\|^2_{\mathcal{H}} := \alpha_H \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)|t-s|^{2H-2} ds dt < \infty.$$

Thus, we regard (7) as the indefinite Wiener integral.

Consider the set $\mathcal{S}$ of smooth functionals of the form

$$F = f\Big(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)\Big) \tag{8}$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all of its derivatives are bounded) and $\varphi_i \in \mathcal{H}$. The derivative operator D^H (the Malliavin derivative) of a functional F of form (8) is defined as

$$D^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\Big(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)\Big)\varphi_j.$$

The derivative operator D^H is a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote the closure of $\mathcal{F}$ by $\mathbb{D}^{1,2}$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^H F\|^2_{\mathcal{H}}}.$$

The divergence integral δ^H is the adjoint of the derivative operator D^H. That is, we say that a random variable u in $L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator δ^H, denoted by $Dom(\delta^H)$, if

$$E\left|\left\langle D^H F, u\right\rangle_{\mathcal{H}}\right| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{F}$. In this case, $\delta^H(u)$ is defined by the duality relationship

$$E\left[F\delta^H(u)\right] = E\left\langle D^H F, u\right\rangle_{\mathcal{H}}$$

for every $u \in \mathbb{D}^{1,2}$. We have $\mathbb{D}^{1,2} \subset Dom(\delta^H)$ and

$$\begin{aligned} E|\delta(u)|^2 &= \alpha_H \int_0^T \int_0^T E[u_s u_t]|s-t|^{2H-2}dsdr \\ &\quad + \alpha_H^2 \int_0^T \int_0^T \int_0^T \int_0^T E\left[(D_x^H u_s)(D_y^H u_t)\right]|s-y|^{2H-2}|t-x|^{2H-2}dsdtdxdy \end{aligned}$$

for all $u \in \mathbb{D}^{1,2}$. We will use the notation

$$\delta^H(u) = \int_0^T u_s \delta B_s^H$$

to express the Skorohod integral of a process u, and the indefinite Skorohod integral is defined as $\int_0^t u_s \delta B_s^H = \delta^H(u 1_{[0,t]})$.

Finally, we recall that the fBm $t \to B_t^H$ admits almost surely a bounded $p > \frac{1}{H}$-variation on any finite interval. As an immediate result, one can define the Young integral

$$\int_0^t u_s dB_s^H$$

as the limit in probability of a *Riemann sum*, and

$$u_t B_t^H = \int_0^t u_s dB_s^H + \int_0^t B_s^H du_s$$

provided the process u is of bounded q-variation on any finite interval with $q > 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. Moreover, if $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ such that

$$\int_0^T \int_0^T |D_s^H u_t||t-s|^{2H-2}dsdt < \infty.$$

then, we have

$$\int_0^T u_s dB_s^H = \int_0^T u_s \delta B_s^H + \alpha_H \int_0^T \int_0^T D_s^H u_t |t-s|^{2H-2}dsdt.$$

3. Large Time Behaviors

The object of this section is to expound and prove the large time behaviors of the linear self-attracting diffusion

$$X_t^H = B_t^H + \sigma \int_0^t X_s^H ds + \nu t - \theta \int_0^t \int_0^s (X_s^H - X_u^H)duds \tag{9}$$

with $\theta > 0$, $\sigma, \nu \in \mathbb{R}$, where B_t^H is a fractional Brownian motion with $\frac{1}{2} \leq H < 1$. For simplicity, throughout this paper, C stand for a positive constant that may depend on H, θ, σ, ν, and its value may be different in appearance. This assumption is also suitable for c.

Proposition 1. *Equation (9) admits a unique solution, and the solution can be expressed as*

$$X_t^H = \int_0^t h_{\theta,\sigma}(t,s)dB_s^H + \nu \int_0^t h_{\theta,\sigma}(t,s)ds, \tag{10}$$

where

$$h_{\theta,\sigma}(t,s) = \left(1 - (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}\int_s^t e^{-\frac{\theta}{2}u^2 + \sigma u}du\right)1_{\{t \geq s\}} \tag{11}$$

for $s, t \geq 0$.

Proof. We can show the result by integration by parts. Of course, we also can regard (9) as a deterministic equation since the diffusion coefficient is equal to a constant. Thus, we solve the equation by the variation of constants method. In fact, Equation (9) is equivalent to

$$\ddot{X}_t^H = \ddot{B}_t^H - \theta t \dot{X}_t^H + \sigma \dot{X}_t^H \tag{12}$$

in distribution, with $X_0^H = 0$ and $\dot{X}_0^H = \dot{B}_0^H + \nu$. Let Z_t be the solution of the equation

$$\dot{Z}_t = -\theta t Z_t + \sigma Z_t.$$

Then, we have

$$Z_t = C_t e^{-\frac{\theta}{2}t^2 + \sigma t}.$$

Through the variation of constants method, we can assume that the process

$$\dot{X}_t^H = C_t^H e^{-\frac{\theta}{2}t^2 + \sigma t} \tag{13}$$

is the solution of Equation (12) with $X_0^H = 0$ and $\dot{X}_0^H = \dot{B}_0^H + \nu$. Then, we have $C_0^H = \dot{B}_0^H + \nu$ and

$$\dot{C}_t^H e^{-\frac{\theta}{2}t^2 + \sigma t} = \ddot{B}_t^H$$

for all $t \geq 0$, which implies that

$$\begin{aligned}
C_t^H &= \int_0^t e^{\frac{\theta}{2}s^2 - \sigma s}\ddot{B}_s^H ds + C_0^H \\
&= \dot{B}_t^H e^{\frac{\theta}{2}t^2 - \sigma t} - \dot{B}_0^H - \int_0^t (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}dB_s^H + C_0^H \\
&= e^{\frac{\theta}{2}t^2 - \sigma t}(\nu + \dot{B}_t^H) - \int_0^t (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}(dB_s^H + \nu ds).
\end{aligned}$$

It follows from $X_0^H = 0$ and (13) that

$$\begin{aligned}
X_t^H &= \int_0^t C_s^H e^{-\frac{\theta}{2}s^2 + \sigma s}ds + X_0^H \\
&= \int_0^t \left(\nu + \dot{B}_u^H - e^{-\frac{\theta}{2}u^2 + \sigma u}\int_0^u (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}(\nu ds + dB_s^H)\right)du \\
&= \nu t + B_t^H - \int_0^t \left(e^{-\frac{\theta}{2}u^2 + \sigma u}\int_0^u (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}(\nu ds + dB_s^H)\right)du \\
&= \nu t + B_t^H - \int_0^t (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}\left(\int_s^t e^{-\frac{\theta}{2}u^2 + \sigma u}du\right)(\nu ds + dB_s^H) \\
&= \int_0^t \left(1 - (\theta s - \sigma)e^{\frac{\theta}{2}s^2 - \sigma s}\int_0^t e^{-\frac{\theta}{2}u^2 + \sigma u}du\right)\left(\nu ds + dB_s^H\right) \\
&= \int_0^t h_{\theta,\sigma}(t,s)dB_s^H + \nu \int_0^t h_{\theta,\sigma}(t,s)ds
\end{aligned}$$

for all $t \geq 0$. □

Lemma 1. *Let $\theta > 0$. Then, the function $(t,s) \mapsto h_{\theta,\sigma}(t,s)$ admits the following properties:*

(1) The limit

$$h_{\theta,\sigma}(s) := \lim_{t\to\infty} h_{\theta,\sigma}(t,s) = 1 - (\theta s - \sigma) e^{\frac{\theta}{2}s^2 - \sigma s} \int_s^\infty e^{-\frac{\theta}{2}u^2 + \sigma u} du \tag{14}$$

exists for all $s \geq 0$.

(2) For all $t \geq s \geq 0$, we have

$$e^{-\frac{1}{2}\theta(t^2 - s^2) + \sigma(t-s)} \leq h_{\theta,\sigma}(t,s) \leq 1 + C_{\theta,\sigma} \tag{15}$$

for all $t \geq s \geq 0$.

(3) When $\sigma \leq 0$, we have

$$h_{\theta,\sigma}(s) \leq h_{\theta,\sigma}(t,s)$$

for all $t \geq s \geq 0$. When $\sigma > 0$, we have

$$h_{\theta,\sigma}(t,s) \leq h_{\theta,\sigma}(s)$$

for all $0 \leq s \leq \frac{\sigma}{\theta}$ and

$$h_{\theta,\sigma}(s) \leq h_{\theta,\sigma}(t,s)$$

for all $s > \frac{\sigma}{\theta}$.

(4) The estimates

$$0 \leq h_{\theta,\sigma}(s) \leq C_{\theta,\sigma} \min\left\{1, \frac{1}{(\theta s - \sigma)^2}\right\} \tag{16}$$

hold for all $s \geq 0$.

(5) For all $t \geq u \geq 0$, we have $h_{\theta,\sigma}(t, \frac{\sigma}{\theta}) = h_{\theta,\sigma}(t,t) = 1$ and

$$\int_u^t h_{\theta,\sigma}(t,s) ds = e^{\frac{1}{2}\theta u^2 - \sigma u} \int_u^t e^{-\frac{1}{2}\theta s^2 + \sigma s} ds.$$

(6) We have

$$\begin{aligned} &|h_{\theta,\sigma}(t,s_2) - h_{\theta,\sigma}(s_2)||h_{\theta,\sigma}(t,s_1) - h_{\theta,\sigma}(s_1)| \\ &\qquad \leq \frac{|\theta s_1 - \sigma||\theta s_2 - \sigma|}{(\theta t - \sigma)^2} e^{\frac{\theta}{2}(s_1^2 + s_2^2) - \sigma(s_1 + s_2)} \cdot e^{-\theta t^2 + 2\sigma t} \end{aligned}$$

for $0 < s_1, s_2 \leq t$. Moreover, we have

$$\left| \int_0^t [h_{\theta,\sigma}(t,s) - h_{\theta,\sigma}(s)] ds \right| \leq \frac{1}{|\theta t - \sigma|} \tag{17}$$

for all $t > 0$.

Proof. The statement (1) is trivial. For the statement (2), we have

$$e^{-\frac{1}{2}\theta(t^2 - s^2) + \sigma(t-s)} \leq h_{\theta,\sigma}(t,s) \leq 1$$

if $\sigma \leq 0$. When $\sigma > 0$, we have

$$h_{\theta,\sigma}(t,s) = 1 + \left(\frac{\sigma}{\theta} - s\right) e^{\frac{1}{2}\theta(\frac{\sigma}{\theta} - s)^2} \int_{\frac{\sigma}{\theta} - s}^{\frac{\sigma}{\theta} - t} e^{-\frac{1}{2}\theta x^2} dx \leq 1 + \frac{\sigma^2}{\theta^2}$$

if $0 \le s \le t \le \frac{\sigma}{\theta}$, and

$$h_{\theta,\sigma}(t,s) = 1 + \left(\frac{\sigma}{\theta} - s\right) e^{\frac{1}{2}\theta(\frac{\sigma}{\theta}-s)^2} \int_{s-\frac{\sigma}{\theta}}^{t-\frac{\sigma}{\theta}} e^{-\frac{1}{2}\theta x^2} dx \le 1 + \frac{\sigma^2}{\theta^2} \le 1 + \frac{\sigma^2}{\theta^2} e^{\frac{\sigma^2}{2\theta}}$$

if $0 \le s \le \frac{\sigma}{\theta} \le t$, and

$$e^{-\frac{1}{2}\theta(t^2-s^2)+\sigma(t-s)} \le h_{\theta,\sigma}(t,s) \le 1$$

if $t \ge s \ge \frac{\sigma}{\theta}$. Similarly, one can show this for the other statements. □

Lemma 2. *Let $\theta > 0$ and denote*

$$\Delta_\theta(t) := \int_0^t h_{\theta,\sigma}(t,s)ds - \int_0^\infty h_{\theta,\sigma}(s)ds \tag{18}$$

for all $t \ge 0$. Then, we have

$$\lim_{t\to\infty} \left(t - \frac{\sigma}{\theta}\right) \Delta_\theta(t) = 0.$$

Proof. This is a simple calculus exercise. In fact, we have that

$$\begin{aligned}
&\int_0^t h_{\theta,\sigma}(t,s)ds - \int_0^\infty h_{\theta,\sigma}(s)ds \\
&= \int_0^t (h_{\theta,\sigma}(t,s) - h_{\theta,\sigma}(s))ds - \int_t^\infty h_{\theta,\sigma}(s)ds \\
&= \left(e^{\frac{1}{2}\theta t^2 - \sigma t} - 1\right) \int_t^\infty e^{-\frac{1}{2}\theta u^2 + \sigma u} du - \int_t^\infty h_{\theta,\sigma}(s)ds
\end{aligned}$$

for all $t \ge 0$ and $\theta > 0$. Thus, the Lemma follows from convergences

$$\begin{aligned}
&\lim_{t\to\infty} (t - \frac{\sigma}{\theta}) \left(e^{\frac{1}{2}\theta t^2 - \sigma t} - 1\right) \int_t^\infty e^{-\frac{1}{2}\theta u^2 + \sigma u} du \\
&= \lim_{t\to\infty} \frac{1}{\left(t - \frac{\sigma}{\theta}\right)^{-1} e^{-\frac{1}{2}\theta t^2 + \sigma t}} \int_t^\infty e^{-\frac{1}{2}\theta u^2 + \sigma u} du = \frac{1}{\theta}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to\infty} \left(t - \frac{\sigma}{\theta}\right) \int_t^\infty h_{\theta,\sigma}(s)ds &= \lim_{t\to\infty} \frac{1}{\left(t - \frac{\sigma}{\theta}\right)^{-1}} \int_t^\infty h_{\theta,\sigma}(s)ds \\
&= \lim_{t\to\infty} \left(t - \frac{\sigma}{\theta}\right)^2 h_{\theta,\sigma}(t) = \frac{1}{\theta}
\end{aligned}$$

for $\theta > 0$. This completes the proof. □

Lemma 3. *Let $\theta > 0$. Then, we have*

$$\lim_{t\to\infty} (\theta t - \sigma)^3 \frac{d}{dt} h_{\theta,\sigma}(t) = -2\theta^2. \tag{19}$$

Proof. Given $\theta > 0$, integration by parts implies that

$$\begin{aligned}
\int_t^\infty e^{-\frac{1}{2}\theta u^2 + \sigma u} du &= -\int_t^\infty \frac{1}{\theta u - \sigma} e^{-\frac{1}{2}\theta u^2 + \sigma u} d\left(-\frac{1}{2}\theta u^2 + \sigma u\right) \\
&= \frac{1}{\theta t - \sigma} e^{-\frac{1}{2}\theta t^2 + \sigma t} - \theta \int_t^\infty \frac{1}{(\theta u - \sigma)^2} e^{-\frac{1}{2}\theta u^2 + \sigma u} du \\
&= \frac{1}{\theta t - \sigma} \left(1 - \frac{\theta}{(\theta t - \sigma)^2} + \frac{3\theta^2}{(\theta t - \sigma)^4} - \frac{15\theta^3}{(\theta t - \sigma)^6} + o(t^{-6})\right) e^{-\frac{1}{2}\theta t^2 + \sigma t}
\end{aligned}$$

for all $t > (0 \vee \frac{\sigma}{\theta})$. It follows that

$$\begin{aligned}\frac{d}{dt}h_{\theta,\sigma}(t) &= -\left(\theta e^{\frac{\theta}{2}t^2-\sigma t} + (\theta t-\sigma)^2 e^{\frac{\theta}{2}t^2-\sigma t}\right)\int_t^\infty e^{-\frac{\theta}{2}u^2+\sigma u}du \\ &\quad + (\theta t-\sigma)e^{\frac{\theta}{2}t^2-\sigma t}e^{-\frac{\theta}{2}t^2+\sigma t} \\ &= -\left(\theta + (\theta t-\sigma)^2\right)e^{\frac{\theta}{2}t^2-\sigma t}\int_t^\infty e^{-\frac{\theta}{2}u^2+\sigma u}du + (\theta t-\sigma) \\ &= -\left(\frac{\theta}{\theta t-\sigma} + (\theta t-\sigma)\right) \\ &\quad \cdot\left(1 - \frac{\theta}{(\theta t-\sigma)^2} + \frac{3\theta^2}{(\theta t-\sigma)^4} - \frac{15\theta^3}{(\theta t-\sigma)^6} + o(t^{-6})\right) + (\theta t-\sigma) \\ &= -\frac{2\theta^2}{(\theta t-\sigma)^3} + \frac{12\theta^3}{(\theta t-\sigma)^5} - \frac{15\theta^5}{(\theta t-\sigma)^7} + o(t^{-7})\end{aligned}$$

for all $t > (0 \vee \frac{\sigma}{\theta})$, which implies that

$$\lim_{t\to\infty}(\theta t-\sigma)^3\frac{\mathrm{d}}{\mathrm{d}t}h_{\theta,\sigma}(t) = -2\theta^2.$$

This completes the proof. □

Lemma 4. *Let $\frac{1}{2} < H < 1$, $\theta > 0$ and $\sigma \in \mathbb{R}$. Then, the supremum*

$$\begin{aligned}\Xi_{H,\theta,\sigma} := \sup_{t\geq 0}\Bigg\{& e^{-H\theta t^2+2H\sigma t}\left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2-2H} \\ &\cdot\int_0^t\int_0^t(\theta u-\sigma)(\theta v-\sigma)e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}\frac{dvdu}{|u-v|^{2-2H}}\Bigg\}\end{aligned} \tag{20}$$

is finite and non-zero.

Proof. By the continuity, the Lemma is equivalent to

$$\begin{aligned}\Lambda_{H,\theta,\lambda} := \lim_{t\to\infty}\Bigg\{& e^{-H\theta t^2+2H\sigma t}\left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2-2H} \\ &\cdot\int_0^t\int_0^t(\theta u-\sigma)(\theta v-\sigma)e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}\frac{dvdu}{|u-v|^{2-2H}}\Bigg\} \in (0,\infty)\end{aligned}$$

for $\frac{1}{2} < H < 1$, $\theta > 0$ and $\sigma \in \mathbb{R}$. According to L'Hospital's rule and making the substitution

$$\frac{1}{2}\theta\left(t^2-v^2\right) - \sigma(t-v) = x,$$

we have

$$
\begin{aligned}
\Lambda_{H,\theta,\sigma} &= \lim_{t\to\infty} \frac{2e^{-\theta t^2+2\sigma t}}{(\theta t-\sigma)^{2-2H}} \int_0^t \int_0^u (\theta u-\sigma)(\theta v-\sigma) e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)} \frac{dvdu}{(u-v)^{2-2H}} \\
&= \lim_{t\to\infty} \frac{1}{(\theta t-\sigma)^{2-2H} e^{\frac{1}{2}\theta t^2-\sigma t}} \int_0^t (\theta v-\sigma) e^{\frac{1}{2}\theta v^2-\sigma v}(t-v)^{2H-2} dv \\
&= \lim_{t\to\infty} \frac{\theta^{2H-1}}{\left(t-\frac{\sigma}{\theta}\right)^{2-2H}} \int_0^t \left(v-\frac{\sigma}{\theta}\right) e^{-\frac{1}{2}\theta(t^2-v^2)+\sigma(t-v)}(t-v)^{2H-2} dv \\
&= \lim_{t\to\infty} \frac{\theta^{2H-2}}{\left(t-\frac{\sigma}{\theta}\right)^{2-2H}} \int_0^{\frac{1}{2}\theta t^2-\sigma t} e^{-x} \left(\left(t-\frac{\sigma}{\theta}\right) - \sqrt{\left(t-\frac{\sigma}{\theta}\right)^2 - \frac{2x}{\theta}}\right)^{2H-2} dx \\
&= \lim_{t\to\infty} \frac{\theta^{2H-2}}{\left(t-\frac{\sigma}{\theta}\right)^{2-2H}} \\
&\quad \cdot \int_0^{\frac{1}{2}\theta t^2-\sigma t} e^{-x} \left(\frac{2x}{\theta}\right)^{2H-2} \left(\left(t-\frac{\sigma}{\theta}\right) + \sqrt{\left(t-\frac{\sigma}{\theta}\right)^2 - \frac{2x}{\theta}}\right)^{2-2H} dx \\
&= 2^{2H-2} \lim_{t\to\infty} \int_0^{\frac{1}{2}\theta t^2-\sigma t} e^{-x} x^{2H-2} \left(1+\sqrt{1-\frac{2x}{\theta(t-\frac{\sigma}{\theta})^2}}\right)^{2-2H} dx.
\end{aligned}
$$

It follows from the dominated convergence theorem that

$$
\Lambda_{H,\theta,\lambda} = \int_0^\infty e^{-x} x^{2H-2} dx = \Gamma(2H-1)
$$

for all $\frac{1}{2} < H < 1$. This completes the proof. □

Lemma 5. *Let $\theta > 0$ and $\frac{1}{2} \le H < 1$. Denote*

$$
Q_t^H := \int_0^t [h_{\theta,\sigma}(t,s) - h_{\theta,\sigma}(s)] dB_s^H
$$

for all $t \ge 0$. Then, we have

$$
E\left[\left|Q_t^H - Q_s^H\right|^2\right] \le C_{H,\theta,\sigma} |t-s|^{2H} \tag{21}
$$

for all $0 < s < t$.

Proof. Let $\frac{1}{2} < H < 1$; this is a simple calculus exercise. In fact, we have

$$
\begin{aligned}
Q_t^H - Q_s^H &= \int_0^t [h_{\theta,\sigma}(t,r) - h_{\theta,\sigma}(r)] dB_r^H - \int_0^s [h_{\theta,\sigma}(s,r) - h_{\theta,\sigma}(r)] dB_r^H \\
&= \int_s^t [h_{\theta,\sigma}(t,r) - h_{\theta,\sigma}(r)] dB_r^H + \int_0^s [h_{\theta,\sigma}(t,r) - h_{\theta,\sigma}(s,r)] dB_r^H \\
&= \left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u} du\right)\left(\int_s^t (\theta r-\sigma) e^{\frac{1}{2}\theta r^2-\sigma r} dB_r^H\right) \\
&\quad - \left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u} du\right)\left(\int_0^s (\theta r-\sigma) e^{\frac{1}{2}\theta r^2-\sigma r} dB_r^H\right) \\
&\equiv Q^H(1) - Q^H(2)
\end{aligned}
$$

for all $0 < s < t$. We estimate the variances of $Q^H(1)$ and $Q^H(2)$ in three cases.

Case I: $\sigma \le 0$ or $\frac{\sigma}{\theta} \le s < t$. By means of the convergence

$$
\lim_{t\to\infty} \frac{1}{t^{-1} e^{-\frac{1}{2}\theta t^2}} \int_t^\infty e^{-\frac{1}{2}\theta x^2} dx = \frac{1}{\theta}, \quad \lim_{t\to 0} \frac{1}{e^{-\frac{1}{2}\theta t^2}} \int_t^\infty e^{-\frac{1}{2}\theta x^2} dx = \sqrt{\frac{\pi}{2\theta}}
$$

and continuity of the functions $t \mapsto \int_t^\infty e^{-\frac{1}{2}\theta x^2}dx$ and $t \mapsto \left(\frac{1}{t} \wedge 1\right)e^{-\frac{1}{2}\theta t^2}$, we obtain the inequality

$$\int_t^\infty e^{-\frac{1}{2}\theta x^2}dx \le C_\theta \left(\frac{1}{t} \wedge 1\right)e^{-\frac{1}{2}\theta t^2} \tag{22}$$

for all $t > 0$. As an immediate result, we see that

$$\begin{aligned}\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du &= e^{\frac{\sigma^2}{2\theta}}\int_{t-\frac{\sigma}{\theta}}^\infty e^{-\frac{1}{2}\theta x^2}dx \\ &\le \frac{1}{t-\frac{\sigma}{\theta}}e^{\frac{\sigma^2}{2\theta}}\int_{t-\frac{\sigma}{\theta}}^\infty xe^{-\frac{1}{2}\theta x^2}dx = \frac{1}{\theta t-\sigma}e^{-\frac{1}{2}\theta t^2+\sigma t}\end{aligned} \tag{23}$$

for all $t > 0$. It follows from Lemma 4 that

$$\begin{aligned}E\left[(Q^H(1))^2\right] &= \left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2 E\left(\int_s^t (\theta r-\sigma)e^{\frac{1}{2}\theta r^2-\sigma r}dB_r^H\right)^2 \\ &= \alpha_H\left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2 \\ &\quad\cdot \int_s^t\int_s^t (\theta r-\sigma)(\theta v-\sigma)|r-v|^{2H-2}e^{\frac{1}{2}\theta(r^2+v^2)-\sigma(r+v)}dvdr \\ &\le (\theta t-\sigma)^2 e^{\theta t^2-2\sigma t}\left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2\left(\alpha_H\int_s^t\int_s^t |r-v|^{2H-2}dvdr\right) \\ &\le C_{\theta,\sigma}(t-s)^{2H}\end{aligned}$$

and

$$\begin{aligned}E\left[(Q^H(2))^2\right] &= \left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2 E\left(\int_0^s (\theta r-\sigma)e^{\frac{1}{2}\theta r^2-\sigma r}dB_r^H\right)^2 \\ &= \left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2H}\left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2-2H} \\ &\quad\cdot \alpha_H\int_0^s\int_0^s (\theta u-\sigma)(\theta v-\sigma)e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}|u-v|^{2H-2}dvdu \\ &\le (t-s)^{2H}e^{-H\theta s^2+2H\sigma s}\left(\int_s^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2-2H} \\ &\quad\cdot \alpha_H\int_0^s\int_0^s (\theta u-\sigma)(\theta v-\sigma)e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}|u-v|^{2H-2}dvdu \\ &\le \alpha_H \Xi_{H,\theta,\sigma}(t-s)^{2H}\end{aligned}$$

for all $t > s \ge \frac{\sigma}{\theta}$.

Case II: $\sigma > 0$ and $0 < s < t \le \frac{\sigma}{\theta}$. We have

$$\begin{aligned}E\left[(Q^H(1))^2\right] &= \alpha_H\left(\int_t^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2 \\ &\quad\cdot \int_s^t\int_s^t (\theta r-\sigma)(\theta v-\sigma)|r-v|^{2H-2}e^{\frac{1}{2}\theta(r^2+v^2)-\sigma(r+v)}dvdr \\ &\le \sigma^2 e^{\frac{\sigma^2}{\theta}}\left(\int_{\frac{\sigma}{\theta}}^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2\left(\alpha_H\int_s^t\int_s^t |r-v|^{2H-2}dvdr\right) \\ &\le C_{\theta,\sigma}(t-s)^{2H}\end{aligned}$$

and

$$
\begin{aligned}
E\left[(Q^H(2))^2\right] &= \left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^2 \\
&\quad \cdot \alpha_H \int_0^s \int_0^s (\theta u-\sigma)(\theta v-\sigma)|u-v|^{2H-2}e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}dvdu \\
&\le 2\sigma^2 e^{\frac{\sigma^2}{\theta}}(t-s)^2\left(\alpha_H\int_0^s\int_0^s |u-v|^{2H-2}dvdu\right) \\
&= 2\sigma^2\left(\frac{\sigma}{\theta}\right)^{2H} e^{\frac{\sigma^2}{\theta}}(t-s)^2 \le C(t-s)^{2H}
\end{aligned}
$$

for all $0<s<t\le\frac{\sigma}{\theta}$.

Case III: $\sigma>0$ and $0<s<\frac{\sigma}{\theta}\le t$. According to the inequality (22), we have

$$
\begin{aligned}
E\left[(Q^H(1))^2\right] &\le \alpha_H\theta^{-2}\Big((t-\frac{\sigma}{\theta})^{-2}\wedge 1\Big)e^{-\theta t^2+2\sigma t} \\
&\quad \cdot \int_s^t\int_s^t (\theta r-\sigma)(\theta v-\sigma)|r-v|^{2H-2}e^{\frac{1}{2}\theta(r^2+v^2)-\sigma(r+v)}dvdr \\
&= \alpha_H\Big((t-\frac{\sigma}{\theta})^{-2}\wedge 1\Big)e^{-\theta(t-\frac{\sigma}{\theta})^2} \\
&\quad \cdot \int_{s-\frac{\sigma}{\theta}}^{t-\frac{\sigma}{\theta}}\int_{s-\frac{\sigma}{\theta}}^{t-\frac{\sigma}{\theta}} uv|u-v|^{2H-2}e^{\frac{1}{2}\theta(u^2+v^2)}dvdu \\
&\le \alpha_H\Big((t-\frac{\sigma}{\theta})^{-2}\wedge 1\Big)e^{-\theta(t-\frac{\sigma}{\theta})^2} \\
&\qquad \cdot \int_0^{t-\frac{\sigma}{\theta}}\int_0^{t-\frac{\sigma}{\theta}} uv|u-v|^{2H-2}e^{\frac{1}{2}\theta(u^2+v^2)}dvdu \\
&\quad + \alpha_H\Big((t-\frac{\sigma}{\theta})^{-2}\wedge 1\Big)e^{-\theta(t-\frac{\sigma}{\theta})^2} \\
&\qquad \cdot \int_{s-\frac{\sigma}{\theta}}^{0}\int_{s-\frac{\sigma}{\theta}}^{0} uv|u-v|^{2H-2}e^{\frac{1}{2}\theta(u^2+v^2)}dvdu \\
&\le \alpha_H\int_0^{t-\frac{\sigma}{\theta}}\int_0^{t-\frac{\sigma}{\theta}}|u-v|^{2H-2}dvdu \\
&\quad + \alpha_H\Big((t-\frac{\sigma}{\theta})^{-2}\wedge 1\Big)(s-\frac{\sigma}{\theta})^2 e^{\frac{\sigma^2}{\theta}}\int_{s-\frac{\sigma}{\theta}}^{0}\int_{s-\frac{\sigma}{\theta}}^{0}|u-v|^{2H-2}dvdu \\
&\le \left(t-\frac{\sigma}{\theta}\right)^{2H} + C_{\theta,\sigma}\left(\frac{\sigma}{\theta}-s\right)^{2H} \le C_{\theta,\sigma}(t-s)^{2H}
\end{aligned}
$$

for all $t\ge\frac{\sigma}{\theta}>s>0$. Similar to case I, we also have

$$
\begin{aligned}
E\left[(Q^H(2))^2\right] &= \left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2H}\left(\int_s^t e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2-2H} \\
&\quad \cdot \alpha_H\int_0^s\int_0^s(\theta u-\sigma)(\theta v-\sigma)e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}|u-v|^{2H-2}dvdu \\
&\le (t-s)^{2H}e^{-H\theta s^2+2H\sigma s}\left(\int_s^\infty e^{-\frac{1}{2}\theta u^2+\sigma u}du\right)^{2-2H} \\
&\quad \cdot \alpha_H\int_0^s\int_0^s(\theta u-\sigma)(\theta v-\sigma)e^{\frac{1}{2}\theta(u^2+v^2)-\sigma(u+v)}|u-v|^{2H-2}dvdu \\
&\le C_{H,\theta,\sigma}(t-s)^{2H}
\end{aligned}
$$

for all $t\ge\frac{\sigma}{\theta}>s>0$. Thus, we complete the proof for $\frac{1}{2}<H<1$. Similarly, we can obtain the case $H=\frac{1}{2}$. □

Theorem 1. *Let $\theta > 0$ and $\frac{1}{2} \leq H < 1$. Then, the solution X_t^H of (9) converges to the random variable*

$$X_\infty^H := \int_0^\infty h_{\theta,\sigma}(s) dB_s^H + \nu \int_0^\infty h_{\theta,\sigma}(s) ds \tag{24}$$

in L^2 and almost surely as t tends to infinity.

Proof. Let $\theta > 0$. We first consider the convergence in L^2. We decompose

$$\begin{aligned} X_t^H - X_\infty^H &= \int_0^t [h_{\theta,\sigma}(t,s) - h_{\theta,\sigma}(s)] dB_s^H - \int_t^\infty h_{\theta,\sigma}(s) dB_s^H \\ &\quad + \nu \left(\int_0^t h_{\theta,\sigma}(t,s) ds - \int_0^\infty h_{\theta,\sigma}(s) ds \right) \\ &= Q_t^H - \int_t^\infty h_{\theta,\sigma}(s) dB_s^H + \nu \Delta_\theta(t) \end{aligned} \tag{25}$$

for all $t \geq 0$ and $\frac{1}{2} \leq H < 1$. When $H = \frac{1}{2}$, by the fact

$$\int_t^\infty e^{-\frac{1}{2}\theta u^2 + \sigma u} du \sim \frac{1}{\theta t - \sigma} e^{-\frac{1}{2}\theta t^2 + \sigma t},$$

as t tends to infinity, we have

$$\begin{aligned} E\left(Q_t^{1/2}\right)^2 &= \left(\int_t^\infty e^{-\frac{1}{2}\theta u^2 + \sigma u} du \right)^2 \int_0^t (\theta s - \sigma)^2 e^{\theta s^2 - 2\sigma s} ds \\ &\sim \frac{1}{(\theta t - \sigma)^2} e^{-\theta t^2 + 2\sigma t} \int_0^t (\theta s - \sigma)^2 e^{\theta s^2 - \sigma s} ds \sim \frac{1}{2(\theta t - \sigma)} \longrightarrow 0 \end{aligned} \tag{26}$$

and

$$E\left(\int_t^\infty h_{\theta,\sigma}(s) dB_s^{1/2} \right)^2 = \int_t^\infty (h_{\theta,\sigma}(s))^2 ds = O\left(\frac{1}{(\theta t - \sigma)^3} \right) \longrightarrow 0, \tag{27}$$

as t tends to infinity. Combining this with (25) and Lemma 2, we show that $X_t^{1/2}$ converges to $X_\infty^{1/2}$ in L^2 for $H = \frac{1}{2}$ as t tends to infinity.

When $\frac{1}{2} < H < 1$, we also have that

$$\begin{aligned} E\left[\left| Q_t^H \right|^2 \right] &= \| h_{\theta,\sigma}(t,\cdot) - h_{\theta,\sigma}(\cdot) \|_{\mathcal{H}}^2 \leq C \| h_{\theta,\sigma}(t,\cdot) - h_{\theta,\sigma}(\cdot) \|_{L^{1/H}([0,t])}^2 \\ &= C \left(\int_0^t |h_{\theta,\sigma}(t,s) - h_{\theta,\sigma}(s)|^{1/H} ds \right)^{2H} \\ &= C \left(\int_t^\infty e^{-\frac{1}{2}\theta s^2 + \sigma s} ds \right)^2 \left(\int_0^t |\theta s - \sigma|^{1/H} e^{\frac{1}{2H}\theta s^2 - \frac{1}{H}\sigma s} ds \right)^{2H} \\ &\sim \frac{C}{(\theta t - \sigma)^{2H}} \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} E\left(\int_t^\infty h_{\theta,\sigma}(s) dB_s^H \right)^2 &= \alpha_H \int_t^\infty \int_t^\infty h_{\theta,\sigma}(s) h_{\theta,\sigma}(r) |s - r|^{2H-2} ds dr \\ &\leq C \int_t^\infty \int_t^\infty \frac{1}{[(\theta s - \sigma)(\theta r - \sigma)]^2} |s - r|^{2H-2} ds dr \\ &= \theta^{-2} C \int_{t - \frac{\sigma}{\theta}}^\infty \int_{t - \frac{\sigma}{\theta}}^\infty \frac{1}{(xy)^2} |x - y|^{2H-2} dx dy \\ &= \frac{C}{(t - \frac{\sigma}{\theta})^{4-2H}} \longrightarrow 0 \end{aligned} \tag{28}$$

as t tends to infinity for all $\frac{1}{2} < H < 1$. Combining this with Lemma 2, we show that X_t^H converges to X_∞^H in L^2 for all $\frac{1}{2} < H < 1$ as t tends to infinity.

We now prove the convergence with probability one. According to the decomposition (25) and Lemma 2, we need to show that the convergence

$$Q_t^H \longrightarrow 0, \tag{29}$$

$$\int_t^\infty h_{\theta,\sigma}(s)dB_s^H \longrightarrow 0 \tag{30}$$

holds almost surely as t tends to infinity.

First, on the grounds of (16), Lemma 3 and the fact that

$$\frac{B_T^H}{T} \longrightarrow 0 \quad (T \to \infty)$$

almost surely for all $\frac{1}{2} \le H < 1$ as T tends to infinity, we prove that

$$\int_t^\infty h_{\theta,\sigma}(s)dB_s^H = -h_{\theta,\sigma}(t)B_t^H - \int_t^\infty B_s^H h'_{\theta,\sigma}(s)ds \longrightarrow 0$$

for all $\frac{1}{2} \le H < 1$ as t tends to infinity.

Now, we consider the convergence (29). When $\frac{1}{2} < H < 1$, for integer numbers n, k with $0 \le k < n$, we set $M_{n,k}^H = Q_{n+\frac{k}{n}}^H$. Then, $M_{n,k}^H$ is Gaussian, and we have

$$\sigma^2 := E\left[\left(M_{n,k}^H\right)^2\right] = E\left[\left(Q_{n+\frac{k}{n}}^H\right)^2\right] \le \frac{C}{\left(\theta(n+\frac{k}{n})-\sigma\right)^{2H}} \le \frac{C}{(\theta n-\sigma)^{2H}} \tag{31}$$

with $n \ge (0 \vee \frac{\sigma}{\theta})$ and

$$\begin{aligned} P\left(\left|M_{n,k}^H\right| > \varepsilon\right) &= \int_\varepsilon^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}dx \le \frac{1}{\varepsilon}\int_\varepsilon^\infty \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}dx \\ &= \frac{\sigma}{\varepsilon}\int_{\varepsilon/\sigma}^\infty \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}dy = \frac{\sigma}{\varepsilon\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2}} \\ &\le \frac{\sqrt{C}}{\varepsilon(\theta n-\sigma)^H} \exp\left\{-C\varepsilon^2(\theta n-\sigma)^{2H}\right\} \end{aligned}$$

for all $\varepsilon > 0$ and $n \ge (0 \vee \frac{\sigma}{\theta})$. Furthermore, for $s \in (0,1)$, we denote $R_s^{n,k} = Q_{n+\frac{k+s}{n}}^H - Q_{n+\frac{k}{n}}^H$. Then, $\{R_s^{n,k}, s \in (0,1)\}$ is a Gaussian process for any n and k, and, on the basis of Lemma 5, we have

$$\begin{aligned} E\left[\left(R_s^{n,k} - R_{s'}^{n,k}\right)^2\right] &\le \frac{C}{(\theta n-\sigma)^{2H}}|s-s'|^{2H} \\ &= \frac{C}{(\theta n-\sigma)^{2H}} E\left[\left(B_s^H - B_{s'}^H\right)^2\right]. \end{aligned}$$

for all $\varepsilon > 0$ and $n \ge (0 \vee \frac{\sigma}{\theta})$. It follows from Slepian's Lemma and Markov's inequality that

$$\begin{aligned} P\left(\sup_{0\le s\le 1}\left|R_s^{n,k}\right| > \varepsilon\right) &\le P\left(\frac{C}{(\theta n-\sigma)^{2H}}\sup_{0\le s\le 1}\left|B_s^H\right| > \varepsilon\right) \\ &\le \frac{C}{\varepsilon^{2\gamma}(\theta n-\sigma)^{2\gamma H}} E\left[\sup_{0\le s\le 1}\left|B_s^H\right|^{2\gamma}\right] \le \frac{C}{\varepsilon^{2\gamma}(\theta n-\sigma)^{2H\gamma}} \end{aligned}$$

for any $\varepsilon > 0$, $n \geq (0 \vee \frac{\sigma}{\theta})$ and $\gamma \geq 1$. Combining this with the Borel–Cantelli Lemma and the inclusion relation,

$$\left\{ \sup_{n+\frac{k}{n}<t<n+\frac{k+1}{n}} \left| Q_t^H \right| > \varepsilon \right\} \subseteq \left\{ \left| M_{n,k}^H \right| > \frac{\varepsilon}{2} \right\} \cup \left\{ \sup_{0\leq s\leq 1} \left| R_s^{n,k} \right| > \frac{\varepsilon}{2} \right\}$$

for all $k, n \geq (0 \vee \frac{\sigma}{\theta})$, we show that the convergence (29) holds almost surely. Similarly, we can also check the case $H = \frac{1}{2}$. This completes the proof. □

Theorem 2. *Let $\frac{1}{2} \leq H < 1$, $\theta > 0$ and $\sigma, \nu \in \mathbb{R}$. As $t \to \infty$, we have*

$$\left(t - \frac{\sigma}{\theta}\right)^H \left(X_t^H - X_\infty^H\right) \to N(0, \zeta(H, \theta)) \tag{32}$$

in distribution, where $N(0, \sigma)$ denotes a central normal random variable with the variance σ and

$$\zeta(H, \theta) = H\theta^{-2H}\Gamma(2H).$$

Proof. Keep the notation of Theorem 1. From the decomposition (25), it follows that

$$\begin{aligned} \left(t - \frac{\sigma}{\theta}\right)^H \left(X_t^H - X_\infty^H\right) = \left(t - \frac{\sigma}{\theta}\right)^H Q_t^H + \nu\left(t - \frac{\sigma}{\theta}\right)^H \Delta_\theta(t) \\ - \left(t - \frac{\sigma}{\theta}\right)^H \int_t^\infty h_{\theta,\sigma}(s) dB_s^H \end{aligned} \tag{33}$$

for all $t \geq (0 \vee \frac{\sigma}{\theta})$. Thus, according to the Slutsky theorem, we only need to check that the following convergence:

$$\left(t - \frac{\sigma}{\theta}\right)^H Q_t^H \longrightarrow N(0, \zeta(H, \theta)) \quad \text{in distribution,} \tag{34}$$

$$\Lambda_t^H := \left(t - \frac{\sigma}{\theta}\right)^H \int_t^\infty h_{\theta,\sigma}(s) dB_s^H \longrightarrow 0 \quad \text{in probability,} \tag{35}$$

$$\left(t - \frac{\sigma}{\theta}\right)^H \Delta_\theta(t) \longrightarrow 0, \tag{36}$$

as t tends to infinity. The convergence (36) is an immediate result of Lemma 2, and the convergence (35) follows from (27) and (28). Finally, for (34), by the normality of Q_t^H, we only need to calculate

$$\begin{aligned} \zeta(H, \theta) &= \lim_{t\to\infty} \left(t - \frac{\sigma}{\theta}\right)^{2H} E\left[(Q_t^H)^2\right] \\ &= \theta^{-2H} \lim_{t\to\infty} \frac{e^{-\theta t^2 + 2\sigma t}}{(\theta t - \sigma)^{2-2H}} E\left(\int_0^t (\theta s - \sigma) e^{\frac{1}{2}\theta s^2 - \sigma s} dB_s^H\right)^2 \\ &= H\theta^{-2H}\Gamma(2H) \end{aligned} \tag{37}$$

for all $\frac{1}{2} \leq H < 1$. In accordance with the proof of Lemma 4, we can obtain (37), which gives (34), and the theorem follows from the Slutsky theorem. □

4. The Laws of Large Numbers

In this section, we check that convergence (5) and (6). In fact, these limits can be seen as the laws of large numbers associated with the self-attracting diffusion. Denote $B = B^H$ when $H = \frac{1}{2}$.

In Section 3, we have shown that the solution can be expressed as

$$X_t^H = \int_0^t h_{\theta,\sigma}(t, s) dB_s^H + \nu \int_0^t h_{\theta,\sigma}(t, s) ds$$

and

$$X_t^H \to X_\infty^H = \int_0^\infty h_{\theta,\sigma}(s)dB_s^H + \nu \int_0^\infty h_{\theta,\sigma}(s)ds$$

in L^2 and almost surely as t tends to infinity. Moreover, the solution X_t^H admits the following estimation.

Lemma 6. *Let $\frac{1}{2} \leq H < 1$ and $\theta > 0$. Then, the solution X_t^H satisfies*

$$E\left[\left(X_t^H - X_s^H\right)^2\right] \leq C(t-s)^{2H} + \nu^2(t-s)^2$$

for all $0 \leq s, t \leq T$.

Proof. Let $\frac{1}{2} < H < 1$. We have

$$\begin{aligned} X_t^H - X_s^H &= \int_s^t h_{\theta,\sigma}(t,r)dB_r^H + \int_0^s [h_{\theta,\sigma}(t,r) - h_{\theta,\sigma}(s,r)]dB_r^H \\ &\quad + \nu \int_s^t h_{\theta,\sigma}(t,r)dr + \nu \int_0^s [h_{\theta,\sigma}(t,r) - h_{\theta,\sigma}(s,r)]dr \\ &\equiv \sum_{j=1}^4 \Lambda_j^H(t,s) \end{aligned}$$

for all $0 < s < t$. Clearly, on the basis of (15), we have

$$\begin{aligned} E\left[\left(\Lambda_1^H(t,s)\right)^2\right] &= E\left(\int_s^t h_{\theta,\sigma}(t,r)dB_r^H\right)^2 \\ &= \alpha_H \int_s^t \int_s^t h_{\theta,\sigma}(t,r)h_{\theta,\sigma}(t,v)|r-v|^{2H-2}drdv \\ &\leq \alpha_H \int_s^t \int_s^t |r-v|^{2H-2}drdv \leq C_H(t-s)^{2H} \end{aligned}$$

and

$$E\left[\left(\Lambda_3^H(t,s)\right)^2\right] = \nu^2 \int_s^t \int_s^t h_{\theta,\sigma}(t,r)h_{\theta,\sigma}(t,v)drdv \leq \nu^2(t-s)^2$$

for all $0 < s < t$. The proof of Lemma 5 implies that

$$E\left[\left(\Lambda_2^H(t,s)\right)^2\right] = E\left[\left(Q^H(2)\right)^2\right] \leq C(t-s)^{2H}$$

for all $0 < s < t$. Finally, we have

$$\begin{aligned} E\left[\left(\Lambda_4^H(t,s)\right)^2\right] &= \nu^2\left(\int_s^t e^{-\frac{1}{2}\theta u^2 + \sigma u}du\right)^2 \\ &\quad \cdot \left(\int_0^s \int_0^s (\theta r - \sigma)(\theta v - \sigma)e^{\frac{1}{2}\theta(r^2+v^2) - \sigma(r+v)}drdv\right) \\ &\leq \nu^2(t-s)^2 e^{-\theta s^2 + 2\sigma s}\left(e^{\frac{1}{2}\theta s^2 - \sigma s} - 1\right)^2 \leq \nu^2(t-s)^2 \end{aligned}$$

for all $0 < s < t$. Thus, we complete the proof for $\frac{1}{2} < H < 1$. Similarly, we can obtain the case $H = \frac{1}{2}$. □

As an immediate corollary, we assert that the process $t \mapsto X_t^H$ is a Hölder function of order H. Thus, the Young integral

$$\int_0^t u_s dX_s^H$$

is well-defined as a limit in probability of a *Riemann sum* and

$$\int_0^t u_s dX_s^H = u_t X_t^H - \int_0^t X_s^H du_s,$$

if u admits a bounded p-variation with $1 \le p < \frac{1}{1-H}$.

Consider the process

$$Y_t^H = \int_0^t \left(r - \frac{\sigma}{\theta}\right) dX_r^H, \quad t \ge 0, \tag{38}$$

where X_t^H is the self-attracting diffusion defined by (9). Then, we have

$$\begin{aligned} X_t^H &= B_t^H + \sigma \int_0^t X_s^H ds - \theta \int_0^t \left(\int_0^s X_s^H - X_r^H\right) drds + \nu t \\ &= B_t^H - \theta \int_0^t \left(\int_0^s \left(r - \frac{\sigma}{\theta}\right) dX_r^H\right) ds + \nu t \\ &= B_t^H - \theta \int_0^t Y_s^H ds + \nu t \end{aligned} \tag{39}$$

for all $t \ge 0$.

Lemma 7. *Let $\frac{1}{2} \le H < 1$ and $\theta > 0$. Then, we have*

$$Y_t^H = e^{-\frac{1}{2}\theta t^2 + \sigma t} \int_0^t \left(s - \frac{\sigma}{\theta}\right) e^{\frac{1}{2}\theta s^2 - \sigma s} dB_s^H + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta t^2 + \sigma t}\right) \tag{40}$$

for all $t \ge 0$.

Proof. By the definition of Y_t^H and (39), we have

$$\begin{aligned} dY_t^H &= \left(t - \frac{\sigma}{\theta}\right) dX_t^H \\ &= -\theta\left(t - \frac{\sigma}{\theta}\right) Y_t^H dt + \left(t - \frac{\sigma}{\theta}\right) dB_t^H + \nu\left(t - \frac{\sigma}{\theta}\right) dt \end{aligned} \tag{41}$$

for all $t \ge 0$. Through the variation of constants method, we can assume that the process

$$Y_t^H = C_t^H e^{-\frac{1}{2}\theta t^2 + \sigma t}$$

is the solution of (41) with $C_0^H = Y_0^H = 0$. Then, according to (41), we have

$$e^{-\frac{1}{2}\theta t^2 + \sigma t} dC_t^H = \left(t - \frac{\sigma}{\theta}\right) dB_t^H + \nu\left(t - \frac{\sigma}{\theta}\right) dt$$

for all $t \ge 0$, which implies that

$$\begin{aligned} C_t^H &= \int_0^t \left(s - \frac{\sigma}{\theta}\right) e^{\frac{1}{2}\theta s^2 - \sigma s} dB_s^H + \nu \int_0^t \left(s - \frac{\sigma}{\theta}\right) e^{\frac{1}{2}\theta s^2 - \sigma s} ds \\ &= \int_0^t \left(s - \frac{\sigma}{\theta}\right) e^{\frac{1}{2}\theta s^2 - \sigma s} dB_s^H + \frac{\nu}{\theta}\left(e^{\frac{1}{2}\theta t^2 + \sigma t} - 1\right) \end{aligned}$$

for all $t \ge 0$. This gives (40). □

Lemma 8. *Let $\frac{1}{2} \le H < 1$ and $\theta > 0$. As $T \to \infty$, we have*

$$\frac{1}{T}\int_0^T Y_t^H dt \longrightarrow \frac{\nu}{\theta}$$

in L^2 and almost surely.

Proof. The theorem is clear. In fact, according to (39) and Theorem 1, we have

$$\frac{1}{T}\int_0^T Y_t^H dt = \frac{B_T^H}{\theta T} - \frac{X_T^H}{\theta T} + \frac{\nu}{\theta} \longrightarrow \frac{\nu}{\theta}$$

in L^2 and almost surely as T tends to infinity. □

Lemma 9. *Let $\frac{1}{2} \le H < 1$ and $\theta > 0$. Define the process $\eta = \{\eta_t, t \ge 0\}$ by*

$$\eta_t = e^{-\frac{1}{2}\theta t^2 + \sigma t}\int_0^t \left(s - \frac{\sigma}{\theta}\right) e^{\frac{1}{2}\theta s^2 - \sigma s} dB_s^H.$$

Then, we have

$$\lim_{T\to\infty} T^{2H-2} E\left(\eta_T^2\right) = H\theta^{-2H}\Gamma(2H).$$

Proof. This Lemma follows from (37). □

Lemma 10. *Let $\frac{1}{2} < H < 1$ and $\theta > 0$. We have*

$$E[\eta_t\eta_s] \le C(t-s)^{2H-2}\left|\left(t-\frac{\sigma}{\theta}\right)^2 - \left(s-\frac{\sigma}{\theta}\right)^2\right|^{\gamma} + C \tag{42}$$

for all $t > s \ge 0$ and $0 \le \gamma \le 2 - 2H$. In particular, we have

$$E[\eta_t\eta_s] \le C(t-s)^{2H-2} + C, \tag{43}$$

for all $t > s \ge 0$.

Proof. When $\frac{1}{2} < H < 1$, consider the decomposition

$$\begin{aligned}
E(\eta_t\eta_s) &= \alpha_H e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2 + (s-\frac{\sigma}{\theta})^2\right]} \\
&\quad \cdot \int_0^t\int_0^s (u-\frac{\sigma}{\theta})(v-\frac{\sigma}{\theta}) e^{\frac{\theta}{2}\left[(u-\frac{\sigma}{\theta})^2 + (v-\frac{\sigma}{\theta})^2\right]} |u-v|^{2H-2} dv du \\
&= \alpha_H e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2 + (s-\frac{\sigma}{\theta})^2\right]} \\
&\quad \cdot \int_s^t (u-\frac{\sigma}{\theta}) e^{\frac{\theta}{2}(u-\frac{\sigma}{\theta})^2}\left(\int_0^s (v-\frac{\sigma}{\theta}) e^{\frac{\theta}{2}(v-\frac{\sigma}{\theta})^2}(u-v)^{2H-2} dv\right) du \\
&\quad + \alpha_H e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2 + (s-\frac{\sigma}{\theta})^2\right]} \\
&\quad \cdot \int_0^s\int_0^s (u-\frac{\sigma}{\theta})(v-\frac{\sigma}{\theta}) e^{\frac{\theta}{2}\left[(u-\frac{\sigma}{\theta})^2 + (v-\frac{\sigma}{\theta})^2\right]} |u-v|^{2H-2} dv du \\
&\equiv \Psi_1(H;t,s) + \Psi_2(H;t,s)
\end{aligned}$$

for all $t > s > 0$. Now, we estimate the above terms $\Psi_1(H;t,s)$ and $\Psi_2(H;t,s)$ by using a method similar to proving Lemma 5.

Case I: $\sigma \le 0$ or $\frac{\sigma}{\theta} \le s < t$. We have that

$$\begin{aligned}
\Psi_1(H;t,s) &\le \alpha_H e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]} \\
&\qquad \cdot \int_s^t (u-\frac{\sigma}{\theta})(u-s)^{2H-2} e^{\frac{\theta}{2}(u-\frac{\sigma}{\theta})^2}\left(\int_0^s (v-\frac{\sigma}{\theta})e^{\frac{\theta}{2}(v-\frac{\sigma}{\theta})^2}dv\right)du \\
&\le \frac{\alpha_H}{\theta} e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]}\left(e^{\frac{\theta}{2}(s-\frac{\sigma}{\theta})^2}-1\right)\int_s^t (u-\frac{\sigma}{\theta})(u-s)^{2H-2}e^{\frac{\theta}{2}(u-\frac{\sigma}{\theta})^2}du \\
&\le \frac{\alpha_H}{\theta} e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2\right]}\int_s^t (u-\frac{\sigma}{\theta})(u-s)^{2H-2}e^{\frac{\theta}{2}\left[(u-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2\right]}du
\end{aligned}$$

for all $\frac{\sigma}{\theta} \le s < t$. Making the substitution

$$(u-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2 = x,$$

to obtain

$$\begin{aligned}
&\int_s^t (u-\frac{\sigma}{\theta})(u-s)^{2H-2}e^{\frac{\theta}{2}\left[(u-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2\right]}du \\
&\qquad = \int_0^{(t-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2} e^{\frac{1}{2}\theta x}\left[\sqrt{(s-\frac{\sigma}{\theta})^2+x}-(s-\frac{\sigma}{\theta})\right]^{2H-2}dx \\
&\qquad = \frac{1}{2}\int_0^{(t-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2} e^{\frac{1}{2}\theta x}x^{2H-2}\left[\sqrt{(s-\frac{\sigma}{\theta})^2+x}+(s-\frac{\sigma}{\theta})\right]^{2-2H}dx \\
&\qquad \le \frac{1}{2}\left(\left(t-\frac{\sigma}{\theta}\right)+\left(s-\frac{\sigma}{\theta}\right)\right)^{2-2H}\int_0^{(t-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2} e^{\frac{1}{2}\theta x}x^{2H-2}dx
\end{aligned}$$

for all $\frac{\sigma}{\theta} \le s < t$. It follows from the fact

$$\int_0^x y^\beta e^y dy \asymp x^\beta(1\wedge x)e^x \tag{44}$$

with $x \ge 0$ and $\beta > -1$ that

$$\begin{aligned}
\Psi_1(H;t,s) &\le C\left(\left(t-\frac{\sigma}{\theta}\right)+\left(s-\frac{\sigma}{\theta}\right)\right)^{2-2H} \\
&\qquad \cdot\left(\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right)^{2H-2}\left\{1\wedge\left(\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right)\right\} \\
&= C(t-s)^{2H-2}\left(1\wedge\left(\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right)\right).
\end{aligned}$$

For the term $\Psi_2(H;t,s)$, according to Lemma 4 and the facts

$$\int_t^\infty e^{-\theta s^2+2\sigma s}ds \asymp \frac{1}{1\vee(t-\frac{\sigma}{\theta})}e^{-\theta t^2+2\sigma t}$$

and $e^{-x} \le \frac{1}{1+x} \le \frac{1}{x^\varrho}$ with $0 < \varrho < 1$ and $x \ge 0$, we obtain

$$\begin{aligned}
\Psi_2(H;t,s) &= 2\alpha_H e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]} \int_{-\frac{\sigma}{\theta}}^{s-\frac{\sigma}{\theta}} \int_{-\frac{\sigma}{\theta}}^{x} xye^{\frac{\theta}{2}\left(x^2+y^2\right)}(x-y)^{2H-2}dydx \\
&\le Ce^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2-(s-\frac{\sigma}{\theta})^2\right]}\left(s-\frac{\sigma}{\theta}\right)^{2-2H} \\
&\le \frac{C(s-\frac{\sigma}{\theta})^{2-2H}}{\left(\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right)^{2-2H-\gamma}} \\
&\le C(t-s)^{2H-2}\left(\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right)^{\gamma}
\end{aligned}$$

for all $t > s \ge \frac{\sigma}{\theta}, \sigma \le 0$ and $0 \le \gamma \le 2-2H$.

Case II: $\sigma > 0$ and $0 < s < t \le \frac{\sigma}{\theta}$. From the forms of $\Psi_1(H;t,s)$ and $\Psi_2(H;t,s)$, it is easy to find that they are bounded uniformly in t and s.

Case III: $\sigma > 0$ and $0 < s < \frac{\sigma}{\theta} \le t$. Clearly, $\Psi_2(H;t,s)$ is bounded uniformly in t and s. For $\Psi_1(H;t,s)$, based on the following estimates

$$\begin{aligned}
\Psi_{11}(H;t,s) :&= e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]} \int_0^{t-\frac{\sigma}{\theta}} \int_{\frac{\sigma}{\theta}-s}^{\frac{\sigma}{\theta}} xy(x+y)^{2H-2}e^{\frac{1}{2}\theta(x^2+y^2)}dxdy \\
&\le \frac{1}{2H}(\frac{\sigma}{\theta})^{2H}e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]}e^{\frac{\sigma^2}{2\theta}}\int_0^{t-\frac{\sigma}{\theta}} xe^{\frac{1}{2}\theta x^2}dx \\
&\le \frac{1}{2\theta H}(\frac{\sigma}{\theta})^{2H}e^{\frac{\sigma^2}{2\theta}}
\end{aligned}$$

and

$$\Psi_{12}(H;t,s) := e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]} \int_{s-\frac{\sigma}{\theta}}^{0} \int_{-\frac{\sigma}{\theta}}^{s-\frac{\sigma}{\theta}} uv(u-v)^{2H-2}e^{\frac{1}{2}\theta(u^2+v^2)}dvdu \le C$$

for all $0 < s < \frac{\sigma}{\theta} \le t$, we have found that

$$\begin{aligned}
|\Psi_1(H;t,s)| &= e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]} \\
&\quad\cdot\left|\int_s^t\int_0^s (u-\frac{\sigma}{\theta})(v-\frac{\sigma}{\theta})(u-v)^{2H-2}e^{\frac{\theta}{2}\left[(u-\frac{\sigma}{\theta})^2+(v-\frac{\sigma}{\theta})^2\right]}dvdu\right| \\
&= e^{-\frac{\theta}{2}\left[(t-\frac{\sigma}{\theta})^2+(s-\frac{\sigma}{\theta})^2\right]}\left|\int_{s-\frac{\sigma}{\theta}}^{t-\frac{\sigma}{\theta}}\int_{-\frac{\sigma}{\theta}}^{s-\frac{\sigma}{\theta}} uv(u-v)^{2H-2}e^{\frac{1}{2}\theta(u^2+v^2)}dvdu\right| \\
&\le \Psi_{11}(H;t,s)+\Psi_{12}(H;t,s)
\end{aligned}$$

is bounded uniformly in t and s. Thus, we have completed the proof. □

Theorem 3. *Let $\frac{1}{2} \le H < 1$ and $\theta > 0$. Then, we have*

$$\frac{1}{T^{3-2H}}\int_0^T \left(Y_t^H\right)^2 dt \longrightarrow \frac{H\theta^{-2H}}{3-2H}\Gamma(2H) \tag{45}$$

in L^2 and almost surely as T tends to infinity.

Let $\frac{1}{2} \le H < 1$ and denote

$$\Delta_t = \frac{v}{\theta}\left(1-e^{-\frac{1}{2}\theta t^2+\sigma t}\right)$$

for all $t \ge 0$. Then, according to $Y_t^H = \eta_t + \Delta_t$ and

$$\lim_{T\to\infty}\frac{1}{T^{3-2H}}\int_0^T (\Delta_t)^2 dt = 0,$$

the convergence (45) is equivalent to

$$\frac{1}{T^{3-2H}}\int_0^T (\eta_t)^2 dt \longrightarrow \frac{H\theta^{-2H}}{3-2H}\Gamma(2H) \tag{46}$$

in L^2 and almost surely as T tends to infinity. We now verify that the convergence (46) holds in L^2 and almost surely, respectively.

Proof of the L^2-convergence. We first show that the convergence (46) holds in L^2. This is equivalent to

$$\begin{aligned}\triangle_T(H,\theta,\sigma) :&= \frac{1}{T^{6-4H}}E\left(\int_0^T \left((\eta_t)^2 - E(\eta_t)^2\right)dt\right)^2 \\ &= \frac{1}{T^{6-4H}}\int_0^T\int_0^T \left\{E\left[(\eta_t)^2(\eta_s)^2\right] - E(\eta_t)^2 E(\eta_s)^2\right\}dsdt \\ &= \frac{2}{T^{6-4H}}\int_0^T\int_0^T (E\eta_t\eta_s)^2 dsdt \longrightarrow 0\end{aligned} \tag{47}$$

as T tends to infinity, according to the fact that

$$E\left(\eta_t^2\eta_s^2\right) = E\left(\eta_t^2\right)E\left(\eta_s^2\right) + 2(E\eta_t\eta_s)^2$$

for all $t, s > 0$. We now check convergence (47) in the four cases.

Case 1: $H = \frac{1}{2}$. On the basis of the fact that

$$\int_0^s x^\alpha e^{x^2}dx \le C_\alpha s^{\alpha-1}e^{s^2} \tag{48}$$

for all $s \ge 0$ and $\alpha > -1$, we have

$$\begin{aligned}E(\eta_t\eta_s) &= e^{-\frac{1}{2}\theta\left(t^2+s^2\right)+\sigma(t+s)}E\left(\int_0^s \left(v-\frac{\sigma}{\theta}\right)e^{\frac{1}{2}\theta v^2-\sigma v}dB_v\right)^2 \\ &= e^{-\frac{1}{2}\theta\left(t^2+s^2\right)+\sigma(t+s)}\int_0^s \left(v-\frac{\sigma}{\theta}\right)^2 e^{\theta v^2-2\sigma v}dv \\ &= e^{-\frac{1}{2}\theta\left[\left(t-\frac{\sigma}{\theta}\right)^2+\left(s-\frac{\sigma}{\theta}\right)^2\right]}\int_0^s \left(v-\frac{\sigma}{\theta}\right)^2 e^{\theta\left(v-\frac{\sigma}{\theta}\right)^2}dv \\ &\le \left|s-\frac{\sigma}{\theta}\right|e^{-\frac{1}{2}\theta\left[\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right]}\end{aligned} \tag{49}$$

for all $t > s > 0$. It follows from (48) that

$$\begin{aligned}\triangle_T\left(\frac{1}{2},\theta,\sigma\right) &\le \frac{1}{T^4}\int_0^T\int_0^t \left(s-\frac{\sigma}{\theta}\right)^2 e^{-\theta\left[\left(t-\frac{\sigma}{\theta}\right)^2-\left(s-\frac{\sigma}{\theta}\right)^2\right]}dsdt \\ &\le \frac{C_{\theta,\sigma}}{T^4}\int_0^T\left(t-\frac{\sigma}{\theta}\right)dt \longrightarrow 0,\end{aligned}$$

as T tends to infinity.

Case 2: $\frac{3}{4} < H < 1$. According to (43), we have

$$\begin{aligned}\lim_{T\to\infty}\triangle_T(H,\theta,\sigma) &\le \lim_{T\to\infty}\frac{C}{T^{6-4H}}\int_0^T\int_0^t \left((t-s)^{4H-4}+1\right)dsdt \\ &\le \lim_{T\to\infty} C\left(T^{8H-8}+T^{4H-4}\right) = 0.\end{aligned}$$

Case 3: $\frac{1}{2} < H < \frac{3}{4}$. According to (42) with $\alpha = \gamma$ and $\frac{3}{2} - 2H < \alpha < 2 - 2H$, we obtain

$$\begin{aligned}
&\int_1^T \int_{\sqrt{t^2-1}}^t [E(\eta_t \eta_s)]^2 dsdt \\
&\le C \int_1^T \int_{\sqrt{t^2-1}}^t \left((t-s)^{4H-4+2\alpha} \left(\left(t - \frac{\sigma}{\theta}\right) + \left(s - \frac{\sigma}{\theta}\right) \right)^{2\alpha} + 1 \right) dsdt \\
&\le 2C \int_1^T \int_{\sqrt{t^2-1}}^t \left((t-s)^{4H-4+2\alpha} \left(t - \frac{\sigma}{\theta}\right)^{2\alpha} + 1 \right) dsdt \\
&\le C \int_1^T \frac{\left(t - \frac{\sigma}{\theta}\right)^{2\alpha}}{\left(t + \sqrt{t^2-1}\right)^{4H-3+2\alpha}} dt + CT \le C \left(T - \frac{\sigma}{\theta}\right)^{4-4H} + CT
\end{aligned}$$

for all $T > 1$ and $\frac{1}{2} < H < \frac{3}{4}$ since $0 < t^2 - s^2 < 1$ for

$$(s,t) \in \left\{ (s,t) \mid t \le T, \sqrt{t^2-1} < s < t \right\}.$$

Similarly, according to (43), we also have

$$\int_1^T \int_0^{\sqrt{t^2-1}} [E(\eta_t \eta_s)]^2 dsdt \le C \int_1^T \int_0^{\sqrt{t^2-1}} \left((t-s)^{4H-4} + 1 \right) dsdt \le C \left(T^{4H-2} + T^2 \right)$$

for all $T > 1$. It follows from the fact

$$\int_0^1 \int_0^t [E(\eta_t \eta_s)]^2 dsdt \le \int_0^1 \int_0^t E(\eta_t)^2 E(\eta_s)^2 dsdt \le C$$

for all $\frac{1}{2} < H < \frac{3}{4}$ that

$$\begin{aligned}
\triangle_T(H,\theta,\sigma) &= \frac{1}{T^{6-4H}} \int_0^1 \int_0^t [E(\eta_t \eta_s)]^2 dsdt \\
&\quad + \frac{1}{T^{6-4H}} \int_1^T \int_0^{\sqrt{t^2-1}} [E(\eta_t \eta_s)]^2 dsdt \\
&\quad + \frac{1}{T^{6-4H}} \int_1^T \int_{\sqrt{t^2-1}}^t [E(\eta_t \eta_s)]^2 dsdt \\
&\le \frac{C}{T^{6-4H}} + \frac{C}{T^{4-4H}} + \frac{C}{T^{6-4H}} \left(T - \frac{\sigma}{\theta}\right)^{4-4H} \longrightarrow 0,
\end{aligned}$$

as T tends to infinity.

Case 4: $H = \frac{3}{4}$. According to (43), we have

$$\begin{aligned}
\int_1^T \int_0^{\sqrt{t^2-1}} [E(\eta_t \eta_s)]^2 dsdt &\le C \int_1^T \int_0^{\sqrt{t^2-1}} \left(\frac{1}{t-s} + 1 \right) dsdt \\
&\le CT \log(T) + CT^2
\end{aligned}$$

for all $T > 1$. It follows from the proof of Case 3 that $\triangle_T(H,\theta,\sigma) \to 0$ as T tends to infinity. Thus, we have obtained the convergence in L^2. □

Proof of the convergence with probability one. Denote $T_n = 2^n n$ for integer number $n \ge 0$. Then, we have

$$\begin{aligned}
\Upsilon_n(H) &= \sup_{T_n \le T \le T_{n+1}} \left| \frac{1}{T^{3-2H}} \int_0^T \eta_t^2 dt - \frac{1}{T^{3-2H}} \int_0^T E(\eta_t^2) dt \right| \\
&\le \frac{1}{T_n^{3-2H}} \sup_{T_n \le T \le T_{n+1}} \left| \int_0^T \left(\eta_t^2 - E(\eta_t^2) \right) dt \right|
\end{aligned}$$

for all $n \geq 1$. In order to prove the convergence with probability one, based on the Borel–Cantelli Lemma, it is sufficient to check that

$$\sum_{n=0}^{\infty} P(\Upsilon_n(H) > \varepsilon) < \infty \tag{50}$$

for all $\varepsilon > 0$. Let

$$\beta(x,y) = \int_0^1 (1-u)^{x-1} u^{y-1} du, \ x,y > 0$$

be the classical Beta function; then,

$$\beta(1-\alpha,\alpha) = \int_t^T (s-t)^{-\alpha}(T-s)^{\alpha-1} ds$$

for all $0 \leq t < T$ and $\alpha \in (0,1)$, and

$$\begin{aligned}
&\int_0^T \left(\eta_t^2 - E(\eta_t^2)\right) dt \\
&\quad = \frac{1}{\beta(1-\alpha,\alpha)} \int_0^T \left(\eta_t^2 - E(\eta_t^2)\right)\left(\int_t^T (s-t)^{-\alpha}(T-s)^{\alpha-1} ds\right) dt \\
&\quad = \frac{1}{\beta(1-\alpha,\alpha)} \int_0^T (T-s)^{\alpha-1}\left(\int_0^s \left(\eta_t^2 - E(\eta_t^2)\right)(s-t)^{-\alpha} dt\right) ds \\
&\quad = \frac{1}{\beta(1-\alpha,\alpha)} \int_0^T (T-s)^{\alpha-1} s^{\frac{1}{2}-\alpha}\left(s^{\alpha-\frac{1}{2}} \int_0^s \left(\eta_t^2 - E(\eta_t^2)\right)(s-t)^{-\alpha} dt\right) ds
\end{aligned}$$

for all $\alpha \in (0,1)$. It follows from Cauchy's inequality that

$$\begin{aligned}
\left|\int_0^T \left(\eta_t^2 - E(\eta_t^2)\right) dt\right|^2 \leq & \frac{\beta(2\alpha-1, 2-2\alpha)}{\beta(1-\alpha,\alpha)^2} \\
& \cdot \int_0^T s^{2\alpha-1}\left(\int_0^s \left(\eta_t^2 - E(\eta_t^2)\right)(s-t)^{-\alpha} dt\right)^2 ds
\end{aligned}$$

for all $\alpha \in (\frac{1}{2}, 1)$. An elementary calculation may check that

$$\begin{aligned}
\int_u^{T_n} (s-u)^{-\alpha}(s-v)^{-\alpha} ds &= (u-v)^{1-2\alpha} \int_0^{\frac{T_n-u}{u-v}} x^{-\alpha}(1+x)^{-\alpha} dx \\
&\leq (u-v)^{1-2\alpha} \int_0^{\infty} x^{-\alpha}(1+x)^{-\alpha} dx = C(u-v)^{1-2\alpha}
\end{aligned}$$

for all $0 < v < u < T_n$ and $\alpha \in (\frac{1}{2}, 1)$. Combining this with the fact that

$$E\left[\left(\eta_t^2 - E(\eta_t^2)\right)\left(\eta_s^2 - E(\eta_s^2)\right)\right] = E\left(\eta_t^2 \eta_s^2\right) - E\left(\eta_t^2\right) E\left(\eta_s^2\right) = 2(E\eta_t \eta_s)^2,$$

we see that

$$
\begin{aligned}
E|\Upsilon_n(H)|^2 &\le \frac{C}{T_n^{6-4H}}\int_0^{T_{n+1}} s^{2\alpha-1}E\left(\int_0^s \left(\eta_t^2 - E(\eta_t^2)\right)(s-t)^{-\alpha}dt\right)^2 ds \\
&= \frac{C}{T_n^{6-4H}}\int_0^{T_{n+1}} s^{2\alpha-1}\left(\int_0^s\int_0^s (s-v)^{-\alpha}(s-u)^{-\alpha}[E(\eta_u\eta_v)]^2 dvdu\right)ds \\
&\le \frac{C(T_{n+1})^{2\alpha-1}}{T_n^{6-4H}} \\
&\quad \cdot \int_0^{T_{n+1}}\int_0^{T_{n+1}}[E(\eta_u\eta_v)]^2\left(\int_{u\vee v}^{T_{n+1}}(s-v)^{-\alpha}(s-u)^{-\alpha}ds\right)dvdu \\
&\le \frac{C(T_{n+1})^{2\alpha-1}}{T_n^{6-4H}}\int_0^{T_{n+1}}\int_0^u [E(\eta_u\eta_v)]^2(u-v)^{1-2\alpha}dvdu
\end{aligned}
$$

for all $n \ge 1$ and $\alpha \in (\frac{1}{2}, 1)$. According to Lemma 10 and (49), one may verify that there exists a constant $\gamma > 0$ depending only on H and α such that

$$E|\Upsilon_n(H)|^2 \le \frac{C}{(T_n)^\gamma} = \frac{C}{n^\gamma(2^\gamma)^n}$$

for all $n \ge 1$ and $\frac{1}{2} \le H < 1$. This shows that

$$\sum_{n=0}^{\infty} P(\Upsilon_n(H) > \varepsilon) < \infty \tag{51}$$

for all $\varepsilon > 0$ and $\frac{1}{2} \le H < 1$, and the convergence with probability one follows. □

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