fractal and fractional

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A Brief Study on Julia Sets in the Dynamics of Entire Transcendental Function Using Mann Iterative Scheme

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Abstract: In this research, we look at the Julia set patterns that are linked to the entire transcendental function $f(z) = ae^{z^n} + bz + c$, where $a, b, c \in \mathbb{C}$ and $n \geq 2$, using the Mann iterative scheme, and discuss their dynamical behavior. The sophisticated orbit structure of this function, whose Julia set encompasses the entire complex plane, is described using symbolic dynamics. We also present bifurcation diagrams of Julia sets generated using the proposed iteration and function, which altogether contain four parameters, and discuss the graphical analysis of bifurcation occurring in the family of this function.

Keywords: fixed point; Mann orbit; Julia set; bifurcation

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1. Introduction

Fractals are infinitely complex identical patterns with many real-life applications. Fractal geometry is a kind of non-euclidean geometry in which, using algorithms, we can model beautiful natural objects that are not possible to easily model using ordinary (Euclidean) geometry. In 1918, French mathematician Gaston Julia coined the term fractal and also initiated the concept of Julia sets [1]. He explored iterated complex polynomials and announced a classical example of fractals in terms of the Julia set. In 1975, French-American mathematician Benoit B. Mandelbrot [2] used the complex polynomial $z^2 + c$ to introduce the notion of the Mandelbrot set and gave some natural examples of fractals such as variations of traffic flow, records of heartbeat, irregular coastal structures, etc. (see [3–5], etc.). There are many techniques to generate and investigate fractals, one of which is the iterated function system (IFS) [6]. IFS is also used to approximate the fixed points of functions under suitable conditions. The theory of fixed points via fractal geometry may be applied to numerous nonlinear phenomena in distinct branches of sciences such as physics, biology, biotechnology, computer science, engineering, etc.(see, [4,7], and so on), as it includes lots of details at each point.

The original Mandelbrot and Julia sets were obtained for the polynomial $f(z) = z^2 + c$, $c \in \mathbb{C}$ by using a simple Picard iterative process. Later on, various researchers ([8–21]) used different iterative processes and obtained variants of these sets to study their behavior and pattern for different polynomials because it is known that the shape, size, color, and other characteristics vary with the iterative procedures for the same function. Shahid et al. [22] introduced a new iterative scheme to study the behavior of orbits and dynamics for a $(k+1)$st degree complex polynomial. Sajid and Kapoor [23] generated Julia sets of a family of transcendental meromorphic functions having rational Schwarzian derivatives. Julia sets of Joukowski-Exponential maps have been discussed in [24].

Thus far, many researchers have worked on complex analytic functions such as polynomial function, rational function, and exponential function without singularities to study the characteristics and dynamics of a Julia set using Picard iteration. In 1981, Misiurewicz [25] examined the mathematical aspects of the complex exponential map $f(z) = e^z$. Devaney (see, [6,26–28]) described the topology and dynamics of the complex exponential family $E_\lambda(z) = \lambda e^z, \lambda > 0$. Baker and Rippon [29] discussed the interesting geometry of Julia sets for different values of λ. Romera et al. [30] studied the complex families $e^{z^2+\lambda}$ and $e^{\frac{z}{\lambda}}$ using the Picard iterative process, while Prasad et al. [31] used the Ishikawa iterative scheme to prove the same results.

A dynamical system's goal is to comprehend the nature of all orbits and to discover the set of asymptotic and periodic orbits. Frequently applied in the study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behavior. These variations may include the periodic point structure, including other changes too. Sajid [32] obtained the chaotic behavior and bifurcation in the real dynamics of a newly proposed family of functions that depends on two parameters in one dimension. A recent review on some advances in dynamics of one-variable complex functions has been given in [33].

In this line, we introduce a more generalized, new transcendental function $f(z) = ae^{z^n} + bz + c$, where $n \geq 2$ and $a, b, c \in \mathbb{C}$. The motivation behind this work is the fact that exponential functions are solutions to simple dynamical systems and appear in physics, chemistry, computer science, engineering, economics, and mathematical biology, etc. For instance, it emerges in bacteria growth models, self-reproducing populations, a fund accruing compound interest, growth or decay of population, or a growing body of manufacturing expertise. The function is found to be very useful in many aspects, which may further open up new frontiers for researchers to study the properties of Julia sets. Furthermore, the exponential map can be used as the evolution function of the discrete nonlinear dynamical system in the theory of dynamical systems. We utilize the Mann iteration to evidence the escape criteria for this function. We establish a novel escape radius and generate the algorithms for the visualization of alluring Julia sets as fractals. The escape criterion [34] is well known as the key to generating Julia sets. Using the obtained escape criteria, we obtain fascinating non-classical variants of Julia sets as fractals using MATLAB software and colormap. Towards the end, we present bifurcation diagrams for a four-parameter family of complex-valued functions using the Mann iteration scheme to perform a bifurcation analysis graphically.

2. Preliminaries

Definition 1 ([1])**.** *If $f : \mathbb{C} \to \mathbb{C}$ is a complex-valued polynomial function. Then, the filled Julia set is*

$$F_f = \{z \in \mathbb{C} : |f^p(z)|_{p=1}^{\infty} \nrightarrow \infty \text{ as } p \to \infty\},$$

where $f^p(z)$ is the p-th iterate of f. The boundary of the filled Julia set, i.e., ∂F_f is simply known as Julia set.

Definition 2 ([35])**.** *Let $\{z_k\}$ be the sequence of iterates and $z_0 \in \mathbb{C}$ be an initial point. The sequence of iterates $\{z_k\}$ is known as Mann orbit (MO) if*

$$\{z_{k+1} : z_{k+1} = (1-\alpha)z_k + \alpha f(z_k)\}, \tag{1}$$

where $\alpha \in (0,1]$ for $k = 0, 1, 2, 3, \ldots$, which is a function of 3-tuple (f, z_0, α).

3. Main Results

First, we derive the escape criterion for the complex function of type $f(z) = ae^{z^n} + bz + c$ utilizing the Mann iteration [35], where $n \geq 2$ and $a, b, c \in \mathbb{C}$.

Theorem 1. *Let* $|z_0| \geq |c| > \left(|b| + \frac{2}{\alpha}\right)^{\frac{1}{(n-1)}}$ *and* $|z_0|^n \leq |a| Re(z_0{}^n)$, *where* $\alpha \in (0,1]$ *and* $a, b, c \in \mathbb{C}$. *Let a sequence* $\{z_k\}_{k \in \mathbb{N}}$ *be defined as:*

$$z_{k+1} = (1-\alpha)z_k + \alpha f(z_k), \tag{2}$$

for $k = 0, 1, 2, \ldots$, *then* $|z_k| \to \infty$ *as* $k \to \infty$.

Proof. Consider

$$\begin{aligned}
|z_1| &= |(1-\alpha)z_0 + \alpha f(z_0)| \\
&= |(1-\alpha)z_0 + \alpha(ae^{z_0{}^n} + bz_0 + c)| \\
&\geq \alpha|ae^{z_0{}^n} + bz_0 + c| - (1-\alpha)|z_0| \\
&\geq \alpha|ae^{z_0{}^n}| - \alpha|bz_0| - \alpha|c| - |z_0| + \alpha|z_0| \\
&\geq \alpha|ae^{z_0{}^n}| - \alpha|bz_0| - \alpha|z_0| - |z_0| + \alpha|z_0|, \text{ since } |z_0| \geq |c| \\
&= \alpha|ae^{z_0{}^n}| - \alpha|bz_0| - |z_0| \\
&= \alpha|a||e^{Re(z_0{}^n)}| - \alpha|bz_0| - |z_0| \\
&\geq \alpha|a|Re(z_0{}^n) - \alpha|bz_0| - |z_0| \\
&\geq \alpha|z_0|^n - \alpha|bz_0| - |z_0| \qquad (|z_0|^n \leq |a|Re(z_0{}^n)) \\
&\geq |z_0|(\alpha|z_0|^{n-1} - \alpha|b| - 1) \\
&> |z_0|, \qquad \left(|z_0| \geq |c| > \left(|b| + \frac{2}{\alpha}\right)^{\frac{1}{(n-1)}}\right)
\end{aligned}$$

which implies that there exists $\lambda > 0$ so that $|z_1| > (1+\lambda)|z_0|$.

On continuing the above procedure, we attain $|z_k| > (1+\lambda)^k |z_0|$. Hence, $|z_k| \to \infty$ as $k \to \infty$, i.e., the orbit of z_0 tends to infinity. □

Corollary 1. *If* $|z_0| > max\left[|c|, \left(|b| + \frac{2}{\alpha}\right)^{\frac{1}{(n-1)}}\right]$ *and* $|z_0|^n \leq |a|Re(z_0{}^n)$, *where* $\alpha \in (0,1]$ *and* $a, b, c \in \mathbb{C}$, *then there exists* $\lambda > 0$ *so that* $|z_k| > (1+\lambda)^k |z_0|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

Corollary 2. *If* $|z_m| > |z_0| > max\left[|c|, \left(|b| + \frac{2}{\alpha}\right)^{\frac{1}{(n-1)}}\right]$, $m \geq 0$ *and* $|z_0|^n \leq |a|Re(z_0{}^n)$. *Then there exists* $\lambda > 0$ *so that* $|z_{m+k}| > (1+\lambda)^k |z_m|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

4. Generation of Julia Sets

The Julia set (J-set) secured an indistinguishable place in the field of dynamical systems and fractal geometry due to its useful characteristics, applications in various fields, and amazing beauty ([3,5,36–39]). In this section, we utilize the escape time algorithm to generate Julia sets for different input parameters by taking a maximum of 30 iterations. The standard 'jet' colormap has been utilized to color the points z_0 (Figure 1). The algorithm used for generating fractals is given in Algorithm 1.

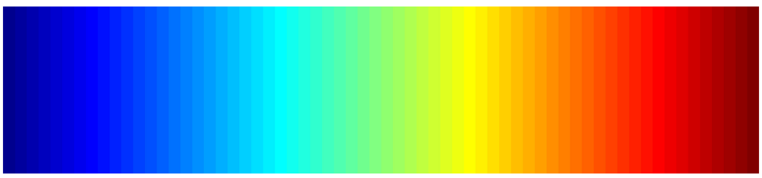


Figure 1. Colormap used for generating fractals.

Algorithm 1 Julia set.

Input: $f(z) = ae^{z^n} + bz + c$, where $a, b, c \in \mathbb{C}$, $n = 2, \dots$; K—maximum number of iterations; $A \subset \mathbb{C}$—area; $\alpha \in (0, 1)$—parameter of the Mann iteration; $colormap[0 \dots C-1]$—color map with C colors.
Output: Julia set for area A.

for $z_0 \in A$ **do**
  $R_1 = \max\{|c|, (|b| + \frac{2}{\alpha})^{(\frac{1}{n-1})}\}$
  $R_2 = (|a| Re({z_0}^n))^{\frac{1}{n}}$
  $k = 0$
  while $k \leq K$ **do**
    $f(z_k) = ae^{{z_k}^n} + bz_k + c$
    $z_{k+1} = (1-\alpha)z_k + \alpha f(z_k)$
    if $R_1 < |z_{k+1}| \leq R_2$ **then**
      break
    end if
    $k = k + 1$
  end while
  $i = (C-1)\frac{k}{K}$
  color z_0 with $colormap[i]$
end for

The J-sets, for $f(z)$, for different values of input parameters are given in Figure 2. Clearly, by changing the signs of real-valued parameters a and c, we obtain the mirror images. The seed-like structure disappears from the J-set when we change the value of parameter a from real to complex. Furthermore, the removal of the real part in the complex-valued parameter a ($3 + 3i$ to $3i$) gives the mirror images. It is surprising to see that all these J-sets occupy the same area. The parameters used in Figure 2 are given in Table 1.

Table 1. Parameters for different values of a and c for Julia sets in MO.

	a	b	c	α	Area	n
(i)	-3	-0.00000001	3	0.1	$[-5, 5] \times [-5, 5]$	2
(ii)	3	-0.00000001	-3	0.1	$[-5, 5] \times [-5, 5]$	2
(iii)	$3 + 3i$	-0.00000001	3	0.1	$[-5, 5] \times [-5, 5]$	2
(iv)	$3i$	-0.00000001	3	0.1	$[-5, 5] \times [-5, 5]$	2

The J-sets for $f(z)$ for various values of parameter α and fixing the remaining parameters are given in Figures 3 and 4. The variation in parameter α affects the shape and color of the J-sets. The parameters used in Figure 3 are given in Table 2.

Table 2. Parameters for different values of α for Julia sets in MO.

	a	b	c	α	Area	n
(i)	3	-0.00000001	-3	0.3	$[-2, 2] \times [-2, 2]$	2
(ii)	3	-0.00000001	-3	0.2	$[-2, 2] \times [-2, 2]$	2

The parameters used in Figure 4 are given in Table 3.

Table 3. Parameters for different values of α for Julia sets in MO.

	a	b	c	α	Area	n
(i)	3	-0.0111	-3	0.46909999	$[-0.8, 0.8] \times [-0.8, 0.8]$	2
(ii)	3	-0.0111	-3	0.909999	$[-0.5, 0.5] \times [-0.5, 0.5]$	2

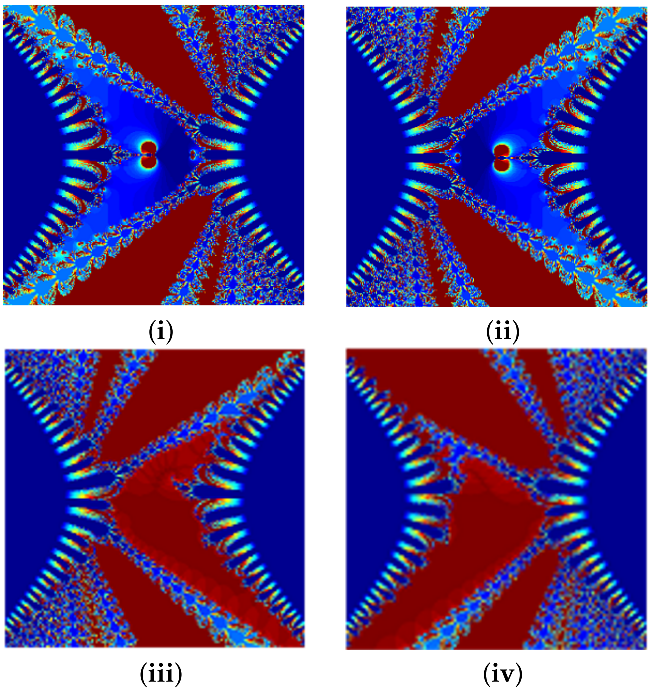


Figure 2. Effect of parameters a and c on the Julia set in MO.

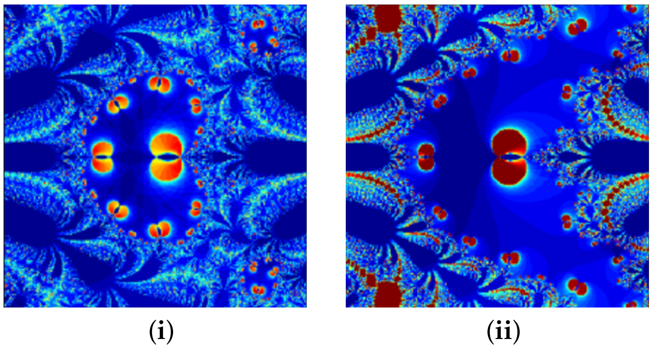


Figure 3. Effect of decrease in the value of parameter α on the Julia set in MO.

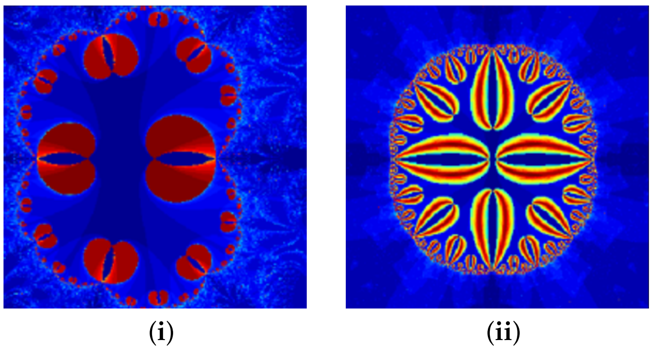


Figure 4. Effect of increase in the value of parameter α on the Julia set in MO.

The J-sets obtained by fixing all the parameters except parameter c are given in Figure 5. As the value of parameter c increases, the basic shape and size remain the same, but the number of points that escape to infinity decreases. Also, a significant change in the shape and color of internal structure is observed in Figure 5i–iv. The parameters used in Figure 5 are given in Table 4:

Table 4. Parameters for different values of c for Julia sets in MO.

	a	b	c	α	**Area**	n
(i)	16	-9	10.75	0.0487654321	$[-10, 10] \times [-10, 10]$	2
(ii)	16	-9	11.75	0.0487654321	$[-10, 10] \times [-10, 10]$	2
(iii)	16	-9	12.75	0.0487654321	$[-10, 10] \times [-10, 10]$	2
(iv)	16	-9	13.75	0.0487654321	$[-10, 10] \times [-10, 10]$	2

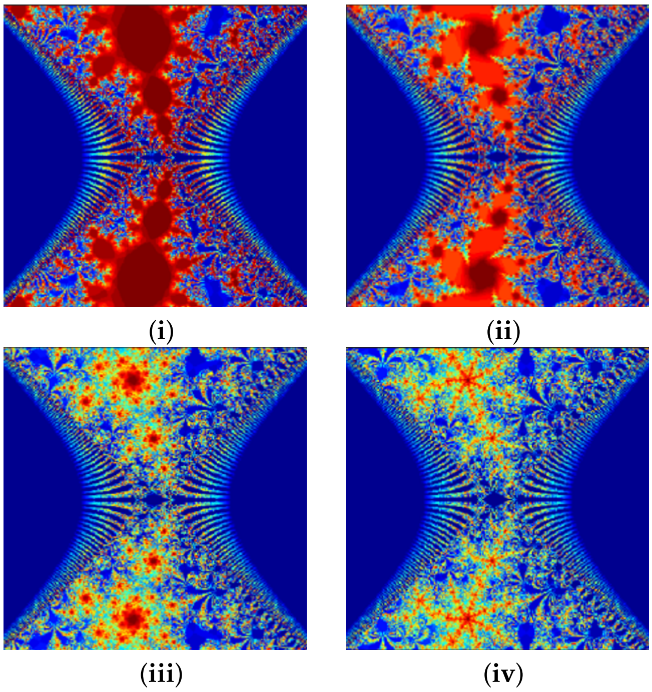


Figure 5. Effect of increase in the value of parameter c on the Julia set in MO.

The dynamics of slight variation in parameter a can be seen in Figure 6. The parameters used in Figure 6 are given in Table 5.

Table 5. Parameters for different values of a for Julia sets in MO.

	a	b	c	α	**Area**	n
(i)	1.000001	1.8	-1	0.3	$[-1.65, 0.35] \times [-1, 1]$	2
(ii)	1	1.8	-1	0.3	$[-1.65, 0.35] \times [-1, 1]$	2

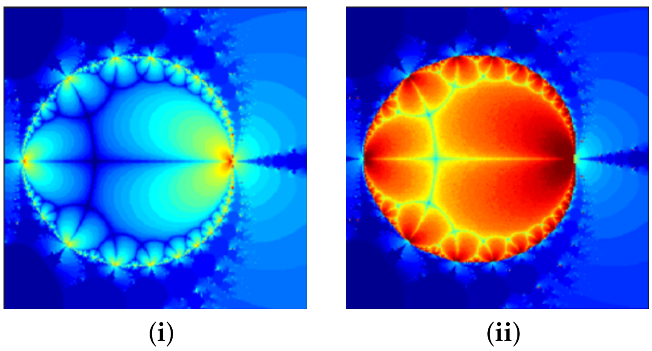


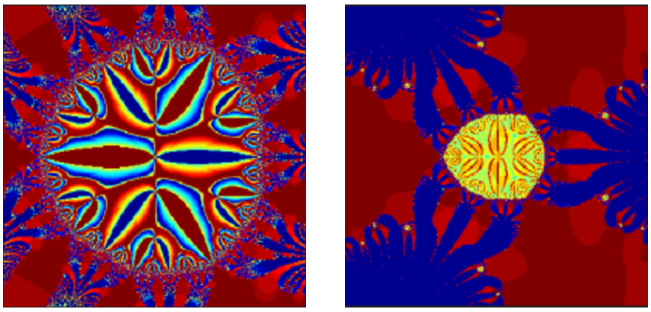
Figure 6. Effect of a slight change in the value of parameter a on the Julia set in MO.

The J-sets for change in parameters b and α are in Figure 7. Clearly, the variation in parameters b and α gives significant variation in the dynamics of the exponential function.

The parameters used in Figure 7 are given in Table 6.

Table 6. Parameters for different values of b and α for Julia sets in MO.

	a	b	c	α	Area	n
(i)	3	-0.0111	-3	0.89999	$[-0.8, 0.8] \times [-0.8, 0.8]$	3
(ii)	3	-0.3	-3	0.9	$[-2, 2] \times [-2, 2]$	3



(i) **(ii)**

Figure 7. Effect of increase in value of parameter b on the Julia set in MO.

The next illustration shows the Julia sets when the parameters a, b, and c are complex numbers. It is observed that conjugate values of a and c (simultaneously) give mirror images (see, Figure 8(ii,iv)) while the conjugate value of a as well as c alone gives a significant change in color and shape (see, Figure 8(i,ii) and (i,iv)). Furthermore, the conjugate values of b and c simultaneously give a significant change in shape and color ((see, Figure 8(ii,iii)). On the other hand, the conjugate value of b alone, which is the coefficient of z, does not give significant change ((see, Figure 8(i,iii)). However, the area occupied by all the J-sets is the same. The parameters used in Figure 8 are given in Table 7.

Table 7. Parameters for different complex values of a, b, and c for Julia sets in MO.

	a	b	c	α	Area	n
(i)	$-1.725 - 1.725i$	$-0.025 + 0.025i$	$-1.725 + 1.725i$	0.165	$[-2, 2] \times [-2, 2]$	3
(ii)	$-1.725 - 1.725i$	$-0.025 + 0.025i$	$-1.725 - 1.725i$	0.165	$[-2, 2] \times [-2, 2]$	3
(iii)	$-1.725 - 1.725i$	$-0.025 - 0.025i$	$-1.725 + 1.725i$	0.165	$[-2, 2] \times [-2, 2]$	3
(iv)	$-1.725 + 1.725i$	$-0.025 + 0.025i$	$-1.725 + 1.725i$	0.165	$[-2, 2] \times [-2, 2]$	3

The J-sets for change in parameter b (real values) are in Figure 9. The geometry of the J-sets in Figure 9 is very distinct and can be used in the fabric industry as its resemblance is with Kashmiri embroidery found in Kashmir, located in the northern part of the Indian subcontinent. The parameters used in Figure 9 are given in Table 8:

Table 8. Parameters for different values of b and α for Julia sets in MO.

	a	b	c	α	Area	n
(i)	3	-0.01	-3	0.99	$[-1, 1] \times [-1, 1]$	5
(ii)	3	-0.0111	-3	0.9999	$[-1, 1] \times [-1, 1]$	5
(iii)	3	-0.02	-3	0.99	$[-1, 1] \times [-1, 1]$	5
(iv)	3	-0.05	-3	0.99	$[-1, 1] \times [-1, 1]$	5

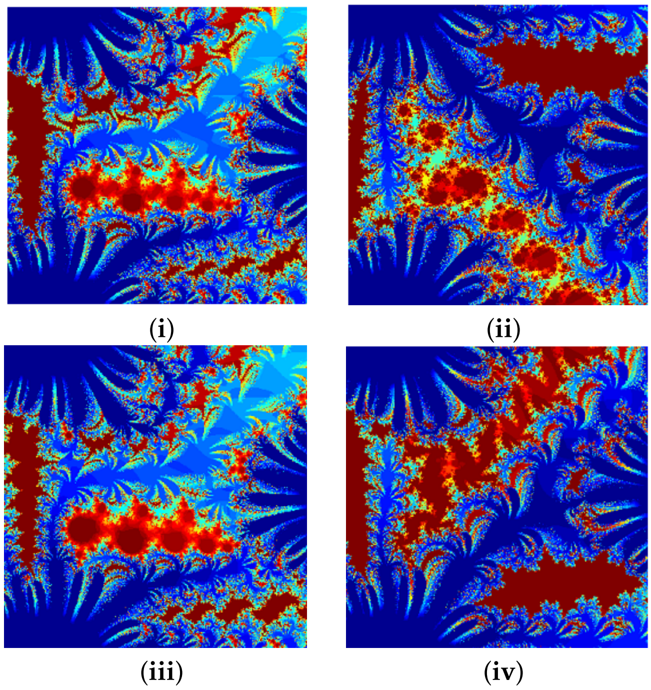


Figure 8. Effect of conjugates of complex parameters a, b, and c on the Julia set in MO.

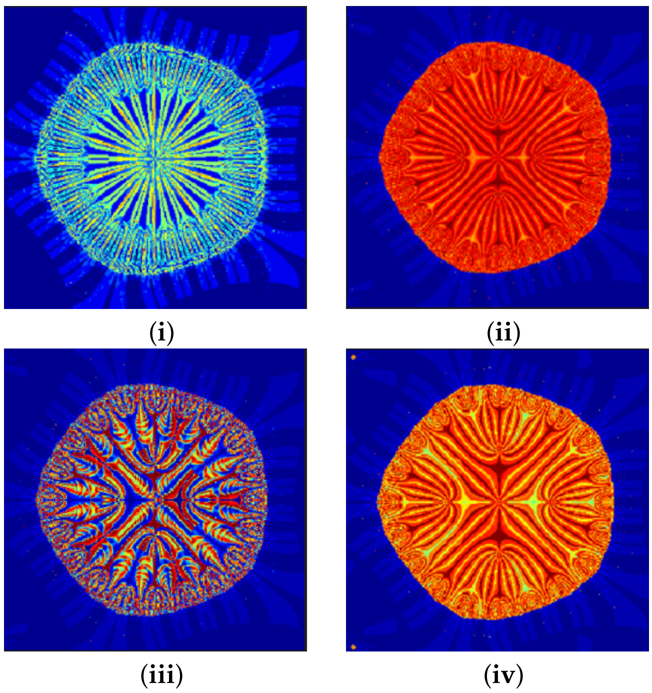


Figure 9. Effect of decrease in the value of parameter b on the Julia set in MO.

Now we generate Julia sets by taking complex values of the parameters b. An increase in the absolute value of b adds beauty to the J-sets and gives resemblance to Kashmiri embroidery. The parameters used in Figure 10 are given in Table 9:

Table 9. Parameters for different complex values of b for Julia sets in MO.

	a	b	c	α	Area	n
(i)	3	$-3i$	-3	0.9	$[-1,1] \times [-1,1]$	7
(ii)	3	$0.7 - 0.3i$	-3	0.9	$[-1,1] \times [-1,1]$	7
(iii)	3	$1 - 0.3i$	-3	0.9	$[-1,1] \times [-1,1]$	7

(i) **(ii)** **(iii)**

Figure 10. Effect of change in the value of parameter b on the Julia set in MO.

Remark 1. *For $a = 0$ and $b, c \neq 0$, the function becomes linear and the conditions of the Theorem 1 fail. Therefore, we do not obtain any J-set.*

5. Bifurcation Analysis

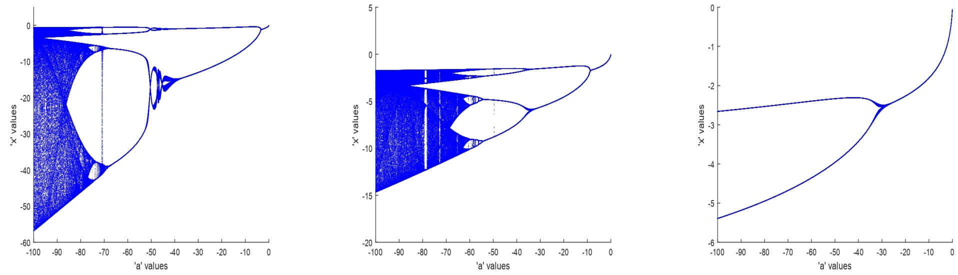
In this section, we present a few bifurcation diagrams to visualize graphically the dynamical behavior of $f(z) = ae^{z^n} + bz + c$, where $a, b, c \in \mathbb{C}$ and $n \in \mathbb{N}$, for various parameter values of a by keeping other parameters fixed and using the Mann iterative scheme. We use bifurcation diagrams to check the qualitative change in global dynamics of the function by making a small change in input parameters and to show the eventual behavior of iterates, such as convergence, periodicity, or unpredictability. The occurrence of period-doubling in the real dynamics of a function leads to chaos.

The bifurcation diagrams for the exponential family are shown in Figures 11–13. We fix the values of parameters b and c while varying parameter a. It can be seen from the bifurcation diagrams that as α decreases from 1 to 0.6 in the steps of 0.2 for $n = 1$ and the values of a as in Table 10, there is a shift in the bifurcation point as well as the period-doubling points (see, Figure 11), and the number of times the attractor undergoes bifurcation decreases by 1. Whereas in Figures 12 and 13, as the value of α decreases from 1 to 0.6 in the steps of 0.2, for $n = 3$ and $n = 5$ and the values of a as in Tables 11 and 12, respectively, the number of times the attractor undergoes bifurcation increases by 1. The period-doubling bifurcation is shown by the forking of curves as seen in Figures 11–13. On the other hand, white regions indicate the presence of nonchaotic windows in the bifurcation diagrams. After the third bifurcation, the numerous curves in the figures scaleup intensively and unite together to generate an almost solid blue area, and this behavior is indicative of the onset of chaos because of an infinite series of period-doubling bifurcations. For large values of odd n the shaded blue region vanishes, but the number of bifurcations remains the same for any particular set of parameters.

The parameters used in Figure 11 are given in Table 10 for $n = 1$:

Table 10. Parameters for bifurcation diagrams of $f(z)$ for $n = 1$.

	a	z	b	c	α
(i)	$[-100, 0]$	$[-60, 0]$	0.06	-0.06	1
(ii)	$[-100, 0]$	$[-20, 5]$	0.06	-0.06	0.8
(iii)	$[-100, 0]$	$[-6, 0]$	0.06	-0.06	0.6



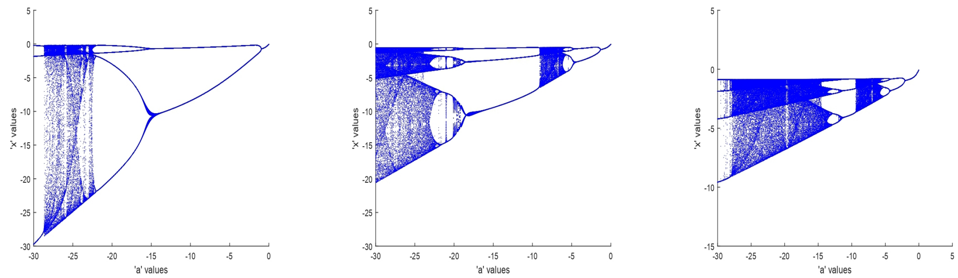
(i) (ii) (iii)

Figure 11. Bifurcation diagrams of $f(z) = ae^{z^n} + bz + c$ for $n = 1$.

The parameters used in Figure 12 are given in Table 11 for $n = 3$:

Table 11. Parameters for bifurcation diagrams of $f(z)$ for $n = 3$.

	a	*z*	*b*	*c*	*α*
(i)	[−30, 0]	[−30, 5]	0.06	−0.06	1
(ii)	[−30, 0]	[−30, 5]	0.06	−0.06	0.8
(iii)	[−30, 0]	[−15, 5]	0.06	−0.06	0.6



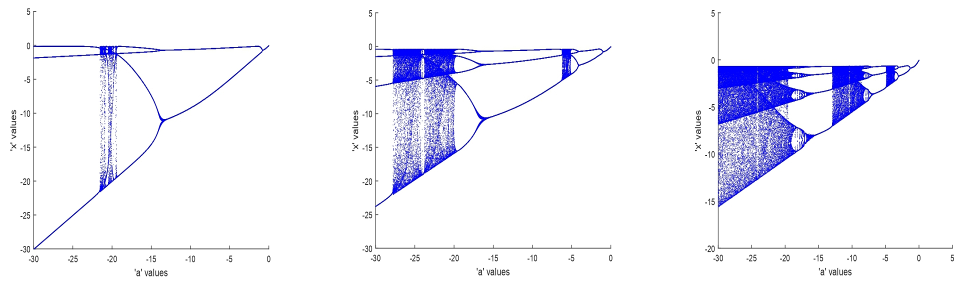
(i) (ii) (iii)

Figure 12. Bifurcation diagrams of $f(z) = ae^{z^n} + bz + c$ for $n = 3$.

The parameters used in Figure 13 are given in Table 12 for $n = 5$:

Table 12. Parameters for bifurcation diagrams of $f(z)$ for $n = 5$.

	a	*z*	*b*	*c*	*α*
(i)	[−30, 0]	[−30, 5]	0.06	−0.06	1
(ii)	[−30, 0]	[−30, 5]	0.06	−0.06	0.8
(iii)	[−30, 0]	[−20, 5]	0.06	−0.06	0.6



(i) (ii) (iii)

Figure 13. Bifurcation diagrams of $f(z) = ae^{z^n} + bz + c$ for $n = 5$.

Remark 2. *For even values of n, we do not obtain bifurcation points or period-doubling points (therefore, we do not obtain the graph). Only for odd values of n do we obtain bifurcation points and period-doubling points for $a < 0$.*

6. Beauty of Julia Sets

Using the escape radius derived in the main results, color map, and applying the escape time algorithm, we have obtained beautiful Julia sets as fractals. Some of these fractals have a resemblance with Kashmiri embroidery (see, Figure 14) found in Kashmir located in the northern part of the Indian subcontinent, which may be utilized in the fabric industry.

Figure 14. The beauty of Julia sets: Resemblance with Kashmiri embroidery, which is useful in fabric industry.

7. Conclusions

In this paper, we have derived the escape radius for the introduced entire transcendental function using the Mann iterative process. We have studied the Julia sets obtained for different values of the parameters a, b, c, n, and α. The fractals thus obtained have a resemblance with Kashmiri embroidery, which may be utilized in the fabric industry. It has been noticed that for odd values of n, the orbit converges rapidly to the fixed points. The Julia sets obtained using the developed algorithm give a variety of novel fractal objects with high resolution compared to the algorithm discussed in [27]. Towards the end, we have performed the bifurcation analysis to demonstrate how a small change in an input parameter can lead to a qualitative change in the behavior of the system.

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References

1. Julia, G. Sur l'itération des functions rationnelles. *J. Math. Pures Appl.* **1918**, *8*, 47–245.
2. Mandelbrot, B.B. *The Fractal Geometry of Nature*; Freeman: New York, NY, USA, 1983; Volume 8, 406p.
3. Captur, G.; Karperien, A.L.; Hughes, A.D.; Francis, D.P.; Moon, J.C. The fractal heart—Embracing mathematics in the cardiology clinic. *Nat. Rev. Cardiol.* **2017**, *14*, 56–64. [CrossRef] [PubMed]
4. May, A.D. *Traffic Flow Fundamentals*; Prentice Hall: Englewood Cliffs, NJ, USA, 1990.
5. Tessier, Y.; Lovejoy, S.; Hubert, P.; Schertzer, D.; Pecknold, S. Multifractal analysis and modeling of rainfall and river flows and scaling, causal transfer functions. *J. Geophys. Res.* **1996**, *101*, 26427–26440. [CrossRef]
6. Devaney, R.L. *A First Course in Chaotic Dynamical Systems: Theory and Experiment*; Addison-Wesley, Reading: Boston, MA, USA, 1992.
7. Malishevsky, A. Applications of fractal analysis in science, technology, and art: A case study on geography of Ukraine. In Proceedings of the IEEE 2nd international conference on system analysis & intelligent computing (SAIC), Kyiv, Ukraine, 5–9 October 2020; pp. 1–6.
8. Antal, S.; Tomar, A.; Parjapati, D.J.; Sajid, M. Fractals as Julia sets of complex sine function via fixed point iterations. *Fractal Fract.* **2021**, *5*, 272. [CrossRef]
9. Antal, S.; Tomar, A.; Prajapati, D.J.; Sajid, M. Variants of Julia and Mandelbrot sets as fractals via Jungck-Ishikawa fixed point iteration system with s-convexity. *AIMS Math.* **2022**, *7*, 10939–10957. [CrossRef]
10. Kang, S.M.; Rafiq, A.; Tanveer, M.; Ali, F.; Kwun, Y.C. Julia and Mandelbrot sets in Jungck Mann and Jungck Ishikawa orbits. *Mitt. Klosterneubg.* **2015**, *65*, 284–312.
11. Li, X.; Tanveer, M.; Abbas, M.; Ahmad, M.; Kwun, Y.C.; Liu, J. Fixed point results for fractal generation in extended Jungck–SP orbit. *IEEE Access* **2019**, *7*, 160472–160481. [CrossRef]
12. Nazeer, W.; Kang, S.M.; Tanveer, M.; Shahid, A.A. Fixed point results in the generation of Julia and Mandelbrot sets. *J. Inequal. Appl.* **2015**, *1*, 1–6. [CrossRef]
13. Negi, A.; Rani, M.; Chugh, R. Julia sets and Mandelbrot sets in Noor orbit. *Appl. Math. Comput.* **2014**, *228*, 615–631.
14. Özgür, N.; Antal, S.; Tomar, A. Julia and Mandelbrot sets of transcendental function via Fibonacci-Mann iteration. *J. Funct. Spaces* **2022**, *2022*, 2592573. [CrossRef]
15. Rani, M. Superior antifractals. *IEEE Proc. ICCAE* **2010**, *1*, 798–802.
16. Rani, M.; Agarwal, R. Effect of stochastic noise on superior Julia sets. *J. Math. Imaging Vis.* **2010**, *36*, 63–68. [CrossRef]
17. Rani, M.; Kumar, V. Superior Julia set. *J. Korea Soc. Math. Educ. Ser. D Res. Math. Educ.* **2004**, *8*, 261–277.
18. Rani, M.; Negi, A. New Julia sets for complex Carotid-Kundalini function. *Chaos Solitons Fractals* **2008**, *36*, 226–236. [CrossRef]
19. Tomar, A.; Prajapati, D.J.; Antal, S.; Rawat, S. Variants of Mandelbrot and Julia fractals for higher-order complex polynomials. *Math. Meth. Appl. Sci.* **2022**, 1–13. [CrossRef]
20. Liu, X.; Zhu, Z.; Wang, G.; Zhu, W. Composed accelerated escape time algorithm to construct the general Mandelbrot set. *Fractals* **2001**, *9*, 149–153. [CrossRef]
21. Yadav, A.; Rani, M. Alternate superior Julia sets. *Chaos Solitons Fractals* **2015**, *73*, 1–9. [CrossRef]
22. Shahid, A.A.; Nazeer, W.; Gdawiec, K. The Picard-Mann iteration with s-convexity in the generation of Mandelbrot and Julia sets. *Monatsh. Math.* **2021**, *195*, 565–584. [CrossRef]
23. Sajid, M.; Kapoor, G.P. Dynamics of a family of transcendental meromorphic functions having rational schwarzian derivative. *J. Math. Anal. Appl.* **2007**, *326*, 1356–1369. [CrossRef]
24. Nayak, T.; Prasad, M.G.P. Julia sets of Joukowski-Exponential maps. *Complex Anal. Oper. Theory* **2014**, *8*, 1061–1076. [CrossRef]
25. Misiurewicz, M. On iterates of e^z. *Ergod. Theory Dyn. Syst.* **1981**, *1*, 103–106. [CrossRef]
26. Devaney, R.L. Complex exponential dynamics. In *Handbook of Dynamical Systems*; Broer, H.W., Hasselblatt, T.B., Eds.; North Holland: Amsterdam, The Netherlands, 2010; pp. 125–223.
27. Devaney, R.L.; Krych, M. Dynamics of exp (z). *Ergod. Theory Dyn. Syst.* **1984**, *4*, 35–52. [CrossRef]
28. Devaney, R.L. Cantor bouquets, explosions, and Knaster continua: Dynamic of complex exponentials. *Publ. Mat.* **1999**, *43*, 27–54. [CrossRef]
29. Baker, N.; Rippon, P.J. Iteration of exponential functions. *Ann. Acad. Sci. Fenn. A* **1984**, *1*, 49–77. [CrossRef]
30. Romera, M.; Pastor, G.; Alvarez, G.; Montoya, F. Growth in complex exponential dynamics. *Comput. Graph.* **2000**, *24*, 115–131. [CrossRef]
31. Prasad, B.; Katiyar, K. Dynamics of Julia Sets for complex exponential functions. In *International Conference on Mathematical Modelling and Scientific Computation*; Springer: Berlin/Heidelberg, Germany, 2012; pp. 185–192.
32. Sajid, M. Chaotic behaviour and bifurcation in real dynamics of two-parameter family of functions including logarithmic map. *Abstr. Appl. Anal.* **2020**, *2020*, 7917184. [CrossRef]
33. Sajid, M. On some advances in dynamics of one variable complex functions. *Int. J. Appl. Math.* **2021**, *34*, 1071–1091. [CrossRef]
34. Barrallo, J.; Jones, D.M. Coloring algorithms for dynamical systems in the complex plane. *Vis. Math.* **1999**, *1*.
35. Mann, W.R. Mean value methods in iteration. *Proc. Amer. Math. Soc.* **1953**, *4*, 506–510. [CrossRef]
36. Ashlock, D.; Jamieson, B. Evolutionary exploration of complex fractals. In *Design by Evolution*; natural computing series; Hingston, P., Barone, L., Michalewicz, Z., Eds.; Springer: Berlin/Heidelberg, Germany, 2008; pp. 121–143.
37. Husain, A.; Nanda, M.N.; Chowdary, M.S.; Sajid, M. Fractals: An eclectic survey, Part-1. *Fractal Fract.* **2022**, *6*, 89. [CrossRef]

38. Lakhtakia, A. Julia sets of switched processes. *Comput. Graph.* **1991**, *15*, 597–599. [CrossRef]
39. Lakhtakia, A.; Varadan, V.; Messier, R.; Varadan, V. On the symmetries of the Julia sets for the process $z \to z^p + c$. *J. Phys. A Math. Gen.* **1987**, *20*, 3533–3535. [CrossRef]