fractal and fractional

Article

Fractional Integral of the Confluent Hypergeometric Function Related to Fuzzy Differential Subordination Theory

Mugur Acu [1], Gheorghe Oros [2,*] and Ancuța Maria Rus [3]

[1] Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, Str. Dr. I. Rațiu No. 5-7, 550012 Sibiu, Romania; acu_mugur@yahoo.com
[2] Department of Mathematics and Computer Science, Faculty of Informatics and Sciences, University of Oradea, 410087 Oradea, Romania
[3] Doctoral School of Engineering Sciences, University of Oradea, 410087 Oradea, Romania; rusancuta4@gmail.com
* Correspondence: gh_oros@yahoo.com

Abstract: The fuzzy differential subordination concept was introduced in 2011, generalizing the concept of differential subordination following a recent trend of adapting fuzzy sets theory to other already-established theories. A prolific tool in obtaining new results related to operators is the fractional integral applied to different functions. The fractional integral of the confluent hypergeometric function was previously investigated using means of the classical theory of subordination. In this paper, we give new applications of this function using the theory of fuzzy differential subordination. Fuzzy differential subordinations are established and their best dominants are also provided. Corollaries are written using particular functions, in which the conditions for the univalence of the fractional integral of the confluent hypergeometric function are given. An example is constructed as a specific application of the results obtained in this paper.

Keywords: fuzzy set; univalent function; fuzzy differential subordination; fuzzy best dominant; confluent hypergeometric function; fractional integral of order α

check for updates

Citation: Acu, M.; Oros, G.; Rus, A.M. Fractional Integral of the Confluent Hypergeometric Function Related to Fuzzy Differential Subordination Theory. *Fractal Fract.* **2022**, *6*, 413. https://doi.org/10.3390/fractalfract6080413

Academic Editors: Peter Massopust and Ahmed I. Zayed

Received: 15 June 2022
Accepted: 25 July 2022
Published: 27 July 2022

1. Introduction

Fuzzy sets theory has its origins in the paper published by Lotfi A. Zadeh in 1965 [1]. At that time, the paper raised many discussions and was regarded as controversial, but it is nowadays considered the foundation of fuzzy logic theory and has reached over 100,000 citations. The concept of fuzzy sets has applications in many domains of the modern technological world. The importance of the fuzzy set notion and certain steps in the evolution of the concept are nicely highlighted in certain review papers [2,3], and the development of different areas of research due to this concept is now obvious.

The basis of fuzzy differential subordination theory was set in 2011 with the use of fuzzy set notion in introducing a generalization of the classical concept of subordination familiar to geometric function theory, called fuzzy subordination [4]. Fuzzy subordination was then extended to fuzzy differential subordination in 2012 [5]. It is seen as a generalization of the differential subordination concept introduced by S.S. Miller and P.T. Mocanu [6,7], developed by many researchers in the following years and synthesized in [8]. The theory of fuzzy differential subordination developed by providing means for obtaining the dominants and best dominants of the fuzzy differential subordinations [9] and by adding operators to the research [10–12], continually following the general theory of differential subordination established in geometric function theory. The dual concept of fuzzy differential superordination was also introduced in 2017 [13] and investigations relating the two concepts continued to provide new results [14,15].

The study of fuzzy differential subordinations continues and interesting results were recently published concerning a Mittag-Leffler-type Borel distribution [16], connecting fuzzy differential subordination to different types of operators [17,18] or for obtaining univalence criteria [19].

Fractional calculus had a powerful impact in recent research, having many applications in different branches of science and engineering. Its importance has been nicely highlighted in a recent review paper [20]. A fractional integral is an important tool for obtaining new, interesting results. Recent papers present new integral inequalities obtained by applying fractional integrals and considering convexity properties [21,22], new extensions involving fractional integrals and the Mittag-Leffler Confluent Hypergeometric Function [23] or tying it to other operators [24,25].

The tremendous development of fractional calculus during the last years also made an impression on the study regarding fuzzy differential subordination theory. New fuzzy differential subordinations were obtained using the Atangana–Baleanu fractional integral [26], the fractional integral of the confluent hypergeometric function [27] and the fractional integral of the Gaussian hypergeometric function [28].

The research presented in this paper involves the fractional integral of the confluent hypergeometric function defined in [29] and is investigated there using the means of classical theories of differential subordination and superordination. Considering the previous applications of this function in obtaining fuzzy differential subordinations and superordinations [30], we continue the study and new fuzzy differential subordinations are obtained in this paper. The best fuzzy dominant is determined for each of them in the two theorems proved in the Main Results part of the paper. As a novelty, the conditions for the univalence of the fractional integral of the confluent hypergeometric function are stated using fuzzy differential subordinations. The first univalence result designed as the corollary of the first original theorem of this paper is obtained by using a certain function as the fuzzy best dominant. Another condition for the univalence of the fractional integral of the confluent hypergeometric function is provided in the second original theorem proved in this paper. An example is constructed in order to illustrate some applications of the newly proved theoretical results.

2. Preliminary Notions and Results

Certain notions specific to geometric function theory are implemented in this study.

Considering $U = \{z \in \mathbb{C} : \ |z| < 1\}$ as the unit disc of the complex plane, the closed unit disc is denoted by $\overline{U} = \{z \in C : \ |z| \leq 1\}$ and $\partial U = \{z \in C : \ |z| = 1\}$.

Let $H(U)$ denote the class of holomorphic functions in the unit disc. An important subclass of $H(U)$ is defined as $A_n = \{f \in H(U) \ : f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \ \in \ U\}$, with $A_1 = A$.

For $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, the subclass of the functions $f \in H(U)$ denoted by $H[a, n]$ consists of functions which can be written as $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, \ z \ \in \ U$.

Let $S = \{f \in A : f(z) = z + a_2 z^2 + \ldots, \ f(0) = 0, \ f'(0) = 1, z \ \in \ U\}$ be the class of the univalent functions in the unit disc and let

$$K = \left\{f \in A : Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \ z \in U\right\},$$

denote the class of the convex functions in the unit disc.

The following notions give the general context of fuzzy differential subordination theory established in [4].

Definition 1 ([4]). *A pair $(A, \ F_A)$, where $F_A \ : \ X \ \to \ [0, \ 1]$ and $A = \{x \in X : 0 < F_A(x) \leq 1\}$, is called the fuzzy subset of X. The set A is called the support of the fuzzy set $(A, \ F_A)$ and F_A is called the membership function of the fuzzy set $(A, \ F_A)$. One can also denote $A = supp(A, \ F_A)$.*

Let

$$\begin{aligned}\Omega = supp(\Omega, \ F_\Omega) = \{z \in \mathbb{C} : 0 < F_\Omega(z) \leq 1\}, \ \Delta = supp(\Delta, \ F_\Delta) \\ = \{z \in \mathbb{C} : 0 < F_\Delta(z) \leq 1\}.\end{aligned}$$

Additionally, let

$$p(U) = supp\Big(p(U), F_{p(U)}(U)\Big) = \Big\{p(z) : 0 < F_{p(U)}(f(z)) \le 1,\ z \in U\Big\}.$$

Definition 2 ([4,5])**.** *Let $D \subset \mathbb{C}$ and let $z_0 \in D$ be a fixed point and let the functions f, g be holomorphic in U. The function f is said to be a fuzzy subordinate to function g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if there exists a function $F : \mathbb{C} \to [0,1]$ such that:*

(i) $f(z_0) = g(z_0)$;
(ii) $F_{f(D)}(f(z)) \le F_{f(D)}(g(z)),\ z \in D.$

Remark 1.

(a)

$$f(D) = supp\Big(f(D), F_{f(D)}(D)\Big) = \Big\{f(z) : 0 < F_{f(D)}(f(z)) \le 1,\ z \in D\Big\},$$
$$g(D) = supp\Big(g(D), F_{g(D)}(D)\Big) = \Big\{g(z) : 0 < F_{g(D)}(g(z)) \le 1,\ z \in D\Big\},$$

and

$$\partial g(D) = supp\Big(\partial g(D), F_{\partial g(D)}(D)\Big) = \Big\{g(z) : 0 < F_{g(D)}(g(z)) = 1,\ z \in D\Big\}.$$

(b) *Relation (ii) is equivalent to $f(D) \subset g(D)$ and $\partial g(D) \not\subset f(D)$.*
(c) *If $D = U$, then relations (i) and (ii) are equivalent to*

$$f(0) = g(0),\ f(U) \subset g(U), \partial g(U) \not\subset f(U).$$

(d) *Such functions $F : \mathbb{C} \to [0,1]$ can be considered:*

$$F_1(z) = \frac{|z|}{1+|z|},\ F_2(z) = \frac{1}{1+|z|},\ F_3(z) = |\sin z|,\ F_4(z) = |\cos z|,\ z \in \mathbb{C}.$$

Definition 3 ([5])**.** *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, $a \in \mathbb{C}$ and let h be univalent in U, with $a \in \mathbb{C}$ and $\psi(a,0,0;0) = h(0) = a$. Let q be univalent in U with $q(0) = a$ and let p be analytic in U with $p(0) = a$. Function $\psi(p(z), zp'(z), z^2p''(z); z)$ is also analytic in U and let $F : \mathbb{C} \to [0,1]$. If p satisfies the (second-order) fuzzy differential subordination*

$$\begin{gathered} F_{\psi(\mathbb{C}^3 \times U)}\big(\psi(p(z), zp'(z), z^2p''(z); z)\big) \le F_{h(U)}(h(z)), \quad \text{i.e.,} \\ \psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), z \in D \end{gathered} \tag{1}$$

then p is called the fuzzy solution of the fuzzy differential subordination. The univalent function q is called the fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simply a fuzzy dominant, if $p(z) \prec_F q(z)$ or $F_{p(U)}(p(z)) \le F_{q(U)}(q(z))$, $z \in U$, for all p satisfying (1). A fuzzy dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z) \prec_F q(z)$ or $F_{p(U)}(\widetilde{q}(z)) \le F_{q(U)}(q(z))$, $z \in U$ for all fuzzy dominants q of (1) is said to be fuzzy best dominant of (1).

The next notions are tools of the classical theory of differential subordination.

Definition 4 ([3])**.** *We denote by Q the set of functions q that are analytic and injective on $\overline{U} \backslash E(q)$, where*

$$E(q) = \Big\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\Big\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$.

Definition 5 ([3]). *Let Ω be a set in $\mathbb{C}$, $q \in Q$, and let n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t; z) = 0, \tag{2}$$

whenever

$$r = q(\zeta),\ s = m\zeta q'(\zeta),\ Re\left(\frac{t}{s} + 1\right) \geq Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),\ z \in U,\ \zeta \in \partial U \backslash E(q),\ m \geq n.$$

The set $\Psi_n[\Omega, q]$ is called the class of admissible functions and condition (2) is called the admissibility condition.

If : $\mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (2) reduces to

$$\psi(r, s; z) = 0, \tag{3}$$

whenever

$$r = q(\zeta),\ s = m\zeta q'(\zeta),\ z \in U,\ \zeta \in \partial U \backslash E(q),\ m \geq n.$$

Remark 2.

(a) *In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is a conformal mapping of U into Ω, the class of admissible functions is denoted by $\Psi_n[h(U), q]$ or $\Psi_n[h, q]$.*

(b) *In the particular case when $\Omega = \Delta = \{w \in \mathbb{C} : Re\, w > 0\}$ and $Re\, a > 0$, the class of admissible functions is denoted by $\Psi_n\{a\}$. Since $Re\, q(\zeta) = 0$, when $\zeta \in \partial U \backslash \{1\}$, $q(z) = \frac{a + \bar{a}z}{1 - z}$, the admissibility condition (2) becomes:*

$$\psi(\rho i, \sigma, \mu + iv; z) = 0 \tag{4}$$

whenever

$$\rho, \sigma, \mu, v \in \mathbb{R},\ \sigma \leq -\frac{n}{2} \cdot \frac{|a - i\rho|^2}{Re\, a},\ \sigma + \mu \leq 0,\ z \in U,\ n \geq 1.$$

In the particular case when $a = 1$, the admissibility condition (4) becomes:

$$\psi(\rho i, \sigma, \mu + iv; z) = 0 \tag{5}$$

whenever

$$\rho, \sigma, \mu, v \in \mathbb{R},\ \sigma \leq -\frac{n}{2} \cdot \left(1 + \rho^2\right),\ \sigma + \mu \leq 0,\ z \in U,\ n \geq 1.$$

One of the first papers where the confluent hypergeometric function was used for research in geometric function theory appeared in 1990 [31] and involved univalence conditions for this function. The confluent hypergeometric function is defined as:

Definition 6 ([31]). *Let a and c be complex numbers with $c \neq 0, -1, -2, \ldots$ and consider*

$$\boldsymbol{\phi(a, c; z) = 1 + \frac{a}{c} \cdot \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \ldots,\ z \in U,} \tag{6}$$

This function is called the confluent (Kummer) hypergeometric function, is analytic in $\mathbb{C}$ and satisfies Kummer's differential equation:

$$z \cdot w''(z) + [c - z] \cdot w'(z) - a \cdot w(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\dots(d+k-1) \text{ and } (d)_0 = 1$$

then (6) can be written in the form

$$\boldsymbol{\phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!}}. \tag{7}$$

Euler's Gamma function is defined for $Re\, z > 0$ as:

$$\Gamma(z) = \int_0^{\infty} e^{-t} \cdot t^{z-1} dt.$$

The gamma function satisfies: $\Gamma(z+1) = z \cdot \Gamma(z)$, $\Gamma(1) = 1$, $\Gamma(n+1) = n! \cdot \Gamma(1) = n!$.

The fractional integral of the confluent hypergeometric function is defined in [29] as follows:

Definition 7 ([29])**.** *Let a and c be complex numbers with $c \neq 0, -1, -2, \dots$ and let $\mu > 1$. We define the fractional integral of the confluent hypergeometric function*

$$\boldsymbol{A_z^{-\mu} \phi(a,c;z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{\phi(a,c;t)}{(z-t)^{1-\mu}} dt = \frac{\Gamma(c)}{\Gamma(a)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)\Gamma(\mu+k+1)} \cdot z^{k+\mu}}. \tag{8}$$

We note that $A_z^{-\mu} \phi(a,c;z) \in H[0,\mu]$. For $\mu = 1$ we can write:

$$\boldsymbol{A_z^{-1} \phi(a,c;z) = z + \frac{a}{c} \cdot \frac{1}{2!} \cdot z^2 + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^3}{3!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)} \cdot \frac{z^4}{4!} + \dots}, \tag{9}$$

$A_z^{-1} \phi(a,c;z) \in H[0,1]$.

We obtain,

$$\left[A_z^{-1} \phi(a,c;z)\right]' = 1 + \frac{a}{c} \cdot z + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)} \cdot \frac{z^3}{8!} + \dots,$$

$\left[A_z^{-1} \phi(a,c;z)\right]' \in H[1,\ 1]$.

In order to prove the original results contained in the next section, we need the following already-established results.

Lemma 1 ([8])**.** *Let $q \in Q$ with $q(0) = a$ and let $p(z) = a + a_n z^n + \dots$ be analytic in U with $p(z) \neq a$ and $n \geq 1$. If p is not subordinate to q, then there exists points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$ and:*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m \zeta_0 q(\zeta_0)$

(iii) $Re\left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right) \geq m Re\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right)$.

Lemma 2 ([8])**.** *Let $p \in H[1,n]$. If $\psi \in \Psi_n\{1\}$, then*

$$Re\left[\psi\left(p(z), zp'(z), z^2 p''(z); z\right)\right] > 0 \text{ implies } Re\, p(z) > 0.$$

3. Main Results

The first theorem investigates a new fuzzy differential subordination involving the fractional integral of the confluent hypergeometric function. The best dominant of the subordiantion is found and this result will be used for stating a condition for the univalence of the fractional integral of the confluent hypergeometric function in the corollary which follows the theorem.

Theorem 1. *Let q be a univalent solution of the equation*

$$\boldsymbol{h}(\boldsymbol{z}) = \boldsymbol{q}(\boldsymbol{z}) + \frac{\boldsymbol{z}\boldsymbol{q}'(\boldsymbol{z})}{\boldsymbol{q}(\boldsymbol{z})},\ \boldsymbol{z} \in \boldsymbol{U}. \tag{10}$$

Function h is convex in U. Let $G : \mathbb{C} \to [0,\ 1]$ be given by:

$$G(z) = \frac{1}{1+|z|}. \tag{11}$$

Let the confluent hypergeometric function $\phi(a, c; z)$ be given by (6) and the fractional integral of the confluent hypergeometric function $A_z^{-\mu}\phi(a, c; z)$ be given by (8).

If the following fuzzy differential subordination is satisfied

$$\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} + \frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]' - \mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)} \prec_F h(z) = q(z) + \frac{zq'(z)}{q(z)}, \tag{12}$$

written equivalently as

$$G\left(\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} + \frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]' - \mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)}\right) \le G(h(z)),$$

or

$$\frac{1}{1+\left|\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} + \frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]' - \mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)}\right|} \le \frac{1}{1+|h(z)|},$$

then the fuzzy differential subordination implies

$$\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} \prec_F q(z),$$

written equivalently as

$$G\left(\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}}\right) \le G(q(z)),$$

or

$$\frac{1}{1+\left|\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}}\right|} \le \frac{1}{1+|q(z)|},\ z \in U.$$

This function q is the best fuzzy dominant.

Proof. Let

$$\boldsymbol{p}(\boldsymbol{z}) = \frac{\boldsymbol{A_z^{-\mu}\phi(a,c;z)}}{\boldsymbol{z^{\mu}}},\ \boldsymbol{z}\in\boldsymbol{U}. \tag{13}$$

Using the expression of the fractional integral of the confluent hypergeometric function given by (8), we can write:

$$p(z)=\frac{\frac{1}{\Gamma(\mu+1)}\cdot z^{\mu}+\frac{a}{c}\cdot\frac{1}{\Gamma(\mu+2)}\cdot z^{\mu+1}+\ldots}{z^{\mu}}=\frac{1}{\Gamma(\mu+1)}+\frac{a}{c}\cdot\frac{1}{\Gamma(\mu+2)}\cdot z+\ldots,$$

hence,

$$p(0)=\frac{1}{\Gamma(\mu+1)}\neq 0.$$

By differentiating relation (13), after some simple calculations, we obtain:

$$\boldsymbol{p(z)+\frac{zp'(z)}{p(z)}=\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}}+\frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]'-\mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)},\ z\in U.} \tag{14}$$

We let the function $\psi:\mathbb{C}^2\times U\to\mathbb{C},\psi\in\Psi_n[h(U),q]$ be given by:

$$\boldsymbol{\psi(r,s;z)=r+\frac{s}{r},\ r\neq 0.} \tag{15}$$

For $r=p(z),\ s=zp'(z)$, relation (15) becomes:

$$\boldsymbol{\psi\left(p(z),zp'(z);z\right)=p(z)+\frac{zp'(z)}{p(z)},p(z)\neq 0,z\in U.} \tag{16}$$

Using relation (16) in (14), we obtain:

$$\boldsymbol{\psi\left(p(z),zp'(z);z\right)=\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}}+\frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]'-\mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)},z\in U.} \tag{17}$$

Using (17), fuzzy differential subordination (12) becomes:

$$\psi\left(p(z),zp'(z);z\right)\prec_F h(z)=q(z)+\frac{zq'(z)}{q(z)},z\in U. \tag{18}$$

Using Definition 2 and Remark 1, we write:

$$\boldsymbol{\left\{z\in U:\psi\left(p(z),zp'(z);z\right)\right\}\subset h(U).} \tag{19}$$

For $z=z_0$, relation (19) is written as:

$$\boldsymbol{\psi\left(p(z_0),z_0p'(z_0);z_0\right)\in h(U).} \tag{20}$$

It is now time to use Lemma 1 and the admissibility condition (3).

Assume that the functions p,q,h satisfy the conditions required by Lemma 1 in the closed unit disc $\overline{U}$. If this is not true, the functions q and h can be replaced by $q_{\rho}(z)=q(\rho z)$ and $h_{\rho}(z)=h(\rho z)$, respectively, which are functions that have the desired properties.

Additionally, assume that $p(z)\not\prec_F q(z)$. Then, by applying Lemma 1, we get that there exist points $z_0=r_0e^{i\theta_0}\in U$ and $\zeta_0\in\partial U\backslash E(q)$ and an $m\geq 1$ such that

$$p(z_0)=\frac{A_z^{-\mu}\phi(a,c;z_0)}{{z_0}^{\mu}}=q(\zeta_0),z_0p'(z_0)=\frac{z_0\cdot\left[A_z^{-\mu}\phi(a,c;z_0)\right]'-\mu\cdot A_z^{-\mu}\phi(a,c;z_0)}{{z_0}^{\mu}}=m\zeta_0q'(\zeta_0).$$

Using these conditions with $r=q(\zeta_0),\ s=m\zeta_0q'(\zeta_0)$ in Definition 5 and considering the admissibility condition (3), we write:

$$\boldsymbol{\psi\left(q(\zeta_0),m\zeta_0q'(\zeta_0);\zeta_0\right)=0.} \tag{21}$$

On the other hand, $p(z_0) = q(\zeta_0), z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$ and we can write

$$\boldsymbol{\psi}(\boldsymbol{p}(\boldsymbol{z_0}), \boldsymbol{z_0}\boldsymbol{p}'(\boldsymbol{z_0}); \boldsymbol{z_0}) = \boldsymbol{\psi}(\boldsymbol{q}(\boldsymbol{\zeta_0}), \boldsymbol{m}\boldsymbol{\zeta_0}\boldsymbol{q}'(\boldsymbol{\zeta_0}); \boldsymbol{\zeta_0}). \tag{22}$$

Using (22) in (21), we obtain:

$$\boldsymbol{\psi}(\boldsymbol{p}(\boldsymbol{z_0}), \boldsymbol{z_0}\boldsymbol{p}'(\boldsymbol{z_0}); \boldsymbol{z}) = \mathbf{0}. \tag{23}$$

Relation (23) contradicts the assumption made when writing relation (20) and we conclude that the assumption is false, hence

$$\boldsymbol{p}(\boldsymbol{z}) \prec_{\boldsymbol{F}} \boldsymbol{q}(\boldsymbol{z}). \tag{24}$$

Using relation (8) in (24), we have:

$$\frac{\boldsymbol{A_z^{-\mu}\phi(a,c;z)}}{\boldsymbol{z^{\mu}}} \prec_{\boldsymbol{F}} \boldsymbol{q}(\boldsymbol{z}), \boldsymbol{z} \in \boldsymbol{U}. \tag{25}$$

Since function q is a univalent solution of Equation (10), we get that function q is the best fuzzy dominant. $\square$

The following corollary can be obtained by using the convex function $q(z) = \frac{z}{1+z}$ as the best fuzzy dominant in Theorem 1. As a result, a condition for the univalence of the fractional integral of the confluent hypergeometric function is stated.

Corollary 1. *Let function*

$$q(z) = \frac{z}{1-z},$$

be a univalent solution of the equation

$$\boldsymbol{h}(\boldsymbol{z}) = \boldsymbol{q}(\boldsymbol{z}) + \frac{\boldsymbol{z}\boldsymbol{q}'(\boldsymbol{z})}{\boldsymbol{q}(\boldsymbol{z})} = \frac{\mathbf{1}+\boldsymbol{z}}{\mathbf{1}-\boldsymbol{z}}, \ \boldsymbol{z} \in \boldsymbol{U}. \tag{26}$$

Function h is convex in U. Let $G : \mathbb{C} \to [0,1]$ be given by:

$$G(z) = \frac{1}{1+|z|}. \tag{27}$$

Let the confluent hypergeometric function $\phi(a,c;z)$ be given by (6) and the fractional integral of the confluent hypergeometric function $A_z^{-\mu}\phi(a,c;z)$ be given by (8).

If the following fuzzy differential subordination is satisfied

$$\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} + \frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]' - \mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)} \prec_F h(z) = \frac{1+z}{1-z}, \tag{28}$$

written equivalently as

$$G\left(\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} + \frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]' - \mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)}\right) \le G\left(\frac{1+z}{1-z}\right),$$

or

$$\frac{1}{1+\left|\frac{A_z^{-\mu}\phi(a,c;z)}{z^{\mu}} + \frac{z\cdot\left[A_z^{-\mu}\phi(a,c;z)\right]' - \mu\cdot A_z^{-\mu}\phi(a,c;z)}{A_z^{-\mu}\phi(a,c;z)}\right|} \le \frac{1}{1+\left|\frac{1+z}{1-z}\right|},$$

then the fuzzy differential subordination implies:

$$(a) \quad \frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu} \prec_F q(z) = \frac{z}{1-z},$$

written equivalently as

$$G\left(\frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu}\right) \le G\left(\frac{z}{1-z}\right),$$

or

$$\frac{1}{1+\left|\frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu}\right|} \le \frac{1}{1+\left|\frac{z}{1-z}\right|},\ z \in U,$$

with function q being the best fuzzy dominant, and also,

$$(b)\ Re\ \frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu} > -\frac{1}{2},$$

written equivalently,

$$\left\{z \in U : \frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu}\right\} \subset \left\{w \in \mathbb{C} : Re\, w > -\frac{1}{2}\right\}.$$

Proof. First, we prove that function $h(z) = \frac{1+z}{1-z}$ is convex in U. In order to achieve that conclusion, we calculate:

$$h'(z) = \frac{2}{(1-z)^2},\ h''(z) = \frac{4}{(1-z)^3}\ ,\ \frac{zh''(z)}{h'(z)} + 1 = \frac{1+z}{1-z}.$$

We now prove that function $q(z) = \frac{z}{1-z}$ is a solution of Equation (26).

$$\frac{z}{1-z} + \frac{\frac{z}{(1-z)^2}}{\frac{z}{1-z}} = \frac{z}{1-z} + \frac{1}{1-z} = \frac{1+z}{1-z}.$$

We now prove that function q is univalent in U. For that we calculate:

$$\begin{aligned} Re\left(\frac{zh''(z)}{h'(z)} + 1\right) &= Re\left(\frac{1+z}{1-z}\right) = Re\ \frac{1+\rho cos\alpha + i\rho sin\alpha}{1-\rho cos\alpha - i\rho sin\alpha} \\ &= Re\frac{(1+\rho cos\alpha + i\rho sin\alpha)(1-\rho cos\alpha - i\rho sin\alpha)}{(1-\rho cos\alpha)^2 + \rho^2 sin\ ^2\alpha} = \frac{1-\rho^2}{(1-\rho cos\alpha)^2+\rho^2 sin\ ^2\alpha} \\ &> 0, \end{aligned} \tag{29}$$

since $0 < \rho < 1$.

Relation (29) states that function h is a convex function in U.

We also prove that function $q(z) = \frac{z}{1-z}$ is convex in U.

We calculate:

$$q'(z) = \frac{1}{(1-z)^2},\ q''(z) = \frac{2}{(1-z)^3}\ ,\ \frac{zq''(z)}{q'(z)} + 1 = \frac{1+z}{1-z}.$$

Using relation (29), we get that

$$Re\left(\frac{zq''(z)}{q'(z)} + 1\right) = Re\left(\frac{1+z}{1-z}\right) = \frac{1-\rho^2}{(1-\rho\cos\alpha)^2 + \rho^2 sin\ ^2\alpha} > 0,$$

Hence, function q is a convex function in U.

Using relation (24) from the proof of Theorem 1, for

$$p(z) = \frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu}\ ,\ q(z) = \frac{z}{1-z}\ ,$$

we obtain:

$$\frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu} \prec_F \frac{z}{1-z}, z \in U. \tag{30}$$

Since function $q(z) = \frac{z}{1-z}$ a univalent solution of Equation (26), we conclude that q is the best fuzzy dominant.

Considering the fact that function $q(z) = \frac{z}{1-z}$ is a convex function, fuzzy differential subordination (30) is equivalent to

$$Re\frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu} > Re\frac{z}{1-z}, z \in U. \tag{31}$$

Because function $q(z) = \frac{z}{1-z}$ is univalent in U, it is a conformal mapping of the unit disc into the half-plane $S = \left\{w \in \mathbb{C} : Re\, w > -\frac{1}{2}\right\}$ and we conclude that $\mathrm{Re}\frac{z}{1-z} > -\frac{1}{2}$, $z \in U$.

We can now state that

$$Re\frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu} > -\frac{1}{2}, z \in U,$$

which is equivalent to

$$\left\{z \in U : \frac{A_z^{-\mu}\phi(a,c;z)}{z^\mu}\right\} \subset \left\{w \in \mathbb{C} : Re\, w > -\frac{1}{2}\right\},\ z \in U.$$

□

The next theorem gives a sufficient condition for the univalence of the fractional integral of the confluent hypergeometric function $A_z^{-1}\phi(a,c;z)$ given by (9).

Theorem 2. *Let q be a univalent solution of the equation*

$$\boldsymbol{h(z) = q(z) + zq'(z),\ z\in U.} \tag{32}$$

Let h be a convex function in U with $Re\, h(z) > 1$. Let $G : \mathbb{C} \to [0,1]$ be given by:

$$G(z) = \frac{1}{1+|z|}. \tag{33}$$

Let the confluent hypergeometric function $\boldsymbol{\phi(a,c;z)}$ be given by (6) and the fractional integral of the confluent hypergeometric function $\boldsymbol{A_z^{-1}\phi(a,c;z)}$ be given by (9).

The fuzzy differential subordination

$$\left[A_z^{-1}\phi(a,c;z)\right]' + z\left[A_z^{-1}\phi(a,c;z)\right]'' + z^2\left[A_z^{-1}\phi(a,c;z)\right]''' \prec_F h(z), z \in U, \tag{34}$$

implies:

(a) $\left[A_z^{-1}\phi(a,c;z)\right]' \prec_F q(z)$, *where q is the best fuzzy dominant.*

(b) $\boldsymbol{A_z^{-1}\phi(a,c;z)\in S}$.

Proof. We first prove part (a) of the theorem.

Let

$$\begin{aligned} \boldsymbol{p(z)} &= \left[\boldsymbol{A_z^{-1}\phi(a,c;z)}\right]' \\ &= \boldsymbol{1 + \frac{a}{c}\cdot z + \frac{a(a+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\cdot\frac{z^3}{8!}} \\ &\quad \boldsymbol{+ \frac{a(a+1)(a+2)(a+3)}{c(c+1)(c+2)(c+3)}\cdot\frac{z^4}{24} + \dots} \end{aligned} \tag{35}$$

We have that $p(0) = \left[A_z^{-1}\phi(a,c;0)\right]' = 1 \neq 0$.
For finalizing the proof, Lemma 1 and Definition 5 will be applied.
Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be given by

$$\boldsymbol{\psi(r,s,t;z) = r + s + t.} \tag{36}$$

For $r = p(z) = \left[A_z^{-1}\phi(a,c;z)\right]'$, $s = zp'(z) = z\left[A_z^{-1}\phi(a,c;z)\right]''$, $t = z^2p''(z) = z^2\left[A_z^{-1}\phi(a,c;z)\right]'''$, relation (32) becomes:

$$\begin{aligned} &\boldsymbol{\psi\left(p(z), zp'(z), z^2p''(z); z\right)} \\ &\quad \boldsymbol{= \left[A_z^{-1}\phi(a,c;z)\right]' + z\left[A_z^{-1}\phi(a,c;z)\right]'' + z^2\left[A_z^{-1}\phi(a,c;z)\right]'''.} \end{aligned} \tag{37}$$

Using (37), the fuzzy differential subordination (34) becomes:

$$\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) \prec_F h(z), z \in U, \tag{38}$$

Since h is a convex function, $h(U)$ is a convex domain and fuzzy differential subordination (38) can be written equivalently as:

$$Re\ \psi\Big(p(z), zp'(z), z^2p''(z); z\Big) > Re\ h(z) > 0, z \in U. \tag{39}$$

For $z = z_0$, relation (39) becomes:

$$Re\ \psi\Big(p(z_0), z_0p'(z_0), z^2p''(z_0); z_0\Big) > Re\ h(z_0) > 0. \tag{40}$$

We shall assume that $p(z) \not\prec_F q(z)$. Then, by applying Lemma 1, we get that there exists points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ and an $m \geq 1$ such that

$$\begin{gathered} \boldsymbol{p(z_0) = \left[A_z^{-1}\phi(a,c;z_0)\right]' = q(\zeta_0), z_0p'(z_0) = z_0\cdot\left[A_z^{-1}\phi(a,c;z_0)\right]'' = m\zeta_0q'(\zeta_0),} \\ \boldsymbol{z_0{}^2p''(z_0) = z_0{}^2\left[A_z^{-1}\phi(a,c;z_0)\right]''' = \zeta_0{}^2q''(\zeta_0).} \end{gathered}$$

By replacing $r = q(\zeta_0),\ s = m\zeta_0q'(\zeta_0),\ t = \zeta_0{}^2q''(\zeta_0),$ in admissibility condition (2), we have:

$$\psi\Big(p(z_0), z_0p'(z_0), z^2p''(z_0); z_0\Big) = \psi\Big(q(\zeta_0), m\zeta_0q'(\zeta_0), \zeta_0{}^2q''(\zeta_0); \zeta_0\Big) = 0 \tag{41}$$

from where we deduce

$$\psi\Big(p(z_0), z_0p'(z_0), z_0{}^2p''(z_0); z_0\Big) = 0. \tag{42}$$

Since relation (42) contradicts relation (40), we conclude that the assumption we have made is false, hence

$$\boldsymbol{p(z)\prec_F q(z).} \tag{43}$$

Using relation (35) in (43), we write:

$$\boldsymbol{\left[A_z^{-1}\phi(a,c;z)\right]'\prec_F q(z).} \tag{44}$$

Since q is the univalent solution of Equation (32), we have that function q is the best fuzzy dominant.

Since $\Omega = \Delta = \{w \in \mathbb{C} : Re\, w > 0\}$ and $p(0) = \left[A_z^{-1}\phi(a,c;0)\right]' = 1 \neq 0,$ in order to prove part b), it suffices to prove that function ψ given by (36) satisfies $\psi \in \Psi_n\{1\}$. For that, we check the admissibility condition (5).

For $r = p(z_0) = \left[A_z^{-1}\phi(a,c;z_0)\right]' = \rho i$, $s = z_0 p'(z_0) = z_0\left[A_z^{-1}\phi(a,c;z_0)\right]'' = \sigma$, $t = {z_0}^2\left[A_z^{-1}\phi(a,c;z_0)\right]''' = \mu + iv$, the admissibility condition *(5)* becomes:

$$\psi\left(\left[A_z^{-1}\phi(a,c;z_0)\right]', z_0\left[A_z^{-1}\phi(a,c;z_0)\right]'', {z_0}^2\left[A_z^{-1}\phi(a,c;z_0)\right]'''; z_0\right) = 0, \tag{45}$$

which is equivalent to $\boldsymbol{\psi} \in \boldsymbol{\Psi_n}\{\mathbf{1}\}$.

Using the conditions $\psi \in \Psi_n\{1\}$ and $p(0) = \left[A_z^{-1}\phi(a,c;0)\right]' = 1$ by applying Lemma 2, we get that

$$Re\, p(z) > 0, z \in U. \tag{46}$$

By replacing, in (46), the expression of the function p given by (35), we obtain:

$$Re\left[A_z^{-1}\phi(a,c;z)\right]' > 0, z \in U. \tag{47}$$

Since $p(0) = \left[A_z^{-1}\phi(a,c;0)\right]' = 1$ and using (47), we conclude that function $A_z^{-1}\phi(a,c;z)$ is univalent in U. □

Example 1. *Let*

$$q(z) = \frac{1}{1+z},\ q'(z) = \frac{-1}{(1+z)^2},\ q''(z) = \frac{2}{(1+z)^3}$$

We have

$$\boldsymbol{Re}\left(\frac{zq''(z)}{q'(z)} + \mathbf{1}\right) = \boldsymbol{Re}\,\frac{1-z}{1+z} = \boldsymbol{Re}\,\frac{1-\rho\cos\alpha - i\rho\sin\alpha}{1+\rho\cos\alpha + i\rho\sin\alpha}$$
$$= \frac{1-\rho^2}{(1+\rho\cos\alpha)^2 + \rho^2\sin^2\alpha} > \mathbf{0},$$

hence, $\boldsymbol{q}$ is a convex function in U.

Let

$$h(z) = q(z) + \frac{zq'(z)}{q(z)} = \frac{1}{1+z} - \frac{z}{1+z} = \frac{1-z}{1+z}.$$

Then, we get:

$$\boldsymbol{h'(z)} = \frac{-2}{(1+z)^2},\ \boldsymbol{h''(z)} = \frac{4}{(1+z)^3},\ \boldsymbol{Re}\left(\frac{zh''(z)}{h'(z)} + \mathbf{1}\right) = \boldsymbol{Re}\,\frac{1-z}{1+z}$$
$$= \frac{1-\rho^2}{(1+\rho\cos\alpha)^2 + \rho^2\sin^2\alpha} > \mathbf{0},$$

*hence, **h** is a convex function in **U**.*

Let $\mu > 0$, $a = -2$, $c = 1+i$; then, we have the fractional integral of the confluent hypergeometric function:

$$A_z^{-\mu}(-2,\ 1+i; z) = z^{\mu} - \frac{1-i}{2}z^{\mu+1} + \frac{1-3i}{30}z^{\mu+2}.$$

Using Theorem 1, we write:

Let $q(z) = \frac{1}{1+z}$ be the univalent solution of the equation

$$h(z) = q(z) + \frac{zq'(z)}{q(z)} = \frac{1-z}{1+z}.$$

Function h is convex in U. Let $G : \mathbb{C} \to [0,\ 1]$ be given by:

$$G(z) = \frac{1}{1+|z|}.$$

If the following fuzzy differential subordination is satisfied

$$\frac{A_z^{-\mu}(-2,\ 1+i;z)}{z^\mu}+\frac{z\cdot\left[A_z^{-\mu}(-2,\ 1+i;z)\right]'-\mu\cdot A_z^{-\mu}(-2,\ 1+i;z)}{A_z^{-\mu}(-2,\ 1+i;z)}\prec_F h(z)=\frac{1-z}{1+z},$$

where

$$\boldsymbol{A_z^{-\mu}(-2,\ 1+i;z)=z^\mu-\tfrac{1-i}{2}z^{\mu+1}+\tfrac{1-3i}{30}z^{\mu+2}},$$
$$\boldsymbol{\tfrac{A_z^{-\mu}(-2,\ 1+i;z)}{z^\mu}=1-\tfrac{1-i}{2}z+\tfrac{1-3i}{30}z^2}$$
$$\boldsymbol{\left[A_z^{-\mu}(-2,\ 1+i;z)\right]'=\mu z^{\mu-1}-\tfrac{1-i}{2}(\mu+1)z^\mu-\tfrac{1-3i}{30}(\mu+2)z^{\mu+1}},$$

written equivalently as

$$G\left(\frac{A_z^{-\mu}(-2,\ 1+i;z)}{z^\mu}+\frac{z\cdot\left[A_z^{-\mu}(-2,\ 1+i;z)\right]'-\mu\cdot A_z^{-\mu}(-2,\ 1+i;z)}{A_z^{-\mu}(-2,\ 1+i;z)}\right)\le G\left(\frac{1-z}{1+z}\right),$$

or

$$\frac{1}{1+\left|\frac{A_z^{-\mu}(-2,\ 1+i;z)}{z^\mu}+\frac{z\cdot\left[A_z^{-\mu}(-2,\ 1+i;z)\right]'-\mu\cdot A_z^{-\mu}(-2,\ 1+i;z)}{A_z^{-\mu}(-2,\ 1+i;z)}\right|}\le\frac{1}{1+\left|\frac{1-z}{1+z}\right|},$$

then it implies:

$$\frac{A_z^{-\mu}(-2,\ 1+i;z)}{z^\mu}=1-\frac{1-i}{2}z+\frac{1-3i}{30}z^2\prec_F q(z)=\frac{1}{1+z},$$

written equivalently as

$$G\left(1-\frac{1-i}{2}z+\frac{1-3i}{30}z^2\right)\le G\left(\frac{1}{1+z}\right),$$

or

$$\frac{1}{1+\left|1-\frac{1-i}{2}z+\frac{1-3i}{30}z^2\right|}\le\frac{1}{1+\left|\frac{1}{1+z}\right|},\ z\in U.$$

Indeed, since $q(z)=\frac{1}{1+z}$ *is a convex function, the fuzzy differential subordination is equivalent to writing:*

$$Re\left(1-\frac{1-i}{2}z+\frac{1-3i}{30}z^2\right)>Re\frac{1}{1+z}.$$

Function q being univalent, we know that it represents a conformal transform of the unit disc U into the half-plane $\left\{w\in\mathbb{C}: Re\ w>\frac{1}{2}\right\}$; *hence, we have:*

$$Re\left(1-\frac{1-i}{2}z^2+\frac{1-3i}{30}z^3\right)>\frac{1}{2},$$

equivalently written as the inclusion of sets:

$$\left\{z\in U:\ 1-\frac{1-i}{2}z+\frac{1-3i}{30}z^2\right\}\subset\left\{w\in\mathbb{C}: Re\ w>\frac{1}{2}\right\}.$$

If we let

$$f(z)=1-\frac{1-i}{2}z+\frac{1-3i}{30}z^2,$$

then we have $f(0)=q(0)=1$ *and* $f(U)\subset q(U)$, *which, according to Definition 2 means that:*

$$f(z)\prec_F q(z),\ z\in U.$$

4. Conclusions

The investigation presented in this paper continues the line of research which has, as its focus, the fractional integral of the confluent hypergeometric function defined in [29]. The studies connecting this function to fuzzy differential subordination and superordination theories [27,30] are enriched in this paper by providing means to obtain the conditions for the univalence of this function. After providing the best fuzzy dominant for a certain fuzzy differential subordination in Theorem 1, in Corollary 1 this theoretical result is applied by using a particular function as the best fuzzy dominant for finding a condition for the univalence of the fractional integral of the confluent hypergeometric function. In Theorem 2, another condition for the univalence of this function is deduced. The example presented at the end of the study shows how fuzzy differential subordinations are obtained using the fractional integral of the confluent hypergeometric function.

In future studies involving the results presented in this paper, researchers could use the properties of the univalence of this function in order to define new subclasses of univalent functions. Other conditions for the univalence of this function, maybe even convexity properties which have many applications, could be investigated following the ideas presented here using the theory of fuzzy differential superordination and the fractional integral of the confluent hypergeometric function.

Author Contributions: Conceptualization, M.A. and G.O.; methodology, M.A., G.O. and A.M.R.; software, A.M.R.; validation, M.A., G.O. and A.M.R.; formal analysis, M.A. and G.O.; investigation, M.A., G.O. and A.M.R.; resources, M.A. and O.G; data curation, M.A. and G.O.; writing—original draft preparation, G.O.; writing—review and editing, M.A., G.O. and A.M.R.; visualization, M.A., G.O. and A.M.R.; supervision, G.O.; project administration, M.A.; funding acquisition, G.O. and A.M.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy Sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Dzitac, I.; Filip, F.G.; Manolescu, M.J. Fuzzy Logic Is Not Fuzzy: World-renowned Computer Scientist Lotfi A. Zadeh. *Int. J. Comput. Commun. Control* **2017**, *12*, 748–789. [CrossRef]
3. Dzitac, S.; Nădăban, S. Soft Computing for Decision-Making in Fuzzy Environments: A Tribute to Professor Ioan Dzitac. *Mathematics* **2021**, *9*, 1701. [CrossRef]
4. Oros, G.I.; Oros, G. The notion of subordination in fuzzy sets theory. *Gen. Math.* **2011**, *19*, 97–103.
5. Oros, G.I.; Oros, G. Fuzzy differential subordination. *Acta Univ. Apulensis* **2012**, *3*, 55–64.
6. Miller, S.S.; Mocanu, P.T. Second order differential inequalities in the complex plane. *J. Math. Anal. Appl.* **1978**, *65*, 289–305. [CrossRef]
7. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–172.
8. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; Marcel Dekker, Inc.: New York, NY, USA; Basel, Switzerland, 2000.
9. Oros, G.I.; Oros, G. Dominants and best dominants in fuzzy differential subordinations. *Stud. Univ. Babes-Bolyai Math.* **2012**, *57*, 239–248.
10. Alb Lupaș, A. A note on special fuzzy differential subordinations using generalized Sălăgean operator and Ruscheweyh derivative. *J. Comput. Anal. Appl.* **2013**, *15*, 1476–1483.
11. Venter, A.O. On special fuzzy differential subordination using Ruscheweyh operator. *Analele Univ. Oradea Fasc. Mat.* **2015**, *XXII*, 167–176.
12. Majeed, A.H. Fuzzy differential subordinations properties of analytic functions involving generalized differential operator. *Sci. Int. Lahore* **2018**, *30*, 297–302.
13. Atshan, W.G.; Hussain, K.O. Fuzzy Differential Superordination. *Theory Appl. Math. Comput. Sci.* **2017**, *7*, 27–38.

14. Ibrahim, R.W. On the subordination and superordination concepts with applications. *J. Comput. Thor. Nanosci.* **2017**, *14*, 2248–2254. [CrossRef]
15. Thilagavathi, K. Fuzzy subordination and superordination results for certain subclasses of analytic functions associated with Srivastava-Attitya operator. *Int. J. Pure Appl. Math.* **2018**, *118*, 921–929.
16. Srivastava, H.M.; El-Deeb, S.M. Fuzzy Differential Subordinations Based upon the Mittag-Leffler Type Borel Distribution. *Symmetry* **2021**, *13*, 1023. [CrossRef]
17. Oros, G.I. New fuzzy differential subordinations. *Commun. Fac.Sci. Univ. Ank. Ser. A1 Math. Stat.* **2021**, *70*, 229–240. [CrossRef]
18. El-Deeb, S.M.; Oros, G.I. Fuzzy differential subordinations connected with the linear operator. *Math. Bohem.* **2021**, *146*, 397–406. [CrossRef]
19. Oros, G.I. Univalence criteria for analytic functions obtained using fuzzy differential subordinations. *Turk. J. Math.* **2022**, *46*, 1478–1491. [CrossRef]
20. Srivastava, H.M. An Introductory Overview of Fractional-Calculus Operators Based Upon the Fox-Wright and Related Higher Transcendental Functions. *J. Adv. Eng. Comput.* **2021**, *5*, 135–166. [CrossRef]
21. Rashid, S.; Khalid, A.; Bazighifan, O.; Oros, G.I. New Modifications of Integral Inequalities via $\wp$-Convexity Pertaining to Fractional Calculus and Their Applications. *Mathematics* **2021**, *9*, 1753. [CrossRef]
22. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Kodamasingh, B.; Shaikh, A.A.; Botmart, T.; El-Shorbagy, M.A. Some Novel Fractional Integral Inequalities over a New Class of Generalized Convex Function. *Fractal Fract.* **2022**, *6*, 42. [CrossRef]
23. Ghanim, F.; Al-Janaby, H.F.; Bazighifan, O. Some New Extensions on Fractional Differential and Integral Properties for Mittag-Leffler Confluent Hypergeometric Function. *Fractal Fract.* **2021**, *5*, 143. [CrossRef]
24. Srivastava, H.M.; Kashuri, A.; Mohammed, P.O.; Nonlaopon, K. Certain Inequalities Pertaining to Some New Generalized Fractional Integral Operators. *Fractal Fract.* **2021**, *5*, 160. [CrossRef]
25. Alb Lupaş, A.; Oros, G.I. On Special Differential Subordinations Using Fractional Integral of Sălăgean and Ruscheweyh Operators. *Symmetry* **2021**, *13*, 1553. [CrossRef]
26. Alb Lupaş, A.; Cătaş, A. Fuzzy Differential Subordination of the Atangana–Baleanu Fractional Integral. *Symmetry* **2021**, *13*, 1929. [CrossRef]
27. Alb Lupaş, A. Fuzzy Differential Sandwich Theorems Involving the Fractional Integral of Confluent Hypergeometric Function. *Symmetry* **2021**, *13*, 1992. [CrossRef]
28. Oros, G.I.; Dzitac, S. Applications of Subordination Chains and Fractional Integral in Fuzzy Differential Subordinations. *Mathematics* **2022**, *10*, 1690. [CrossRef]
29. Alb Lupaş, A.; Oros, G.I. Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function. *Symmetry* **2021**, *13*, 327. [CrossRef]
30. Alb Lupaş, A. Applications of the Fractional Calculus in Fuzzy Differential Subordinations and Superordinations. *Mathematics* **2021**, *9*, 2601. [CrossRef]
31. Miller, S.S.; Mocanu, P.T. Univalence of Gaussian and confluent hypergeometric Functions. *Proc. Am. Math. Soc.* **1990**, *110*, 333–342. [CrossRef]