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On Sliding Mode Control for Singular Fractional-Order Systems with Matched External Disturbances

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Abstract: In this paper, we investigate the problem of sliding mode control for singular fractional-order systems that have matched uncertainties. We design an innovative integral sliding mode function and controller based on the normalizable condition. A strict linear matrix inequality-based sufficient condition is obtained for the system's stability. The normalizable condition is eliminated by updating and developing the control method, and a sufficient and necessary condition is developed for the admissibility of the system. Lastly, verification of our method's effectiveness is numerically conducted in two instances.

Keywords: singular fractional-order systems; sliding mode control; normalization; stability; admissibility

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1. Introduction

Singular systems (SS) accurately describe a class of dynamic models with a broadly practical background, and are widely used in mechanical, power, and biological systems [1–4]. Unlike typical systems, continuous SS should focus on stability and regularity and be impulse free. In recent decades, SS studies have obtained many significant results. In [5], the authors presented the sufficient and necessary condition (SANC) of regular, impulse-free, and stable SS. Several robust stabilization methods for SS were discussed in [6]. A sliding mode control (SMC) method for SS was studied using output information in [7].

Recently, singular fractional-order systems (SFOS), a type of SS with long memory characteristics, have attracted much attention. To date, several valuable conclusions have been reported on the theory and practical application of SFOS. In [8], they authors presented an SANC for SFOS admissibility analysis through strict linear matrix inequality (LMI). In [9], the authors proposed a new SANC for regular SFOS admissibility to reduce the number of decision matrix variables. The authors of [10] discussed three SANCs of admissibility in terms of LMI, which need not verify the assumption of regularity. The sufficient conditions for normalization and asymptotic stability were provided in [11] based on the output feedback. In [12], the authors discussed the stabilization problem of uncertain SFOS through the matrix singular value decomposition and the state and static output feedback without normalization; however, this method requires additional constraints in controller design. In [13], this constraint was removed by presenting a new static output feedback controller design method. Because all states might not be available in practical applications, the authors of [14] designed an observer-based controller for SFOS with fractional order $1 \leq \alpha < 2$ and presented an SANC for admissibility. In [15], the authors studied the observer-based output feedback stabilization problem for a case where $0 < \alpha < 2$. In [16], the authors proposed a sufficient condition for SFOS to describe a circuit system for practical application; however, their system exhibited various external disturbances. Therefore, ways of dealing with external disturbances must be considered.

As a popular control method, an SMC can effectively resist external disturbances [17–19]. In [20], the authors studied the SMC problem for impulsive reaction–diffusion systems, whereas the research on SMCs for SFOS is limited. In [21], the authors discussed a sliding mode fault tolerant control for two T-S fuzzy SFOSs with mismatched uncertainties. The authors of [22] considered the adaptive SMC of SFOS with unmatched uncertainty. The sliding mode equation of their designed SMS has a normal form. The sufficient condition of system stability is provided in the form of LMIs, although the calculation is complicated. In [23], the authors proposed a necessary and sufficient condition for normalization and asymptotic stability of normalized systems by designing linear sliding mode functions (SMF) and SMCs. The authors of [24] presented an integral sliding mode surface (SMS) and a super-twisting SMC for SFOS with matched external disturbances, and provided a sufficient condition for asymptotic stability. Although the method in [24] can effectively reduce chattering, it must satisfy several assumptions, and cannot be applied to multiple-input systems. The SMC for multiple-input SFOS with matched disturbances remains an open problem.

Therefore, we consider an SMC problem for SFOS with matched external disturbances. The present study focuses on the following:

(1) A novel integral SMF and SMC with derivative terms are proposed for SFOS stabilization. The difference with [23] is that the system state is in the SMS from the initial moment, reducing the time required to reach the SMS. A sufficient condition for asymptotic stability is provided for normalization.
(2) An innovative integral SMF and SMC for a class of SFOS that did not satisfy the normalization criterion and obtained an SANC for the admissibility of such closed-loop systems. Compared with [24], the results are more general.
(3) We numerically verified the effectiveness of our scheme in two separate instances.

The rest of this manuscript includes Section 2, which introduces the problem formulation; Sections 3 and 4, which present the primary results and numerical examples, respectively; and Section 5, which concludes the paper.

Notation: $det(\cdot)$ denotes the determinant of a matrix in the sequel, I denotes the corresponding identity matrix, and $\|\cdot\|$ denotes the matrix norm; $rank(\cdot)$ is the ranking of the matrix, $sym(M)$ denotes $M+M^T$, and ${}_{t_0}\mathscr{D}_t^{\alpha}$ denotes the fractional calculus operator, where t_0 and t denote the operational boundaries and $\alpha \in R$, respectively. The operator $\alpha = 0$ is the identity operator. ${}_{t_0}^{C}\mathscr{D}_t^{\alpha}(\cdot)$ represents Caputo fractional-order derivatives with $\alpha > 0$, and ${}_{t_0}^{RL}\mathscr{D}_t^{\alpha}$ represents Riemann–Liouville (RL) fractional-order derivatives ($\alpha > 0$) and integrals ($\alpha < 0$). The fractional-order positive definite matrix set $\mathbb{P}_{\alpha}^{n\times n}$ can be described by

$$\mathbb{P}_{\alpha}^{n\times n} = \{sin(\frac{\alpha\pi}{2})X + cos(\frac{\alpha\pi}{2})Y : X, Y \in \mathbb{R}^{n\times n}, \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0\}.$$

2. Preliminaries and Formulation of the Problem

We consider SFOS with Caputo fractional-order derivatives and matched disturbance, provided by

$$E\,{}_{t_0}^{C}\mathscr{D}_t^{\alpha}x(t) = Ax(t) + B(u(t) + d(t)), \tag{1}$$

where the system matrices E, A, and B are constant, $A \in \mathbb{R}^{n\times n}$, B is the full column rank, $B \in \mathbb{R}^{n\times m}$, $u(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^n$, and $x(t) \in \mathbb{R}^n$ denote the input, disturbance input, and pseudosemistate vectors, respectively, and α denotes the system's fractional order. In practical applications, most singular E meet $rank(E) = r < n$. The fractional-order derivative of this paper is chosen as a Caputo derivative, and satisfies $0 < \alpha < 1$. When $u(t) = d(t) = 0$, system (1) becomes autonomous, and is denoted as (E, A, α).

Assumption 1. *Suppose* rank$(E) = r$*; then, there are two nonsingular matrices,* $\mathcal{M}$ *and* $\mathcal{N}$*, satisfying*

$$\mathcal{M}E\mathcal{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{M}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{2}$$

where $B_1 \in \mathbb{R}^{r\times m}$ *and* $B_2 \in \mathbb{R}^{(n-r)\times m}$.

Assumption 2. *Suppose* $rank(E, B) = n$.

Lemma 1 provides an SANC for the admissibility of a system (E, A, α).

Lemma 1 ([10])**.** *The system* (E, A, α) *is admissible if and only if there are matrices* $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}^{r\times r}$, $\mathcal{X}_3 \in \mathbb{R}^{(n-r)\times r}$, *and* $\mathcal{X}_4 \in \mathbb{R}^{(n-r)\times(n-r)}$ *satisfying*

$$\begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\mathcal{X}_2 & \mathcal{X}_1 \end{bmatrix} > 0, \tag{3}$$

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 & 0 \\ \mathcal{X}_3 & \mathcal{X}_4 \end{bmatrix}, \tag{4}$$

$$\mathcal{Y} = \begin{bmatrix} \mathcal{X}_2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

$$\mathrm{sym}(\mathcal{M}A\mathcal{N}(a\mathcal{X} - b\mathcal{Y})) < 0, \tag{6}$$

where $r = rank(E)$, $\mathcal{M}$ *and* $\mathcal{N}$ *satisfy Assumption 1, and* a *and* b *are the sine and cosine of* $\frac{\pi}{2}\alpha$*, respectively.*

Lemma 2 ([25])**.** *For the fractional-order derivative and integral, the following conclusion holds:*

$${}^{RL}_{t_0}\mathscr{D}^{\alpha}_{t}(f(t) - f(t_0)) = {}^{C}_{t_0}\mathscr{D}^{\alpha}_{t} f(t),$$

where $\alpha, \beta > 0$, $f(t)$ *is the piecewise continuous function.*

Next, we discuss the stabilization problem of a class of SFOS with matched disturbance using the SMC strategy.

3. Synthesis of SMC of SFOS with Matched Uncertainties

3.1. Normalized-Based Integral SMC Method

The controller for system (1) is designated as

$$u = -K_1 {}^{C}_{t_0}\mathscr{D}^{\alpha}_{t} x(t) + v(t), \tag{7}$$

where K_1 denotes the gain matrix designed for normalizing system (1) and $v(t)$ denotes the new or virtual input asymptotically stabilizing the normalized system. Using controller (7), we have

$$(E + BK_1) {}^{C}_{t_0}\mathscr{D}^{\alpha}_{t} x(t) = Ax(t) + B(v(t) + d(t)). \tag{8}$$

If $det(E + BK_1) \neq 0$, (8) can be rewritten as

$${}^{C}_{t_0}\mathscr{D}^{\alpha}_{t} x(t) = (E + BK_1)^{-1} Ax(t) + (E + BK_1)^{-1} B(v(t) + d(t)). \tag{9}$$

Thus, system (1) is normalized. The next lemma presents the normalization criterion.

Lemma 3. *Per [23], system (1) can be normalized if and only if* $rank(E, B) = n$.

When system (1) satisfies Assumption 2, there is always a matrix K_1 meeting $det(E + BK_1) \neq 0$. Let $\check{A} = (E + BK_1)^{-1}A$ and $\check{B} = (E + BK_1)^{-1}B$; then, (9) becomes

$${}^{C}_{t_0}\mathscr{D}^{\alpha}_{t} x(t) = \check{A}x(t) + \check{B}(v(t) + d(t)). \tag{10}$$

Remark 1. *For the design of matrix* K_1*, we can choose* $K_1 = [0, B_2^T]\mathcal{N}^{-1}$*, as shown in [23], where* B_2 *and* $\mathcal{N}$ *satisfy Assumption 1. We can additionally obtain matrix* K_1 *by solving the LMI in [26].*

The selected integral SMF is

$$\sigma(t) = G{}_{t_0}^{RL}\mathscr{D}_t^{\alpha-1}(x(t) - x(t_0)) - G\int_{t_0}^{t}(\check{A} - \check{B}K_2)x(\tau)d\tau, \tag{11}$$

where matrices $G \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times n}$ are designed later.

Remark 2. *Unlike the linear SMF proposed after normalization, as in [23], an integral SMF $\sigma(t)$ is proposed in the above discussion. In this way, the integral term in the SMF can provide a greater degree of freedom, and the parameters can be provided analytically.*

Using the derivation of $\sigma(t)$,

$$\begin{aligned}\dot{\sigma}(t) &= G{}_{t_0}^{C}\mathscr{D}_t^{\alpha}x(t) - G(\check{A} - \check{B}K_2)x(t) \\ &= G(\check{A}x(t) + \check{B}(v(t) + d(t))) - G(\check{A} - \check{B}K_2)x(t) \\ &= G\check{B}(v(t) + d(t) + K_2x(t)).\end{aligned}$$

If matrix $G\check{B}$ is nonsingular, from $\dot{\sigma}(t) = 0$ we have an equivalent controller,

$$v_{eq}(t) = -K_2x(t) - d(t). \tag{12}$$

The sliding mode motion equation of (10) under $v_{eq}(t)$ is described as

$${}_{t_0}^{C}\mathscr{D}_t^{\alpha}x(t) = (\check{A} - \check{B}K_2)x(t). \tag{13}$$

We provide a sufficient condition for the asymptotic stability of system (13), as follows:

Theorem 1. *For $P \in \mathbb{P}_{\alpha}^{n\times n}$, if there is an intermediate matrix K_2 meeting*

$$sym(\check{A}P - \check{B}K_2P) < 0, \tag{14}$$

then system (13) is asymptotically stable. For $K_2 = -QP^{-1}$, we can rewrite inequality (14) as

$$sym(\check{A}P + \check{B}Q) < 0, \tag{15}$$

where $Q \in \mathbb{R}^{m\times n}$.

The proof is similar to that in [23], and thus the full proof process is omitted here.

Considering the reachability, we provide a switching control law, $v_{sw}(t) = -k_1\text{sgn}(\sigma(t))$, where $\text{sgn}(\cdot)$ and k_1 denote the sign vector function and positive real number, respectively. The selected SMC can be written as

$$v(t) = v_{eq}(t) + v_{sw}(t). \tag{16}$$

Theorem 2. *For system (10), if there is an SMC (16), we can drive the state trajectories onto SMS (11) in finite time, where $G = \check{B}^T X$, $X \in \mathbb{R}^{n\times n}$ denotes an arbitrary positive definite matrix.*

Proof. With the Lyapunov function

$$V(t) = \frac{1}{2}\sigma^T(t)(G\check{B})^{-1}\sigma(t),$$

we set the derivative of $V(t)$ along with system (13):

$$\begin{aligned}\dot{V}(t) &= \sigma^T(t)(G\check{B})^{-1}\dot{\sigma}(t) \\ &= \sigma^T(t)(G\check{B})^{-1}G\check{B}(v(t)+d(t)+Kx(t)) \\ &= \sigma^T(t)(-k_1\mathrm{sgn}(\sigma(t))) \\ &= -k_1\|\sigma(t)\|_1.\end{aligned}$$

According to $\|\sigma(t)\|_1 \geqslant \|\sigma(t)\|_2$, we obtain

$$\dot{V}(t) \leqslant -k_1\|\sigma(t)\|_2. \tag{17}$$

Because $G\check{B}$ is a positive definite matrix, the following inequality holds:

$$V(t) \leqslant \frac{1}{2}\lambda_{max}(G\check{B})^{-1}\|\sigma(t)\|_2^2,$$

where $\lambda_{max}(\cdot)$ denotes the eigenvalue of the matrix maximum. The following conclusions can be drawn from the above formula:

$$\|\sigma(t)\|_2 \geqslant \sqrt{\frac{2V(t)}{\lambda_{max}(G\check{B})^{-1}}}. \tag{18}$$

Combining (17) and (18), we obtain

$$\dot{V}(t) \leqslant -k_1\sqrt{\frac{2V(t)}{\lambda_{max}(G\check{B})^{-1}}}. \tag{19}$$

Integrating t_0 to t for (19) yields

$$\sqrt{V(t)} - \sqrt{V(t_0)} \leqslant -\frac{k_1}{\sqrt{2\lambda_{max}(G\check{B})^{-1}}}(t-t_0).$$

Readily, we obtain

$$\begin{aligned}t &\leqslant t_0 + \frac{\sqrt{2\lambda_{max}(G\check{B})^{-1}}}{k_1}(\sqrt{V(t_0)} - \sqrt{V(t)}) \\ &\leqslant t_0 + \frac{\sqrt{2\lambda_{max}(G\check{B})^{-1}}}{k_1}\sqrt{V(t_0)}.\end{aligned}$$

According to

$$V(t_0) \leqslant \frac{1}{2}\lambda_{max}(G\check{B})^{-1}\|\sigma(t_0)\|_2^2,$$

we finally have

$$t \leqslant t_0 + \frac{\lambda_{max}(G\check{B})^{-1}\|\sigma(t_0)\|_2}{k_1}.$$

Hence, the system state trajectories are driven onto the SMS within finite time. □

Remark 3. *When the SMF has an initial value $\sigma(t_0) = 0$, i.e., the initial system state is on the SMS, the SMC can be realized from the initial moment. The above discussion includes this special case.*

It follows from the above discussion that an algorithm to determine the design matrices can be proposed.

Step 1: Check Assumption 2 and stop if it is not satisfied; otherwise, determine a matrix K_1 such that the $det(E + BK_1) \neq 0$;

Step 2: Check the conditions in Theorem 1 and stop if they are not satisfied; otherwise, solve Ps and Qs to satisfy (15);

Step 3: Output the design matrices $K_2 = -QP^{-1}$ and $G = \check{B}^T X$, where X is an arbitrary positive definite matrix.

From the above discussion, the SMC can effectively solve the stabilization problem of normalized SFOS.

Next, we focus on the SMC problem for SFOS without normalization in order to remove the normalization condition.

3.2. Integral SMC Method

This section presents an integral SMF and SMC for system (1).

The SMF is proposed as follows:

$$\sigma(t) = G E_{t_0}^{RL}\mathscr{D}_t^{\alpha-1}(x(t) - x(t_0)) - G\int_{t_0}^{t}(A - BK)x(\tau)d\tau, \tag{20}$$

where $x(t_0)$ is the initial state of $x(t)$ and matrices $G \in \mathbb{R}^{m\times n}$ and $K \in \mathbb{R}^{m\times n}$ denote the sliding surface parameters to be designed later.

Remark 4. *The system state is on the SMS from the initial moment; therefore, no reaching stage is in motion. Unlike in [24], this SMF can be used in multiple-input systems.*

The derivative of $\sigma(t)$ is derived as follows:

$$\begin{aligned}\dot{\sigma}(t) &= G E_{t_0}^{RL}\mathscr{D}_t^{\alpha}(x(t) - x(t_0)) - G(A - BK)x(t)\\ &= G E_{t_0}^{C}\mathscr{D}_t^{\alpha}x(t) - G(A - BK)x(t)\\ &= G(Ax(t) + B(u(t) + d(t))) - G(A - BK)x(t)\\ &= GB(u(t) + d(t) + Kx(t)).\end{aligned} \tag{21}$$

If matrix GB is nonsingular, the equivalent controller $u_{eq}(t) = -Kx(t) - d(t)$. Combining system (1) with $u_{eq}(t)$ yields the sliding mode motion equation

$$E_{t_0}^{C}\mathscr{D}_t^{\alpha}x(t) = (A - BK)x(t). \tag{22}$$

Here, we have the LMI stabilization criterion for a closed-loop system (22).

Theorem 3. *Let two nonsingular matrices $\mathcal{M}$ and $\mathcal{N}$ satisfy Assumption 1 and the design gain matrix $K = -\mathcal{Z}(a\mathcal{X} - b\mathcal{Y})^{-1}\mathcal{N}^{-1}$. If and only if there exist matrices $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$, and $\mathcal{Z}$ such that (3)–(5) hold and*

$$\mathrm{sym}(\mathcal{M}A\mathcal{N}(a\mathcal{X} - b\mathcal{Y}) + \mathcal{M}B\mathcal{Z}) < 0, \tag{23}$$

then the closed-loop continuous SFOS (22) is admissible.

Proof. Let $\bar{A} = A - BK$. We can rewrite system (22) as

$$E_{t_0}^{C}\mathscr{D}_t^{\alpha}x(t) = \bar{A}x(t). \tag{24}$$

According to Lemma 1, if and only if

$$\mathrm{sym}(\mathcal{M}\bar{A}\mathcal{N}(a\mathcal{X} - b\mathcal{Y})) < 0, \tag{25}$$

system (24) is admissible.

Substituting $\bar{A} = A - BK$ into (25), we obtain,

$$\mathrm{sym}(\mathcal{M}A\mathcal{N}(a\mathcal{X} - b\mathcal{Y}) - \mathcal{M}BK\mathcal{N}(a\mathcal{X} - b\mathcal{Y})) < 0. \tag{26}$$

If we choose $K = -\mathcal{Z}(a\mathcal{X} - b\mathcal{Y})^{-1}N^{-1}$, (23) can be obtained using (26). $\square$

Remark 5. *Theorem 3's conclusion is provided in terms of strict LMI, and can thus be conveniently calculated using MATLAB. When $\alpha = 1$ or $E = I$, Theorem 3 becomes an extension of the Lyapunov stability theory. K can be obtained according to the method in [27].*

Remark 6. *System (1) is discussed in [24], and an integral SMC method is proposed that, however, must satisfy the assumption $rank(E, B) = n$. As there is no need to satisfy this assumption in the above discussion, the conclusion reached here is more general.*

In order to satisfy SMS reachability, we design a switching control law, $u_{sw}(t) = -k_2\mathrm{sgn}(\sigma(t))$, where k_2 is a positive real number. We consider the sliding mode controller as

$$u(t) = u_{eq}(t) + u_{sw}(t). \tag{27}$$

We define the Lyapunov function by

$$V(t) = \frac{1}{2}\sigma^T(t)(GB)^{-1}\sigma(t). \tag{28}$$

For system (1) under SMC (27), we can calculate the derivative of $V(t)$ as

$$\begin{aligned}
\dot{V}(t) &= \sigma^T(t)(GB)^{-1}\dot{\sigma}(t) \\
&= \sigma^T(t)(GB)^{-1}GB(u(t) + d(t) + Kx(t)) \\
&= \sigma^T(t)(-k_2\mathrm{sgn}(\sigma(t))) \\
&= -k_2\|\sigma(t)\|_1.
\end{aligned}$$

The SMS reachability can be obtained as in Theorem 2.

A block diagram of the proposed normalization-based integral SMC method and integral SMC method is shown in Figure 1.

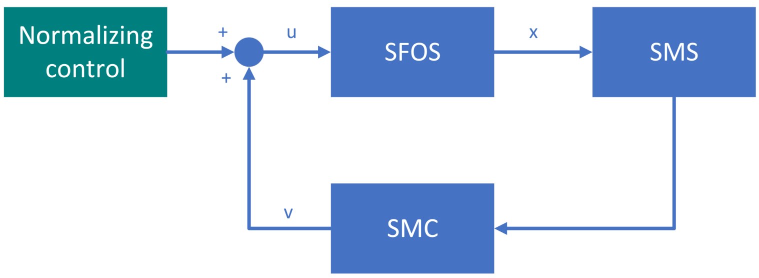

Figure 1. Block diagram of the control mechanism.

4. Numerical Simulations

We now show the usefulness of the proposed control strategy using two separate instances.

Example 1

Consider System (1) with parameters

$$
\begin{aligned}
E &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\
A &= \begin{bmatrix} -1 & 1 & 0 \\ 2 & 2 & -2 \\ 2 & 2 & -4 \end{bmatrix}, \\
B &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \\
d(t) &= 0.8 \sin t,
\end{aligned} \tag{29}
$$

where the fractional-order $\alpha = 0.95$, $\det(s^{\alpha}E - A) = 6s^{1.9.} + 8s^{0.95} - 8 = 0$ is not identity zero and $\text{rank}(E) = 2$ and $\det(sE - A) = 6s^2 + 8s - 8 = 0$ satisfy $\text{rank}(E) = \deg(\det(sE - A)) = 2$. However, $|\arg(\text{spec}(E, A, 0.95))| < 0.475\pi$; then, system (1) with $u = d = 0$ is regular, unstable, and impulse-free. Now, we can easily obtain $m = \text{rank}(B) = 1$. According to Assumption 1, there exist two matrices:

$$
\begin{aligned}
\mathcal{M} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\mathcal{N} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.
\end{aligned}
$$

Using control separation, we can decompose System (1) as

$$
\begin{cases}
{}_{t_0}^{C}\mathscr{D}_t^{0.95}\bar{x}_1(t) = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}\bar{x}_1(t) + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\bar{x}_2(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(u(t) + d(t)) \\
0 = \begin{bmatrix} 2 & -4 \end{bmatrix}\bar{x}_1(t) + 6\bar{x}_2 - u(t) - d(t)
\end{cases}
$$

where $\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \mathcal{N}^{-1}x(t)$, $\bar{x}_1(t) \in \mathbb{R}^{2\times 1}$, $\bar{x}_2(t) \in \mathbb{R}^{1\times 1}$.

When solving the LMI in Theorem 3, we have

$$
\begin{aligned}
\mathcal{X} &= \begin{bmatrix} 0.5072 & -0.0571 & 0 \\ -0.0571 & 0.3707 & 0 \\ -0.3226 & 0.2106 & -0.0946 \end{bmatrix}, \\
\mathcal{Y} &= \begin{bmatrix} 0 & -0.3707 & 0 \\ 0.3707 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathcal{Z} &= \begin{bmatrix} -0.1906 & -0.6158 & 0.0053 \end{bmatrix}.
\end{aligned}
$$

We obtain the gain matrix $K = [0.6995, 1.6873, 0.0560]$.

Then, $\bar{A} = \begin{bmatrix} -3.0000 & 2.0000 & -3.0000 \\ 1.3005 & -3.6873 & 3.9440 \\ 2.6995 & -2.3127 & 6.0560 \end{bmatrix}$, and thus $(E, \bar{A}, \alpha)$ is admissible.

Figures 2 and 3 show the state trajectories of system (29) using different control methods. Figure 2 shows that the integral SMC regulates the SFOS (29) states to zero. Figure 3 is a simulation of System (29) using the normalized-based integral SMC method.

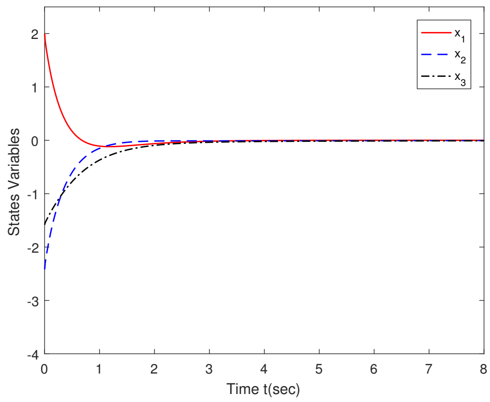

Figure 2. State responses of the integral sliding mode control (SMC).

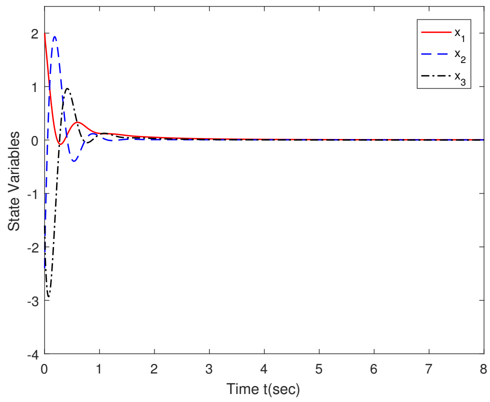

Figure 3. State responses of the normalization-based integral sliding mode control (SMC).

Example 2

Consider System (1) with parameters

$$
\begin{aligned}
E &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
A &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 5 \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \\
d(t) &= \sin(0.5t) + \sin t,
\end{aligned} \tag{30}
$$

where the fractional-order $\alpha = 0.95$, $\det(s^{\alpha}E - A) = -5s^{1.9} + s^{0.95} - 1 = 0$ is not identity zero and $\text{rank}(E) = 2$ and $\det(sE - A) = 6s^2 + 8s - 8 = 0$ satisfy $\text{rank}(E) = \deg(\det(sE - A)) = 2$. However, if $|\arg(\text{spec}(E, A, 0.95))| < 0.475\pi$, then System (1) with $u = d = 0$ is regular, unstable, and impulse-free. We can then easily obtain $rank(E, B) = 2 \neq n = 3$.

Therefore, the normalized-based integral SMC method in Section 3.1 is unfeasible, as is the method proposed in [24]. However, the method proposed in Section 3.2 of this paper can be applied to System (30); in other words, this method is more general. Figure 4 shows the simulation results.

The applicability of the two methods to both examples is shown in Table 1.

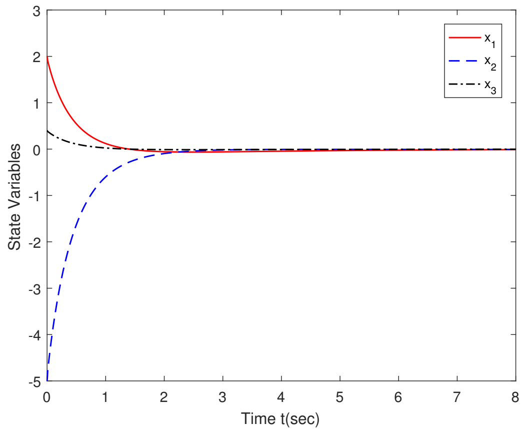

Figure 4. State responses of the integral sliding mode control (SMC).

Table 1. Comparison between Example 1 and Example 2.

	$rank(E\ B) = n$	Normalized-Based Intrgral SMC	Integral SMC
Example 1	✓	✓	✓
Example 2	×	×	✓

5. Conclusions

This paper has examined SMC methods for SFOS with matched disturbance inputs and fractional orders $0 < \alpha < 1$. Based on normalization, we designed an SMC to effectively solve the stabilization problem. An integral SMF and SMC are proposed for systems without normalization, and an SANC for the admissibility of such closed-loop systems is obtained. Several results are provided using LMI, which can be easily applied directly. Our method's effectiveness has been further illustrated using two examples. We intend continue to study the SMC for SFOS with fractional orders ranging from 1 to 2.

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Abbreviations

The following abbreviations are used in this manuscript:

SS	singular systems
SANC	sufficient and necessary conditions
SMC	sliding mode control
SFOSs	singular fractional-order systems
LMI	linear matrix inequality
SMF	sliding mode functions
SMS	sliding mode surface

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