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Article

Generalized Fractional Integral Inequalities for p-Convex Fuzzy Interval-Valued Mappings

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Abstract: The fuzzy order relation ($\succcurlyeq$) and fuzzy inclusion relation ($\supseteq$) are two different relations in fuzzy-interval calculus. Due to the importance of p-convexity, in this article we consider the introduced class of nonconvex fuzzy-interval-valued mappings known as p-convex fuzzy-interval-valued mappings (p-convex f-i-v-ms) through fuzzy order relation. With the support of a fuzzy generalized fractional operator, we establish a relationship between p-convex f-i-v-ms and Hermite–Hadamard ($\mathcal{H}$–$\mathcal{H}$) inequalities. Moreover, some related $\mathcal{H}$–$\mathcal{H}$ inequalities are also derived by using fuzzy generalized fractional operators. Furthermore, we show that our conclusions cover a broad range of new and well-known inequalities for p-convex f-i-v-ms, as well as their variant forms as special instances. The theory proposed in this research is shown, with practical examples that demonstrate its usefulness. These findings and alternative methodologies may pave the way for future research in fuzzy optimization, modeling, and interval-valued mappings (i-v-m).

Keywords: p-convex fuzzy-interval-valued mapping; fuzzy generalized fractional integral operator; Hermite–Hadamard type inequality; Hermite–Hadamard–Fejér type inequality

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1. Introduction

G.W. Leibniz first proposed the concept of fractional derivatives in 1695, and this theory has motivated more and more scholars. The Riemann–Liouville calculus technique, Caputo differential approach, and Grunwald–Letnikov differential approach are the most extensively utilized fractional calculus approaches in engineering application research and basic mathematics research, respectively [1]. Fractional calculus has played an important part in the development of pure and applied mathematics over the last two decades. Because of its applicability in numerous domains such as image processing, signal processing, physics, biology, control theory, computer networking, and fluid dynamics [2,3], it receives considerable attention in continuing research. Recently, investigations have proceeded to generalize current variants via imaginative concepts and innovative fractional calculus approaches. Perhaps the most popular technique among analysts is the use of fractional integral operators. Because of their ability to be studied for the existence and uniqueness of solutions for various classes of differential fractional integral equations and fractional integrals, including integral inequalities, they are highly significant [4]. In 1993, Samko et al. introduced the representation of the extended derivative called the generalized derivative [5]. In 2006, Kilbas et al. proposed a new fractional integral operator that generalizes the integral element of Riemann–Liouville and Hadamard into a single form. When a parameter was specified at different values, it constructed the abovementioned integrals as exceptional cases [6].

Convex sets and convex mappings have been introduced to remarkable varieties of convexities over the years, including harmonic convexity [7], quasi convexity [8], Schur convexity [9,10], strong convexity [11,12], p-convexity [13], fuzzy convexity [14], fuzzy preinvexity [15], generalized convexity [16], p-convexity [17], and so on. The definition of convexity in integral problems is an interesting subject of research. As a result, a large number of equalities and inequalities have been recognized as convex mapping applications by various authors. The Gagliardo–Nirenberg-type inequality [18], Hardy-type inequality [19], Ostrowski-type inequality [20], Olsen-type inequality [21], and the $\mathcal{H}$–$\mathcal{H}$-inequality [22] are all examples of typical outcomes. Many authors have also focused on fractional integral inequalities for single-valued and interval-valued mappings [23–28].

A great deal of research work on fuzzy sets and systems has been dedicated to the development of different fields [29]. Recently, fuzzy interval analysis and fuzzy interval-valued differential equations have been put forward to deal with the ambiguity originated by insufficient data in some mathematical or computer models that apply to real-world phenomena [30–36]. There are some integrals to deal with fuzzy-interval-valued mappings, where the integrands are f-i-v-ms. For instance, Oseuna-Gómez et al. [37] and Costa et al. [38] constructed Jensen's integral inequality for f-i-v-ms through Kulisch–Miranker order relation [39]. By using same approach, Costa and Flores also presented Minkowski and Beckenbach's inequalities, where the integrands are f-i-v-ms. This paper is motivated by [37,38,40], and especially by Costa et al. [41], because they established a relation between elements of fuzzy-interval space and interval space, and introduced level-wise fuzzy order relation on fuzzy-interval space through Kulisch–Miranker order relation defined on interval space. For more information, see [42–46] and the references therein.

Our goal is to use the generalization of the fractional integral operator of Kilbas et al. [6] (which is known as a fuzzy generalized fractional integral operator [47]) as an extension of an n-fold integral that has many applications in variational calculus [48], numerical analysis [49], Langevin equations and probability theory [50], and so on. When a parameter was fixed at different values, it constructed the abovementioned integrals as exceptional cases [6]. These integrals correspond to infinite memory effects and are reduced to the Riemann–Liouville fractional integral operator, Hadamard fractional integral operator, Weyl fractional integral operator, and Liouville fractional integral operator, respectively.

The current paper is motivated by the abovementioned studies, in particular the findings developed in [27,40]. The fuzzy-interval-valued convexity is used to create certain fractional integral fuzzy order relations that are bound up with the extraordinary Hermite–Hadamard as well as Hermite-Hadamard-Fejér-type inequalities. We also use introduced fuzzy-interval-valued generalized integrals to create Hermite–Hadamard-type inequalities in fuzzy order relations to produce two fuzzy interval-valued p-convex mappings.

2. Preliminaries

Let $\mathfrak{X}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$, and $\mathcal{Q} \in \mathfrak{X}_C$ be defined by

$$\mathcal{Q} = [\mathcal{Q}_*,\ \mathcal{Q}^*] = \{\varkappa \in \mathbb{R} |\ \mathcal{Q}_* \leq \varkappa \leq \mathcal{Q}^*\}, (\mathcal{Q}_*,\ \mathcal{Q}^* \in \mathbb{R}).$$

If $\mathcal{Q}_* = \mathcal{Q}^*$, then $\mathcal{Q}$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\mathcal{Q}_* \geq 0$, then $[\mathcal{Q}_*,\ \mathcal{Q}^*]$ is called a positive interval. The set of all positive intervals is denoted by $\mathfrak{X}_C^+$ and defined as $\mathfrak{X}_C^+ = \{[\mathcal{Q}_*,\ \mathcal{Q}^*] : [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathfrak{X}_C \text{ and } \mathcal{Q}_* \geq 0\}$.

Let $\lambda \in \mathbb{R}$ and $\lambda \cdot \mathcal{Q}$ be defined by

$$\lambda \cdot \mathcal{Q} = \begin{cases} [\lambda \mathcal{Q}_*,\ \lambda \mathcal{Q}^*] & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ [\lambda \mathcal{Q}^*, \lambda \mathcal{Q}_*] & \text{if } \lambda < 0. \end{cases} \tag{1}$$

Then the Minkowski difference $\mathcal{Z} - \mathcal{Q}$, and $\mathcal{Q} + \mathcal{Z}$ and $\mathcal{Q} \times \mathcal{Z}$ for $\mathcal{Q}, \mathcal{Z} \in \mathfrak{X}_C$ are defined by

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] + [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* + \mathcal{Q}_*,\ \ \mathcal{Z}^* + \mathcal{Q}^*], \tag{2}$$

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] \times [\mathcal{Q}_*,\ \mathcal{Q}^*] = \\ [\min\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\},\ \max\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\}]. \tag{3}$$

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] - [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* - \mathcal{Q}^*,\ \ \mathcal{Z}^* - \mathcal{Q}_*], \tag{4}$$

The inclusion " $\subseteq$ " means that $\mathcal{Z} \subseteq \mathcal{Q}$, when, and only when, $[\mathcal{Z}_*,\ \mathcal{Z}^*] \subseteq [\mathcal{Q}_*,\ \mathcal{Q}^*]$, when, and only when $\mathcal{Q}_* \leq \mathcal{Z}_*$, $\mathcal{Z}^* \leq \mathcal{Q}^*$.

Remark 1 ([39]). *The relation " $\leq_I$ " defined on $\mathfrak{X}_C$ by*

$[\mathcal{Z}_*,\ \mathcal{Z}^*] \leq_I [\mathcal{Q}_*,\ \mathcal{Q}^*]$*, when, and only when,* $\mathcal{Z}_* \leq \mathcal{Q}_*,\ \mathcal{Z}^* \leq \mathcal{Q}^*$*,*

For every $[\mathcal{Z}_*,\ \mathcal{Z}^*]$*,* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathfrak{X}_C$*, it is an order relation.*

For $[\mathcal{Z}_*,\ \mathcal{Z}^*]$*,* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathfrak{X}_C$*, the Hausdorff–Pompeiu distance between intervals* $[\mathcal{Z}_*,\ \mathcal{Z}^*]$ *and* $[\mathcal{Q}_*,\ \mathcal{Q}^*]$ *is defined by*

$$d_H([\mathcal{Z}_*,\ \mathcal{Z}^*],\ [\mathcal{Q}_*,\ \mathcal{Q}^*]) = max\{|\mathcal{Z}_* - \mathcal{Q}_*|,\ |\mathcal{Z}^* - \mathcal{Q}^*|\}. \tag{5}$$

It is a well-known fact that $(\mathfrak{X}_C, d_H)$ *is a complete metric space* [33,43,44].

Definition 1 ([28,33]). A *fuzzy subset L of* $\mathbb{R}$ *is distinguished by a mapping of* $\widetilde{\psi} : \mathbb{R} \to [0,1]$*, called the membership function of L. That is, a fuzzy subset L of* $\mathbb{R}$ *is a mapping of* $\widetilde{\psi} : \mathbb{R} \to [0,1]$*. Therefore, we have chosen this notation for further study. The family of all fuzzy subsets of* $\mathbb{R}$ *is represented as* $\mathbb{E}$*. We appoint* $\mathbb{E}$ *to denote the set of all fuzzy subsets of* $\mathbb{R}$*.*

Let $\widetilde{\psi} \in \mathbb{E}$*. Then,* $\widetilde{\psi}$ *is known as a fuzzy number or fuzzy interval if the following properties are satisfied by* $\widetilde{\psi}$*:*

(1) $\widetilde{\psi}$ *should be normal if* $\varkappa \in \mathbb{R}$ *and* $\widetilde{\psi}(\varkappa) = 1$*;*
(2) $\widetilde{\psi}$ *should be upper-semicontinuous on* $\mathbb{R}$ *if for given* $\varkappa \in \mathbb{R}$*,* $\varepsilon > 0$ *and* $\delta > 0$ *such that* $\widetilde{\psi}(\varkappa) - \widetilde{\psi}(y) < \varepsilon$ *for all* $y \in \mathbb{R}$ *with* $|\varkappa - y| < \delta$*;*
(3) $\widetilde{\psi}$ *should be fuzzy-convex, that is* $\widetilde{\psi}((1-\nu)x + \nu y) \geq min\left(\widetilde{\psi}(x),\ \widetilde{\psi}(y)\right)$*, for all* $x, y \in \mathbb{R}$ *and* $\nu \in [0,\ 1]$
(4) $\widetilde{\psi}$ *should be compactly supported, that is* $cl\left\{u \in \mathbb{R} \middle| \ \widetilde{\psi}(\varkappa)\rangle 0\right\}$ *is compact.*

We appoint $\mathbb{E}_C$ *to denote the set of all fuzzy intervals or fuzzy numbers of* $\mathbb{R}$*.*

Definition 2 ([28,33]). *Given* $\widetilde{\psi} \in \mathbb{E}_C$*, the level sets or cut sets are given by* $\left[\widetilde{\psi}\right]^\theta = \left\{\varkappa \in \mathbb{R} \middle| \ \widetilde{\psi}(\varkappa) > \theta\right\}$ *for all* $\theta \in [0,\ 1]$ *and by* $\left[\widetilde{\psi}\right]^0 = \left\{\varkappa \in \mathbb{R} \middle| \ \widetilde{\psi}(\varkappa) > 0\right\}$*. These sets are known as* θ*-level sets or* θ*-cut sets of* $\widetilde{\psi}$*.*

Proposition 1 ([41]). *Let* $\widetilde{\psi}, \widetilde{\varphi} \in \mathbb{E}_C$*. Then relation " $\preccurlyeq$ " given on* $\mathbb{E}_C$ *by* $\widetilde{\psi} \preccurlyeq \widetilde{\varphi}$ *when, and only when,* $\left[\widetilde{\psi}\right]^\theta \leq_I \left[\widetilde{\varphi}\right]^\theta$*, for every* $\theta \in [0,\ 1]$*, it is a partial order relation.*

Remember the approaching notions, which are offered in the literature. If $\widetilde{\psi}, \widetilde{\varphi} \in \mathbb{E}_C$ *and* $\lambda \in \mathbb{R}$*, then, for every* $\theta \in [0,\ 1]$*, the arithmetic operations are defined by*

$$\left[\widetilde{\psi} \widetilde{+} \widetilde{\varphi}\right]^\theta = \left[\widetilde{\psi}\right]^\theta + \left[\widetilde{\varphi}\right]^\theta, \tag{6}$$

$$\left[\widetilde{\psi} \widetilde{\times} \widetilde{\varphi}\right]^\theta = \left[\widetilde{\psi}\right]^\theta \times \left[\ \widetilde{\varphi}\right]^\theta, \tag{7}$$

$$\left[\lambda.\widetilde{\psi}\right]^\theta = \lambda.\left[\widetilde{\psi}\right]^\theta. \tag{8}$$

These operations follow directly from Equations (1), (2) and (3), respectively.

Theorem 1 ([33]). *The space* $\mathbb{E}_C$ *dealing with a supremum metric, i.e., for* $\widetilde{\psi},\ \widetilde{\varphi} \in \mathbb{E}_C$

$$d_\infty\left(\widetilde{\psi},\ \widetilde{\varphi}\right) = \sup_{0 \leq \theta \leq 1} d_H\left(\left[\widetilde{\psi}\right]^\theta,\ \left[\widetilde{\varphi}\right]^\theta\right), \tag{9}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on the space of intervals.

2.1. Fractional Integral Operators of Interval- and Fuzzy-Interval-Valued Mappings

Now we will define and discuss some properties of fractional integral operators of interval- and fuzzy-interval-valued mappings.

Theorem 2 ([33,42]). *If $Y : [\rho,\zeta] \subset \mathbb{R} \to \mathfrak{X}_C$ is an interval-valued mapping (i-v-m) satisfying that $Y(\varkappa) = [Y_*(\varkappa),\ Y^*(\varkappa)]$, then Y is Aumann integrable (IA-integrable) over $[\rho,\zeta]$ when, and only when, $Y_*(\varkappa)$ and $Y^*(\varkappa)$ both are integrable over $[\rho,\zeta]$ such that*

$$(IA)\int_{\rho}^{\zeta} Y(\varkappa)d\varkappa = \left[\int_{\rho}^{\zeta} Y_*(\varkappa)d\varkappa,\ \int_{\rho}^{\zeta} Y^*(\varkappa)d\varkappa\right]. \tag{10}$$

Definition 3 ([38]). *Let $Y : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ be called a fuzzy-interval-valued mapping (f-i-v-m). Then, every $\theta \in [0,\ 1]$ as well as θ-levels define the family of i-v-ms $Y_\theta : \mathbb{I} \subset \mathbb{R} \to \mathfrak{X}_C$, satisfying that $Y_\theta(\varkappa) = [Y_*(\varkappa,\theta),\ Y^*(\varkappa,\theta)]$ for every $\varkappa \in \mathbb{I}$. Here, for every $\theta \in [0,\ 1]$ the end point real valued mappings $Y_*(\cdot,\theta),\ Y^*(\cdot,\theta) : \mathbb{I} \to \mathbb{R}$ are called lower and upper mappings of Y.*

Definition 4 ([38]). *Let $Y : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ be an f-i-v-m. Then $Y(\varkappa)$ is said to be continuous at $\varkappa \in \mathbb{I}$, if for every $\theta \in [0,\ 1]$, $Y_\theta(\varkappa)$ is continuous when, and only when, both end point mappings $Y_*(\varkappa,\theta)$ and $Y^*(\varkappa,\theta)$ are continuous at $\varkappa \in \mathbb{I}$.*

From the above literature review, the following results can be concluded (see [1,4,5,19]):

Definition 5 ([42]). *Let $Y : [\rho,\ \zeta] \subset \mathbb{R} \to \mathbb{E}_C$ be an f-i-v-m. The fuzzy Aumann integral ((FA)-integral) of Y over $[\rho,\ \zeta]$, denoted by $(FA)\int_{\rho}^{\zeta} Y(\varkappa)d\varkappa$, is defined level-wise by*

$$\left[(FA)\int_{\rho}^{\zeta} Y(\varkappa)d\varkappa\right]^{\theta} = (IA)\int_{\rho}^{\zeta} Y_\theta(\varkappa)d\varkappa = \left\{\int_{\rho}^{\zeta} Y(\varkappa,\theta)d\varkappa : Y(\varkappa,\theta) \in S(Y_\theta)\right\}, \tag{11}$$

where $S(Y_\theta) = \{Y(.,\theta) \to \mathbb{R} : Y(.,\theta)$ is integrable and $Y(\varkappa,\theta) \in Y_\theta(\varkappa)\}$, for every $\theta \in [0,\ 1]$. Y is (FA)-integrable over $[\rho,\ \zeta]$ if $(FA)\int_{\rho}^{\zeta} Y(\varkappa)d\varkappa \in \mathbb{E}_C$.

Theorem 3 ([41]). *Let $Y : [\rho,\ \zeta] \subset \mathbb{R} \to \mathbb{E}_C$ be an f-i-v-m as well as the θ-levels define the family of i-v-ms $Y_\theta : [\rho,\ \zeta] \subset \mathbb{R} \to \mathfrak{X}_C$, satisfying that $Y_\theta(\varkappa) = [Y_*(\varkappa,\theta),\ Y^*(\varkappa,\theta)]$ for every $\varkappa \in [\rho,\ \zeta]$ and for every $\theta \in [0,\ 1]$. Then Y is (FA)-integrable over $[\rho,\ \zeta]$ when, and only when, $Y_*(\varkappa,\theta)$ and $Y^*(\varkappa,\theta)$ both are integrable over $[\rho,\ \zeta]$. Moreover, if Y is (FA)-integrable over $[\rho,\ \zeta]$, then*

$$\left[(FA)\int_{\rho}^{\zeta} Y(\varkappa)d\varkappa\right]^{\theta} = \left[\int_{\rho}^{\zeta} Y_*(\varkappa,\theta)d\varkappa,\ \int_{\rho}^{\zeta} Y^*(\varkappa,\theta)d\varkappa\right] = (IA)\int_{\rho}^{\zeta} Y_\theta(\varkappa)d\varkappa, \tag{12}$$

for every $\theta \in [0,\ 1]$.

Definition 6 ([5,6]). *Let $g : [\rho,\zeta] \to \mathbb{R}$ be an increasing and positive function on $[\rho,\zeta]$, having a continuous derivative $g'(\varkappa)$ on (ρ,ζ). The left-sided and right-sided fractional integrals of complex-valued Lebesgue measurable mapping Y with respect to the function $g(\varkappa)$ on $[\rho,\zeta]$ of order $\beta > 0$ are defined respectively by*

$$\mathcal{I}_{\rho^+}^{g,\beta} Y(\varkappa) = \frac{1}{\Gamma(\beta)}\int_{\rho}^{\varkappa} \frac{g'(\nu)}{(g(\varkappa) - g(\nu))^{\beta-1}} Y(\nu)d\nu (\varkappa > \rho), \tag{13}$$

and

$$\mathcal{I}_{\zeta^-}^{g,\beta}\, Y(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\varkappa}^{\zeta} \frac{g'(\nu)}{(g(\nu) - g(\varkappa))^{\beta-1}} Y(\nu) d\nu \quad (\varkappa < \zeta), \tag{14}$$

where $\Gamma(\varkappa) = \int_0^\infty \nu^{\varkappa-1} e^{-\nu} d\nu$ *is the Euler gamma mapping.*

If one takes $g(\varkappa) = \frac{1}{p}\varkappa^p$, $p > 0$, *then from Definition 6, one acquires the following left-sided and right-sided generalized fractional integrals:*

The left and right generalized fractional integrals of order $\beta > 0$ *and* $p > 0$ *of* Y *are defined by*

$$\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\varkappa) = \frac{p^{1-\beta}}{\Gamma(\beta)} \int_{\rho}^{\varkappa} (\varkappa^p - \nu^p)^{\beta-1} \nu^{p-1} Y(\nu) d\nu \quad (\varkappa > \rho), \tag{15}$$

and

$$\mathcal{I}_{\zeta^-}^{p,\beta}\, Y(\varkappa) = \frac{p^{1-\beta}}{\Gamma(\beta)} \int_{\varkappa}^{\zeta} (\nu^p - \varkappa^p)^{\beta-1} \nu^{p-1} Y(\nu) d\nu \quad (\varkappa < \zeta) \tag{16}$$

respectively.

Definition 7 ([47]). *Let* $p, \beta > 0$ *and* $L([\rho,\ \zeta], \mathbb{E})$ *be the collection of all Lebesgue measurable f-i-v-ms on* $[\rho, \zeta]$. *Then the fuzzy interval left and right generalized fractional integrals of* $Y \in L([\rho,\ \zeta], \mathbb{E})$ *with order* $\beta > 0$ *are defined by*

$$\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\varkappa) = \frac{p^{1-\beta}}{\Gamma(\beta)} \int_{\rho}^{\varkappa} (\varkappa^p - \nu^p)^{\beta-1} \nu^{p-1} Y(\nu) d\nu, (\varkappa > \rho), \tag{17}$$

and

$$\mathcal{I}_{\zeta^-}^{p,\beta}\, Y(\varkappa) = \frac{p^{1-\beta}}{\Gamma(\beta)} \int_{\varkappa}^{\zeta} (\nu^p - \varkappa^p)^{\beta-1} \nu^{p-1} Y(\nu) d\nu, \quad (\varkappa < \zeta), \tag{18}$$

respectively. The fuzzy interval left- and right-generalized fractional integral based on end point mappings can be defined, that is

$$\begin{aligned} \left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\varkappa)\right]^\theta &= \tfrac{p^{1-\beta}}{\Gamma(\beta)} \int_{\rho}^{\varkappa} (\varkappa^p - \nu^p)^{\beta-1} \nu^{p-1} Y_\theta(\nu) d\nu \\ &= \tfrac{p^{1-\beta}}{\Gamma(\beta)} \int_{\rho}^{\varkappa} (\varkappa^p - \nu^p)^{\beta-1} \nu^{p-1} [Y_*(\nu,\ \theta), Y^*(\nu,\ \theta)] d\nu,\ (\varkappa > \rho), \end{aligned}$$

where

$$\mathcal{I}_{\rho^+}^{p,\beta}\, Y_*(\varkappa,\ \theta) = \frac{p^{1-\beta}}{\Gamma(\beta)} \int_{\rho}^{\varkappa} (\varkappa^p - \nu^p)^{\beta-1} \nu^{p-1} Y_*(\nu,\ \theta) d\nu, \quad (\varkappa > \rho),$$

and

$$\mathcal{I}_{\rho^+}^{p,\beta}\, Y^*(\varkappa,\ \theta) = \frac{p^{1-\beta}}{\Gamma(\beta)} \int_{\rho}^{\varkappa} (\varkappa^p - \nu^p)^{\beta-1} \nu^{p-1} Y^*(\nu,\ \theta) d\nu, \ \ (\varkappa > \rho).$$

Similarly, we can define right-generalized fractional integral Y *of* $\varkappa$ *based on end point mappings.*

2.2. Fuzzy-Interval-Valued Convexities

Definition 8 ([17]). *A mapping of* $Y : [\rho,\ \zeta] \to \mathbb{R}^+$ *is said to be P-mapping if*

$$Y(\nu\varkappa + (1-\nu)y) \le Y(\varkappa) + Y(y), \tag{19}$$

for every $\varkappa,\ y \in [\rho,\ \zeta]$ *together with* $\nu \in [0,\ 1]$. *If (19) is reversed, then* Y *is called P-concave.*

Definition 9 ([14]). *Let* $\mathbb{I}$ *be a convex set. Then f-i-v-m* $Y : \mathbb{I} \to \mathbb{E}_C$ *is said to be convex on* $\mathbb{I}$ *if*

$$Y(\nu\varkappa + (1-\nu)y\ ) \preccurlyeq \nu Y(\varkappa) \tilde{+} (1-\nu) Y(y), \tag{20}$$

for every $\varkappa,\ y \in \mathbb{I}$ *together with* $\nu \in [0,\ 1]$, *where* $Y(\varkappa) \succcurlyeq \widetilde{0}$, *for every* $\varkappa \in \mathbb{I}$.

Definition 10 ([13]). *Let* $p \in \mathbb{R}$ *with* $p \neq 0$. *Then the interval* $\mathbb{I}$. *is said to be p-convex if*

$$[\nu\varkappa^p + (1-\nu)y^p]^{\frac{1}{p}} \in \mathbb{I}, \tag{21}$$

for every $\varkappa,\ y \in \mathbb{I}$ *together with* $\nu \in [0,\ 1]$, *where* $p = 2n+1$. *and* $n \in N$.

Definition 11 ([13]). *Let* $p \in \mathbb{R}$ *with* $p \neq 0$ *and* $\mathbb{I} = [\rho,\ \zeta] \subset \mathbb{R}^+$. *Then, the mapping* $Y : [\rho,\ \zeta] \to \mathbb{R}^+$ *is said to be p-convex mapping if*

$$Y\left([\nu\varkappa^p + (1-\nu)y^p]^{\frac{1}{p}}\right) \leq \nu Y(\varkappa) + (1-\nu)Y(y), \tag{22}$$

for every $\varkappa,\ y \in [\rho,\ \zeta]$ *together with* $\nu \in [0,\ 1]$. *If the inequality (22) is reversed, then* Y *is called p-concave mapping.*

Definition 12 ([47]). *Let* $p \in \mathbb{R}\backslash\{0\}$. *A mapping of* $\mathfrak{C} : [\rho,\ \zeta] \subset (0,\ \infty) \to \mathbb{R}$ *is said to be p-symmetric with respect to* $\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}$, *if* $\mathfrak{C}(\varkappa) = \mathfrak{C}\left([\rho^p + \zeta^p - \varkappa^p]^{\frac{1}{p}}\right)$ *holds for every* $\varkappa \in [\rho,\ \zeta]$.

Remark 2. *In Definition 12, one can see the following:*

If one takes $p = 1$, *one has definitions for a mapping defined on* $(0,\ \infty)$ *(becomes symmetric with respect to* $\frac{\rho+\zeta}{2}$ *).*

Example 1. *Let* $p \in \mathbb{R}\backslash\{0\}$. *Assume that* $\mathfrak{C}_1,\ \mathfrak{C}_2 : [\rho,\ \zeta] \subset (0,\ \infty) \to \mathbb{R}$, $\mathfrak{C}_1(\varkappa) = c$ *for* $c \in \mathbb{R}$, $\mathfrak{C}_2(\varkappa) = \left(\varkappa^p - \frac{\rho^p+\zeta^p}{2}\right)^2$, *then* $\mathfrak{C}_1,\ \mathfrak{C}_2$ *are p-symmetric with respect to* $\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}$.

Definition 13 ([27]). *Let* $\mathbb{I}$ *be a p-convex set. Then f-i-v-m* $Y : \mathbb{I} \to \mathbb{E}_C$ *is said to be:*

- *p-convex on* $\mathbb{I}$ *if*

$$Y\left([\nu\varkappa^p + (1-\nu)y^p]^{\frac{1}{p}}\right) \preccurlyeq \nu Y(\varkappa)\widetilde{+}(1-\nu)Y(y), \tag{23}$$

for every $\varkappa,\ y \in \mathbb{I}$, $\nu \in [0,\ 1]$, *where* $Y(\varkappa) \succcurlyeq \widetilde{0}$.

- *p-concave on*$\mathbb{I}$ *if inequality (23) is reversed.*

We now discuss some new and known special cases of p-convex f-i-v-ms:

Remark 3. *If one takes* $p = 1$, *then p-convex f-i-v-m reduces to convex f-i-v-m, see* [14].

If one takes $p = -1$ *then we obtain the class of harmonically convex mappings, which is new.*

Theorem 4 ([27]). *Let* $\mathbb{I}$ *be a convex set, and let* $Y : \mathbb{I} \to \mathbb{E}_C$ *be an f-i-v-m, as well as θ-levels define the family of i-v-ms* $Y_\theta : \mathbb{I} \subset \mathbb{R} \to \mathfrak{X}_C^+ \subset \mathfrak{X}_C$, *satisfying that*

$$Y_\theta(\varkappa) = [Y_*(\varkappa,\theta),\ Y^*(\varkappa,\theta)],\ \forall\ \varkappa \in \mathbb{I}, \tag{24}$$

for every $\varkappa \in \mathbb{I}$ *and for every* $\theta \in [0,\ 1]$. *Then* Y *is p-convex on* $\mathbb{I}$, *when, and only when, for every* $\theta \in [0,\ 1]$, $Y_*(\varkappa,\ \theta)$ *and* $Y^*(\varkappa,\ \theta)$ *both are p-convex mappings.*

Remark 4. *If* $\mathcal{T}_*(\varkappa,\ \theta) = \mathcal{T}^*(\varkappa,\ \theta)$ *with* $\theta = 1$, *then the p-convex f-i-v-m reduces to the classical p-convex mapping, see* [13].

If $\mathcal{T}_*(\varkappa,\ \theta) = \mathcal{T}^*(\varkappa,\ \theta)$ *with* $\theta = 1$ *and* $p = 1$, *then the p-convex f-i-v-m reduces to the classical convex mapping.*

Example 2. *Let p be an odd number and the f-i-v-m $Y : [\rho,\ \zeta] = [1,\ 3] \to \mathbb{E}_C,$ defined by*

$$Y(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{4-\varkappa^{\frac{p}{2}}} & \sigma \in \left[0,\ 4-\varkappa^{\frac{p}{2}}\right], \\ \frac{2\left(4-\varkappa^{\frac{p}{2}}\right)-\sigma}{4-\varkappa^{\frac{p}{2}}} & \sigma \in \left(4-\varkappa^{\frac{p}{2}},\ 2\left(4-\varkappa^{\frac{p}{2}}\right)\right], \\ 0 & \text{otherwise}, \end{cases} \tag{25}$$

Then, for every $\theta \in [0,\ 1]$, we have $Y_\theta(\varkappa) = \left[\theta\left(4-\varkappa^{\frac{p}{2}}\right), (2-\theta)\left(4-\varkappa^{\frac{p}{2}}\right)\right]$. Since end point mappings $Y_*(\varkappa,\theta) = \theta\left(4-\varkappa^{\frac{p}{2}}\right)$ and $Y^*(\varkappa,\ \theta) = (2-\theta)\left(4-\varkappa^{\frac{p}{2}}\right)$ are 3-convex mappings for every $\theta \in [0,\ 1]$, then $Y(\varkappa)$ is a 3-convex f-i-v-m.

For the rest of the next study, we will discuss all results for positive fuzzy intervals.

3. Fuzzy Fractional-Interval-Valued Hermite–Hadamard Inequalities

Our first key finding about the $\mathcal{H}$–$\mathcal{H}$- and $\mathcal{H}$–$\mathcal{H}$–Fejér-type inequalities is given below, and these are dependent on interval-valued fractional integrals.

Theorem 5. *Let $Y : [\rho,\ \zeta] \to \mathbb{E}_C$ be a p-convex f-i-v-m on $[\rho,\ \zeta]$, as well as θ-levels define the family of i-v-ms $Y_\theta : [\rho,\ \zeta] \subset \mathbb{R} \to \mathfrak{X}_C^+$, satisfying that $Y_\theta(\varkappa) = [Y_*(\varkappa,\theta),\ Y^*(\varkappa,\theta)]$ for every $\varkappa \in [\rho,\ \zeta]$ and for every $\theta \in [0,\ 1]$. If $Y \in L([\rho,\ \zeta], \mathbb{E}_C)$, then*

$$Y\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p^\beta\Gamma(\beta+1)}{2(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\ Y(\zeta)\widetilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\ Y(\rho)\right] \preccurlyeq \frac{Y(\rho)\widetilde{+}Y(\zeta)}{2}. \tag{26}$$

If $Y(\varkappa)$ is a p-concave f-i-v-m, then

$$Y\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right) \succcurlyeq \frac{p^\beta\Gamma(\beta+1)}{2(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\ Y(\zeta)\widetilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\ Y(\rho)\right] \succcurlyeq \frac{Y(\rho)\widetilde{+}Y(\zeta)}{2}. \tag{27}$$

Proof. Let $Y : [\rho,\ \zeta] \to \mathbb{E}_C$ be a p-convex f-i-v-m. Then, for $a, b \in [\rho,\ \zeta]$, we have

$$Y\left([\nu a^p + (1-\nu)b^p]^{\frac{1}{p}}\right) \preccurlyeq \nu Y(a)\widetilde{+}(1-\nu)Y(b).$$

If $\nu = \frac{1}{2}$, then we have

$$2Y\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq Y(a)\widetilde{+}Y(b).$$

Let $a^p = \nu\rho^p + (1-\nu)\zeta^p$ and $b^p = (1-\nu)\rho^p + \nu\zeta^p$. Then, in the above inequality we have

$$2Y\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq Y\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}}\right)\widetilde{+}Y\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right).$$

Therefore, for every $\theta \in [0,\ 1]$, we have

$$2Y_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right) \leq Y_*\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\ \theta\right) + Y_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\ \theta\right),$$

$$2Y^*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right) \leq Y^*\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\ \theta\right) + Y^*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\theta\right).$$

Multiplying both sides by $\nu^{\beta-1}$ and integrating the obtained result with respect to ν over $(0,1)$, we have

$$\begin{aligned}2\int_0^1 \nu^{\beta-1} \quad & Y_*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}, \theta\right)d\nu \\ & \leq \int_0^1 \nu^{\beta-1}Y_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}}, \theta\right)d\nu + \int_0^1 \nu^{\beta-1}Y_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}, \theta\right)d\nu, \\ 2\int_0^1 \nu^{\beta-1} \quad & Y^*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}, \theta\right)d\nu \\ & \leq \int_0^1 \nu^{\beta-1}Y^*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}}, \theta\right)d\nu + \int_0^1 \nu^{\beta-1}Y^*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}, \theta\right)d\nu.\end{aligned}$$

Let $\varkappa^p = (1-\nu)\rho^p + \nu\zeta^p$ and $y^p = \nu\rho^p + (1-\nu)\zeta^p$. Then, we have

$$\begin{aligned}2\frac{1}{\beta}Y_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}, \theta\right) &\leq \frac{p}{(\zeta^p-\rho^p)^\beta}\int_\rho^\zeta (\zeta^p-y^p)^{\beta-1}\frac{Y_*(y,\theta)}{y^{1-p}}dy + \frac{p}{(\zeta^p-\rho^p)^\beta}\int_\rho^\zeta (\varkappa^p-\rho^p)^{\beta-1}\frac{Y_*(\varkappa,\theta)}{\varkappa^{1-p}}d\varkappa, \\ &\leq \frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y_*(\zeta,\theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y_*(\rho,\theta)\right].\end{aligned}$$

Analogously, for $Y^*(\varkappa,\theta)$, we have

$$2\frac{1}{\beta}Y_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}, \theta\right) \leq \frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y^*(\zeta,\theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y^*(\rho,\theta)\right].$$

That is,

$$\begin{aligned}2\tfrac{1}{\beta}\left[Y_*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}, \right.\right. & \left.\left.\theta\right),\ Y^*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}, \theta\right)\right] \leq_I \tfrac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}[\mathcal{I}_{\rho^+}^{p,\beta}\, Y_*(\zeta,\theta) \\ & +\mathcal{I}_{\zeta^-}^{p,\beta}\, Y_*(\rho,\theta),\ \mathcal{I}_{\rho^+}^{p,\beta}\, Y^*(\zeta,\theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y^*(\rho,\theta)].\end{aligned}$$

Thus,

$$2\frac{1}{\beta}\, Y\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\zeta)\widetilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\, Y(\rho)\right]. \tag{28}$$

In a similar way as above, we have

$$\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\zeta)\widetilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\, Y(\rho)\right] \preccurlyeq \frac{Y(\rho)\widetilde{+}Y(\zeta)}{\beta}. \tag{29}$$

Combining (28) and (29), we have

$$Y\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p^\beta\Gamma(\beta+1)}{2(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\zeta)\widetilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\, Y(\rho)\right] \preccurlyeq \frac{Y(\rho)\widetilde{+}Y(\zeta)}{2}.$$

Hence, the required result. □

Remark 5. *From Theorem 5 we can clearly see that:*

If one attempts to require $Y_*(\varkappa,\theta) = Y^*(\varkappa,\theta)$ and $\theta = 1$, then one gets Theorem 2.1, see [26].

Let one attempt to require $p = 1 = \theta$ and $Y_*(\varkappa,\theta) = Y^*(\varkappa,\theta)$. Then one acquires Theorem 5, which becomes the result given in [25].

Let one attempt to require $\beta = p = 1 = \theta$ and $Y_*(\varkappa,\theta) = Y^*(\varkappa,\theta)$. Then one achieves Theorem 5, which reduces to the result in [21].

Example 3. *Let p be an odd number,* $\beta = \frac{1}{2}$, $\varkappa \in [2,3]$, *and the f-i-v-m* $Y : [\rho,\ \zeta] = [2,\ 3] \to \mathbb{E}_C$, *defined by*

$$Y(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{2-\varkappa^{\frac{p}{2}}} & \sigma \in \left[0,\ 2-\varkappa^{\frac{p}{2}}\right], \\ \frac{2\left(2-\varkappa^{\frac{p}{2}}\right)-\sigma}{2-\varkappa^{\frac{p}{2}}} & \sigma \in \left(2-\varkappa^{\frac{p}{2}},\ 2\left(2-\varkappa^{\frac{p}{2}}\right)\right], \\ 0 & \text{otherwise}, \end{cases}$$

Then, for every $\theta \in [0,\ 1]$, *we have* $Y_\theta(\varkappa) = \left[\theta\left(2-\varkappa^{\frac{p}{2}}\right), (2-\theta)\left(2-\varkappa^{\frac{p}{2}}\right)\right]$. *Since end point mappings* $Y_*(\varkappa,\theta) = \theta\left(2-\varkappa^{\frac{p}{2}}\right)$ *and* $Y^*(\varkappa,\ \theta) = (2-\theta)\left(2-\varkappa^{\frac{p}{2}}\right)$ *are* 1*-convex mappings for every* $\theta \in [0,\ 1]$, *then* $Y(\varkappa)$ *is* 1*-convex f-i-v-m. We can clearly see that* $Y \in L([\rho,\ \zeta], \mathbb{E}_C)$ *and*

$$Y_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right) = Y_*\left(\frac{5}{2},\ \theta\right) = \theta\frac{4-\sqrt{10}}{2},$$

$$Y^*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right) = Y^*\left(\frac{5}{2},\ \theta\right) = (2-\theta)\frac{4-\sqrt{10}}{2},$$

$$\frac{Y_*(\rho,\theta)+Y_*(\zeta,\ \theta)}{2} = \theta\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right),$$

$$\frac{Y^*(\rho,\theta)+Y^*(\zeta,\ \theta)}{2} = (2-\theta)\left(\frac{4-\sqrt{2}+\sqrt{3}}{2}\right).$$

Note that

$$\begin{aligned} &\frac{p^\beta\Gamma(\beta)}{2(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y_*(\zeta,\ \theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y_*(\rho,\theta)\right] \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (3^p-\varkappa^p)^{\frac{-1}{2}}\ \varkappa^{p-1}\,\theta\left(2-\varkappa^{\frac{p}{2}}\right)d\varkappa \\ &+\frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (\varkappa^p-2^p)^{\frac{-1}{2}}\ \varkappa^{p-1}\,\theta\left(2-\varkappa^{\frac{p}{2}}\right)d\varkappa \\ &= \frac{1}{4}\theta\left[\frac{7393}{10,000}+\frac{9501}{10,000}\right] = \theta\frac{8447}{20,000}. \end{aligned}$$

$$\begin{aligned} &\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y^*(\zeta,\ \theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y^*(\rho,\theta)\right] \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (3^p-\varkappa^p)^{\frac{-1}{2}}\ \varkappa^{p-1}\,(2-\theta)\left(2-\varkappa^{\frac{p}{2}}\right)d\varkappa \\ &+\frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3 (\varkappa^p-2^p)^{\frac{-1}{2}}\ \varkappa^{p-1}\,(2-\theta)\left(2-\varkappa^{\frac{p}{2}}\right)d\varkappa = \frac{1}{4}(2-\theta)\left[\frac{7393}{10,000}+\frac{9501}{10,000}\right] = (2-\theta)\frac{8447}{20,000}. \end{aligned}$$

Therefore

$$\left[\theta\frac{4-\sqrt{10}}{2}, (2-\theta)\frac{4-\sqrt{10}}{2}\right] \leq_I \left[\theta\frac{8447}{20,000}, (2-\theta)\frac{8447}{20,000}\right] \leq_I \left[\theta\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right), (2-\theta)\left(\frac{4-\sqrt{2}+\sqrt{3}}{2}\right)\right],$$

and Theorem 5 is verified.

The following two theorems, which are linked with the well-known Hermite–Hadamard–Fejér-type inequalities, were obtained using p-symmetric mappings of one-variable forms.

Theorem 6. *Let $\Upsilon : [\rho, \zeta] \to \mathbb{E}_C$ be a p-convex f-i-v-m together with $\rho < \zeta$, as well as θ-levels define the family of i-v-ms $\Upsilon_\theta : [\rho, \zeta] \subset \mathbb{R} \to \mathfrak{X}_C^+$, satisfying that $\Upsilon_\theta(\varkappa) = [\Upsilon_*(\varkappa,\theta), \Upsilon^*(\varkappa,\theta)]$ for every $\varkappa \in [\rho, \zeta]$ and for every $\theta \in [0, 1]$. If $\Upsilon \in L([\rho, \zeta], \mathbb{E}_C)$ and $\mathfrak{C} : [\rho, \zeta] \to \mathbb{R}$, $\mathfrak{C}(\varkappa) \geq 0$ are p-symmetric with respect to $\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}$, then*

$$\left[\mathcal{I}^{p,\beta}_{\rho^+} (\Upsilon \circ \mathfrak{C})(\zeta) \widetilde{+} \mathcal{I}^{p,\beta}_{\zeta^-} (\Upsilon \circ \mathfrak{C})(\rho)\right] \preccurlyeq \frac{\Upsilon(\rho) \widetilde{+} \Upsilon(\zeta)}{2} \left[\mathcal{I}^{p,\beta}_{\rho^+} \mathfrak{C}(\zeta) + \mathcal{I}^{p,\beta}_{\zeta^-} \mathfrak{C}(\rho)\right]. \quad (30)$$

If Υ is a p-concave f-i-v-m, then inequality (19) is reversed.

Proof. Let Υ be a p-convex f-i-v-m and $\nu^{\beta-1}\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) \geq 0$. Then, for every $\theta \in [0, 1]$, we have

$$\begin{aligned} \nu^{\beta-1}\Upsilon_*([\nu\rho^p &+ (1-\nu)\zeta^p]^{\frac{1}{p}}, \theta\Big)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) \\ &\leq \nu^{\beta-1}(\nu\Upsilon_*(\rho, \theta) + (1-\nu)\Upsilon_*(\zeta, \theta))\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right), \\ \nu^{\beta-1}\Upsilon^*([\nu\rho^p &+ (1-\nu)\zeta^p]^{\frac{1}{p}}, \theta\Big)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) \\ &\leq \nu^{\beta-1}(\nu\Upsilon^*(\rho, \theta) + (1-\nu)\Upsilon^*(\zeta, \theta))\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right). \end{aligned} \quad (31)$$

In addition,

$$\begin{aligned} \nu^{\beta-1}\Upsilon_* &\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}, \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) \\ &\leq \nu^{\beta-1}((1-\nu)\Upsilon_*(\rho, \theta) + \nu\Upsilon_*(\zeta, \theta))\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right), \\ \nu^{\beta-1}\Upsilon^* &\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}, \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) \\ &\leq \nu^{\beta-1}((1-\nu)\Upsilon^*(\rho, \theta) + \nu\Upsilon^*(\zeta, \theta))\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right). \end{aligned} \quad (32)$$

Firstly, we discuss left endpoint mapping $\Upsilon_*(\varkappa, \theta)$ of fuzzy-interval-valued mapping $\Upsilon(\varkappa)$. After adding (31) and (32), and integrating over $[0, 1]$, we get

$$\begin{aligned} \int_0^1 \nu^{\beta-1}\Upsilon_* &\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}}, \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &+ \int_0^1 \nu^{\beta-1}\Upsilon_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}, \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &\leq \int_0^1 \begin{bmatrix} \nu^{\beta-1}\Upsilon_*(\rho, \theta)\left\{\nu\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) + (1-\nu)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)\right\} \\ +\nu^{\beta-1}\Upsilon_*(\zeta, \theta)\left\{(1-\nu)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right) + \nu\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)\right\} \end{bmatrix} d\nu, \\ = \Upsilon_*(\rho, \theta) &\int_0^1 \nu^{\beta-1}C\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu + \Upsilon_*(\zeta, \theta)\int_0^1 \nu^{\beta-1}C\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu. \end{aligned} \quad (33)$$

Taking the right side of the above inequality and putting $\varkappa^p = (1-\nu)\rho^p + \nu\zeta^p$, since $\mathfrak{C}$ is p-symmetric, then we have

$$\begin{aligned} &[\Upsilon_*(\rho, \theta) + \Upsilon_*(\zeta, \theta)]\int_0^1 \nu^{\beta-1}C\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &= \frac{\Upsilon_*(\rho, \theta) + \Upsilon_*(\zeta, \theta)}{2}\frac{p^\beta\Gamma(\beta)}{(\zeta^p - \rho^p)^\beta}\left[\mathcal{I}^{p,\beta}_{\rho^+} \mathfrak{C}(\zeta) + \mathcal{I}^{p,\beta}_{\zeta^-} \mathfrak{C}(\rho)\right]. \end{aligned} \quad (34)$$

Now, taking the left side of the inequality (33) and putting $\varkappa^p = (1-\nu)\rho^p + \nu\zeta^p$, we have

$$\begin{aligned}
\int_0^1 \nu^{\beta-1}\Upsilon_* & \left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\, \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\
& + \int_0^1 \nu^{\beta-1}\Upsilon_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\, \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\
& = \tfrac{p}{(\zeta^p-\rho^p)^\beta} \int_\rho^\zeta (\zeta^p - \varkappa^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*\left([\rho^p + \zeta^p - \varkappa^p]^{\frac{1}{p}}, \theta\right)\mathfrak{C}(\varkappa)d\varkappa \\
& + \tfrac{p}{(\zeta^p-\rho^p)^\beta} \int_\rho^\zeta (\varkappa^p - \rho^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*(\varkappa, \theta)\mathfrak{C}(\varkappa)d\varkappa. \\
& = \tfrac{p}{(\zeta^p-\rho^p)^\beta} \int_\rho^\zeta (\zeta^p - \varkappa^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*(\varkappa, \theta)\mathfrak{C}\left([\rho^p + \zeta^p - \varkappa^p]^{\frac{1}{p}}\right)d\varkappa \\
& + \tfrac{p}{(\zeta^p-\rho^p)^\beta} \int_\rho^\zeta (\varkappa^p - \rho^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*(\varkappa, \theta)\mathfrak{C}(\varkappa)d\varkappa, \\
& = \tfrac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, (\Upsilon_*\mathfrak{C})(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon_*\mathfrak{C})(\rho)\right].
\end{aligned} \tag{35}$$

Then from (34) and (35), (33) we have

$$\begin{aligned}
& \frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, (\Upsilon_*\mathfrak{C})(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon_*\mathfrak{C})(\rho)\right] \\
\leq\ & \frac{\Upsilon_*(\rho,\, \theta) + \Upsilon_*(\zeta,\, \theta)}{2}\,\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, \mathfrak{C}(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, \mathfrak{C}(\rho)\right].
\end{aligned}$$

In a similar way as above, for $\Upsilon^*(\varkappa)$, we have

$$\begin{aligned}
\tfrac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\Big[\mathcal{I}_{\rho^+}^{p,\beta} & \quad (\Upsilon^*\mathfrak{C})(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon^*\mathfrak{C})(\rho)\Big] \\
& \leq \tfrac{\Upsilon^*(\rho,\, \theta)+\Upsilon^*(\zeta,\, \theta)}{2}\,\tfrac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, \mathfrak{C}(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, \mathfrak{C}(\rho)\right].
\end{aligned}$$

That is,

$$\begin{aligned}
& \frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, (\Upsilon_*\mathfrak{C})(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon_*\mathfrak{C})(\rho),\, \mathcal{I}_{\rho^+}^{p,\beta}\, (\Upsilon^*\mathfrak{C})(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon^*\mathfrak{C})(\rho)\right] \\
\leq_I\ & \frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\frac{\Upsilon_*(\rho,\, \theta) + \Upsilon_*(\zeta,\, \theta)}{2},\ \frac{\Upsilon^*(\rho,\, \theta) + \Upsilon^*(\zeta,\, \theta)}{2}\right]\left[\mathcal{I}_{\rho^+}^{p,\beta}\, \mathfrak{C}(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, \mathfrak{C}(\rho)\right],
\end{aligned}$$

hence

$$\left[\mathcal{I}_{\rho^+}^{p,\beta}\, (\Upsilon\circ\mathfrak{C})(\zeta)\widetilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon\circ\mathfrak{C})(\rho)\right] \preccurlyeq \frac{\Upsilon(\rho)\widetilde{+}\Upsilon(\zeta)}{2}\left[\mathcal{I}_{\rho^+}^{p,\beta}\, \mathfrak{C}(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, \mathfrak{C}(\rho)\right].$$

□

Next, we first construct the $\mathcal{H}$–$\mathcal{H}$–Fejér inequality for the p-convex f-i-v-m, which generalizes the $\mathcal{H}$–$\mathcal{H}$–Fejér inequalities for convex mapping (see [45]).

Theorem 7. *Let $\Upsilon : [\rho,\, \zeta] \to \mathbb{E}_C$ be a p-convex f-i-v-m together with $\rho < \zeta$, as well as θ-levels define the family of i-v-ms $\Upsilon_\theta : [\rho,\, \zeta] \subset \mathbb{R} \to \mathfrak{X}_C^+$, satisfying that $\Upsilon_\theta(\varkappa) = [\Upsilon_*(\varkappa, \theta),\ \Upsilon^*(\varkappa, \theta)]$ for every $\varkappa \in [\rho,\, \zeta]$ and for every $\theta \in [0,\, 1]$. If $\Upsilon \in\ L([\rho,\, \zeta], \mathbb{E}_C)$ and $\mathfrak{C} : [\rho,\, \zeta] \to \mathbb{R}$, $\mathfrak{C}(\varkappa) \geq 0$ is p-symmetric with respect to $\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}$, then*

$$\Upsilon\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right)\left[\mathcal{I}_{\rho^+}^{p,\beta}\, \mathfrak{C}(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, \mathfrak{C}(\rho)\right] \preccurlyeq \left[\mathcal{I}_{\rho^+}^{p,\beta}\, (\Upsilon\circ\mathfrak{C})(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, (\Upsilon\circ\mathfrak{C})(\rho)\right]. \tag{36}$$

If Υ is a p-concave f-i-v-m, then inequality (36) is reversed.

Proof. Since Υ is a p-convex f-i-v-m, then for $\theta \in [0,\ 1]$, we have

$$\Upsilon_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right) \leq \frac{1}{2}\left(\Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\ \theta\right)+\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\ \theta\right)\right). \quad (37)$$

Since $\mathfrak{C}\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}}\right)=\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)$, then by multiplying (37) by $\nu^{\beta-1}\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)$ and integrating it with respect to ν over $[0,\ 1]$, we obtain

$$\begin{gathered}\Upsilon_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right)\int_0^1\nu^{\beta-1}\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ \leq \frac{1}{2}\left(\begin{array}{c}\int_0^1\nu^{\beta-1}\Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\ \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ +\int_0^1\nu^{\beta-1}\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\ \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu\end{array}\right).\end{gathered} \quad (38)$$

Let $\varkappa^p=(1-\nu)\rho^p+\nu\zeta^p$. Then, by taking the right side of the above inequality, we have

$$\begin{aligned}&\int_0^1\nu^{\beta-1}\Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\ \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &+\int_0^1\nu^{\beta-1}\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\ \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &=\frac{p}{(\zeta^p-\rho^p)^\beta}\int_\rho^\zeta(\zeta^p-\varkappa^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*\left([\rho^p+\zeta^p-\varkappa^p]^{\frac{1}{p}},\theta\right)\mathfrak{C}(\varkappa)d\varkappa \\ &+\frac{p}{(\zeta^p-\rho^p)^\beta}\int_\rho^\zeta(\varkappa^p-\rho^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*(\varkappa,\theta)\mathfrak{C}(\varkappa)d\varkappa. \\ &=\frac{p}{(\zeta^p-\rho^p)^\beta}\int_\rho^\zeta(\zeta^p-\rho^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*(\varkappa,\theta)\mathfrak{C}\left([\rho^p+\zeta^p-\varkappa^p]^{\frac{1}{p}}\right)d\varkappa \\ &+\frac{p}{(\zeta^p-\rho^p)^\beta}\int_\rho^\zeta(\varkappa^p-\rho^p)^{\beta-1}\varkappa^{p-1}\Upsilon_*(\varkappa,\theta)\mathfrak{C}(\varkappa)d\varkappa,\end{aligned}$$

Since $\mathfrak{C}$ is p-symmetric mapping, then from $\mathfrak{C}(\varkappa)=\mathfrak{C}\left([\rho^p+\zeta^p-\varkappa^p]^{\frac{1}{p}}\right)$, we have

$$\begin{aligned}&\int_0^1\nu^{\beta-1}\Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\ \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &+\int_0^1\nu^{\beta-1}\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\ \theta\right)\mathfrak{C}\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}}\right)d\nu \\ &=\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\,(\Upsilon_*\mathfrak{C})(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,(\Upsilon_*\mathfrak{C})(\rho)\right].\end{aligned} \quad (39)$$

Then, from (39), we have

$$\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\Upsilon_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right)\left[\mathcal{I}_{\rho^+}^{p,\beta}\,\mathfrak{C}(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,\mathfrak{C}(\rho)\right]\leq\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\,(\Upsilon_*\mathfrak{C})(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,(\Upsilon_*\mathfrak{C})(\rho)\right].$$

Analogously, for $\Upsilon^*(\varkappa,\theta)$, we have

$$\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\Upsilon^*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\ \theta\right)\left[\mathcal{I}_{\rho^+}^{p,\beta}\,\mathfrak{C}(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,\mathfrak{C}(\rho)\right]\leq\frac{p^\beta\Gamma(\beta)}{(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\,(\Upsilon^*\mathfrak{C})(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,(\Upsilon^*\mathfrak{C})(\rho)\right],$$

from which, we have

$$\frac{p^{\beta}\Gamma(\beta)}{(\zeta^{p}-\rho^{p})^{\beta}}\left[Y_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\,\theta\right),Y^*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\,\theta\right)\right]\left[\mathcal{I}^{p,\beta}_{\rho^+}\,\mathfrak{C}(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,\mathfrak{C}(\rho)\right]$$
$$\leq_I \frac{p^{\beta}\Gamma(\beta)}{(\zeta^{p}-\rho^{p})^{\beta}}\left[\mathcal{I}^{p,\beta}_{\rho^+}\,(Y_*\mathfrak{C})(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,(Y_*\mathfrak{C})(\rho),\mathcal{I}^{p,\beta}_{\rho^+}\,(Y^*\mathfrak{C})(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,(Y^*\mathfrak{C})(\rho)\right].$$

That is

$$\frac{p^{\beta}\Gamma(\beta)}{(\zeta^{p}-\rho^{p})^{\beta}}Y\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right)\left[\mathcal{I}^{p,\beta}_{\rho^+}\,\mathfrak{C}(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,\mathfrak{C}(\rho)\right]$$
$$\preccurlyeq \frac{p^{\beta}\Gamma(\beta)}{(\zeta^{p}-\rho^{p})^{\beta}}\left[\mathcal{I}^{p,\beta}_{\rho^+}\,(Y\circ\mathfrak{C})(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,(Y\circ\mathfrak{C})(\rho)\right]$$

This completes the proof. □

Remark 6. *Theorems 6 and 7 lead to the conclusion that*

If $\mathfrak{C}(\varkappa)=1$, then we get Theorem 5.

If $Y_*(\varkappa,\,\theta)=Y^*(\varkappa,\,\theta)$ and $\beta=1=\theta$, then we get Theorem5 of [46].

If $Y_*(\varkappa,\,\theta)=Y^*(\varkappa,\,\theta)$ and $\mathfrak{C}(\varkappa)=p=\beta=1=\theta$, then we get the classical $\mathcal{H}$–$\mathcal{H}$ inequality [45].

If $Y_*(\varkappa,\,\theta)=Y^*(\varkappa,\,\theta)$ and $\beta=1$, then we obtain the classical $\mathcal{H}$–$\mathcal{H}$–Fejér-type inequality [22].

Theorem 8. *Let $Y,\mathfrak{G}:[\rho,\,\zeta]\to\mathbb{E}_C$ be two p-convex f-i-v-ms on $[\rho,\,\zeta]$, as well as θ-levels $Y_\theta,\,\mathfrak{G}_\theta:[\rho,\,\zeta]\subset\mathbb{R}\to\mathfrak{X}^+_C$ be defined by $Y_\theta(\varkappa)=[Y_*(\varkappa,\theta),\,Y^*(\varkappa,\theta)]$ and $\mathfrak{G}_\theta(\varkappa)=[\mathfrak{G}_*(\varkappa,\theta),\mathfrak{G}^*(\varkappa,\theta)]$ for every $\varkappa\in[\rho,\,\zeta]$ and for every $\theta\in[0,\,1]$. If $Y,\,\mathfrak{G}$ and $Y\widetilde{\times}\mathfrak{G}\in L([\rho,\,\zeta],\mathbb{E}_C)$, then*

$$\frac{p^{\beta}\Gamma(\beta)}{2(\zeta^{p}-\rho^{p})^{\beta}}\left[\mathcal{I}^{p,\beta}_{\rho^+}\,Y(\zeta)\widetilde{\times}\mathfrak{G}(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}Y(\rho)\widetilde{\times}\mathfrak{G}(\rho)\right]$$
$$\preccurlyeq\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}(\rho,\zeta)\widetilde{+}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}(\rho,\zeta).$$

where $\mathcal{M}(\rho,\zeta)=Y(\rho)\widetilde{\times}\mathfrak{G}(\rho)\;\widetilde{+}\;Y(\zeta)\widetilde{\times}\mathfrak{G}(\zeta)$, $\mathcal{N}(\rho,\zeta)=Y(\rho)\widetilde{\times}\mathfrak{G}(\zeta)\;\widetilde{+}\;Y(\zeta)\widetilde{\times}\mathfrak{G}(\rho)$, and $\mathcal{M}_\theta(\rho,\zeta)=[\mathcal{M}_*((\rho,\zeta),\,\theta),\,\mathcal{M}^*((\rho,\zeta),\,\theta)]$ and $\mathcal{N}_\theta(\rho,\zeta)=[\mathcal{N}_*((\rho,\zeta),\,\theta),\,\mathcal{N}^*((\rho,\zeta),\,\theta)]$.

Proof. Since $Y,\,\mathfrak{G}$ both are p-convex f-i-v-ms, then for every $\theta\in[0,\,1]$ we have

$$Y_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right)\leq\nu Y_*(\rho,\theta)+(1-\nu)Y_*(\zeta,\,\theta).$$

and

$$\mathfrak{G}_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right)\leq\nu\mathfrak{G}_*(\rho,\theta)+(1-\nu)\mathfrak{G}_*(\zeta,\,\theta).$$

From the definition of p-convex f-i-v-ms, it follows that $\widetilde{0}\preccurlyeq Y(\varkappa)$ and $\widetilde{0}\preccurlyeq\mathfrak{G}(\varkappa)$, then by (6), (7) and (8), we obtain

$$\begin{aligned}Y_*&\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right)\times\mathfrak{G}_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right)\\&\leq\left(\nu Y_*(\rho,\theta)+(1-\nu)Y_*(\zeta,\,\theta)\right)\left(\nu\mathfrak{G}_*(\rho,\theta)+(1-\nu)\mathfrak{G}_*(\zeta,\,\theta)\right)\\&=\nu^2Y_*(\rho,\theta)\times\mathfrak{G}_*(\rho,\theta)+(1-\nu)^2Y_*(\zeta,\,\theta)\times\mathfrak{G}_*(\zeta,\,\theta)\\&+\nu(1-\nu)Y_*(\rho,\theta)\times\mathfrak{G}_*(\zeta,\,\theta)+\nu(1-\nu)Y_*(\zeta,\,\theta)\times\mathfrak{G}_*(\rho,\theta).\end{aligned}\tag{40}$$

Analogously, we have

$$\begin{aligned}Y_*&\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right)\times\mathfrak{G}_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right)\\&\leq(1-\nu)^2Y_*(\rho,\theta)\times\mathfrak{G}_*(\rho,\theta)+\nu^2Y_*(\zeta,\,\theta)\times\mathfrak{G}_*(\zeta,\,\theta)\\&+\nu(1-\nu)Y_*(\rho,\theta)\times\mathfrak{G}_*(\zeta,\,\theta)+\nu(1-\nu)Y_*(\zeta,\,\theta)\times\mathfrak{G}_*(\rho,\theta).\end{aligned}\tag{41}$$

Adding (29) and (30), we have

$$\begin{aligned} Y_* &\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\, \theta \right) \times \mathfrak{G}_*\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\, \theta \right) \\ &+ Y_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\, \theta \right) \times \mathfrak{G}_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\, \theta \right) \\ &\leq \left[\nu^2 + (1-\nu)^2 \right] [Y_*(\rho,\theta) \times \mathfrak{G}_*(\rho,\theta) + Y_*(\zeta,\, \theta) \times \mathfrak{G}_*(\zeta,\, \theta)] \\ &+ 2\nu(1-\nu)[Y_*(\zeta,\, \theta) \times \mathfrak{G}_*(\rho,\theta) + Y_*(\rho,\theta) \times \mathfrak{G}_*(\zeta,\, \theta)]. \end{aligned} \tag{42}$$

Taking multiplication of (31) by $\nu^{\beta-1}$ and integrating the obtained result with respect to ν over (0,1), we have

$$\begin{aligned} \int_0^1 \nu^{\beta-1} Y_* &\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\, \theta \right) \times \mathfrak{G}_*\left([\nu\rho^p + (1-\nu)\zeta^p]^{\frac{1}{p}},\, \theta \right) \\ &+ \nu^{\beta-1} Y_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\, \theta \right) \times \mathfrak{G}_*\left([(1-\nu)\rho^p + \nu\zeta^p]^{\frac{1}{p}},\, \theta \right) d\nu \\ &\leq \mathcal{M}_*((\rho,\zeta),\, \theta) \int_0^1 \nu^{\beta-1}\left[\nu^2 + (1-\nu)^2 \right] d\nu + 2\mathcal{N}_*((\rho,\zeta),\, \theta) \int_0^1 \nu^{\beta-1}\nu(1-\nu) d\nu. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{p^\beta \Gamma(\beta)}{(\zeta^p - \rho^p)^\beta} &\left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y_*(\zeta,\, \theta) \times \mathfrak{G}_*(\zeta,\, \theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y_*(\rho,\theta) \times \mathfrak{G}_*(\rho,\theta) \right] \\ &\leq \frac{2}{\beta}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \mathcal{M}_*((\rho,\zeta),\, \theta) + \frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)} \right) \mathcal{N}_*((\rho,\zeta),\, \theta). \end{aligned}$$

In a similar way as above, for $Y^*(\varkappa,\, \theta)$ and $\mathfrak{G}^*(\varkappa,\, \theta)$ we have

$$\begin{aligned} &\frac{p^\beta \Gamma(\beta)}{(\zeta^p - \rho^p)^\beta} \left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y^*(\zeta,\, \theta) \times \mathfrak{G}^*(\zeta,\, \theta) + \mathcal{I}_{\zeta^-}^{p,\beta}\, Y^*(\rho,\theta) \times \mathfrak{G}^*(\rho,\theta) \right] \\ &\leq \frac{2}{\beta}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \mathcal{M}^*((\rho,\zeta),\, \theta) + \frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)} \right) \mathcal{N}^*((\rho,\zeta),\, \theta). \end{aligned}$$

That is,

$$\begin{aligned} \frac{p^\beta \Gamma(\beta)}{(\zeta^p - \rho^p)^\beta} \Big[\mathcal{I}_{\rho^+}^{p,\beta}\, Y_* &(\zeta,\, \theta) \times \mathfrak{G}_*(\zeta,\, \theta) \\ &+ \mathcal{I}_{\zeta^-}^{p,\beta}\, Y_*(\rho,\theta) \times \mathfrak{G}_*(\rho,\theta), \mathcal{I}_{\rho^+}^{p,\beta}\, Y^*(\zeta,\, \theta) \times \mathfrak{G}^*(\zeta,\, \theta) \\ &+ \mathcal{I}_{\zeta^-}^{p,\beta}\, Y^*(\rho,\theta) \times \mathfrak{G}^*(\rho,\, \theta) \Big] \\ &\leq_I \frac{2}{\beta}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [\mathcal{M}_*((\rho,\zeta),\, \theta),\, \mathcal{M}^*((\rho,\zeta),\, \theta)] + \\ &\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)} \right) [\mathcal{N}_*((\rho,\zeta),\, \theta),\, \mathcal{N}^*((\rho,\zeta),\, \theta)]. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{p^\beta \Gamma(\beta)}{2(\zeta^p - \rho^p)^\beta} \left[\mathcal{I}_{\rho^+}^{p,\beta}\, Y(\zeta) \widetilde{\times} \mathfrak{G}(\zeta) + \mathcal{I}_{\zeta^-}^{p,\beta} Y(\rho) \widetilde{\times} \mathfrak{G}(\rho) \right] \\ &\preccurlyeq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \mathcal{M}(\rho,\zeta) \widetilde{+} \left(\frac{\beta}{(\beta+1)(\beta+2)} \right) \mathcal{N}(\rho,\zeta). \end{aligned}$$

and the theorem has been established. □

Example 4. *Let p be an odd number, $[\rho,\zeta] = [0,2]$, $\beta = \frac{1}{2}$, $Y(\varkappa) = [\varkappa^p, 2\varkappa^p]$, and $\mathfrak{G}(\varkappa) = [\varkappa^p, 3\varkappa^p]$.*

$$Y(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{\varkappa^p} & \sigma \in [0,\, \varkappa^p], \\ \frac{2\varkappa^p - \sigma}{\varkappa^p} & \sigma \in (\varkappa^p,\, 2\varkappa^p], \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathfrak{G}(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{2\varkappa^p} & \sigma \in [0,\, 2\varkappa^p], \\ \frac{4\varkappa^p - \sigma}{2\varkappa^p} & \sigma \in (2\varkappa^p,\, 4\varkappa^p], \\ 0 & \text{otherwise}. \end{cases}$$

Then, for every $\theta \in [0, 1]$, we have $\Upsilon_\theta(\varkappa) = [\theta\varkappa^p, (2-\theta)\varkappa^p]$ and $\mathfrak{G}_\theta(\varkappa) = [2\theta\varkappa^p, 2(2-\theta)\varkappa^p]$. Since end point mappings $\Upsilon_*(\varkappa, \theta) = \theta\varkappa^p$, $\Upsilon^*(\varkappa, \theta) = (2-\theta)\varkappa^p$, $\mathfrak{G}_*(\varkappa, \theta) = 2\theta\varkappa^p$, and $\mathfrak{G}^*(\varkappa, \theta) = 2(2-\theta)\varkappa^p$ are p-convex mappings for every $\theta \in [0, 1]$, then $\Upsilon(\varkappa)$ and $\mathfrak{G}(\varkappa)$ both are p-convex f-i-v-ms. We can clearly see that $\Upsilon(\varkappa)\widetilde{\times}\mathfrak{G}(\varkappa) \in L([\rho, \zeta], \mathbb{E}_C)$ and

$$\frac{p^\beta\Gamma(1+\beta)}{2(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}^{p,\beta}_{\rho^+}\,\Upsilon_*(\zeta)\times\mathfrak{G}_*(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,\Upsilon_*(\rho)\times\mathfrak{G}_*(\rho)\right]$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(2^p-\varkappa^p)^{\frac{-1}{2}}\,\varkappa^{p-1}\left(2\theta^2\varkappa^{2p}\right)d\varkappa+\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(\varkappa^p)^{\frac{-1}{2}}\,\varkappa^{p-1}\left(2\theta^2\varkappa^{2p}\right)d\varkappa\approx 2.9332\theta^2,$$

$$\frac{p^\beta\Gamma(1+\beta)}{2(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}^{p,\beta}_{\rho^+}\,\Upsilon^*(\zeta)\times\mathfrak{G}^*(\zeta)+\mathcal{I}^{p,\beta}_{\zeta^-}\,\Upsilon^*(\rho)\times\mathfrak{G}^*(\rho)\right]=\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(2^p-\varkappa^p)^{\frac{-1}{2}}\,\varkappa^{p-1}\left(2(2-\theta)^2\varkappa^{2p}\right)d\varkappa+$$

$$\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(\varkappa^p)^{\frac{-1}{2}}\,\varkappa^{p-1}\left(2(2-\theta)^2\varkappa^{2p}\right)d\varkappa\approx 2.9332(2-\theta)^2.$$

Note that

$$\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}_*(\rho,\zeta)=[\Upsilon_*(\rho)\times\mathfrak{G}_*(\rho)+\Upsilon_*(\zeta)\times\mathfrak{G}_*(\zeta)]=\frac{11}{30}\cdot 8\theta^2,$$

$$\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}^*(\rho,\zeta)=[\Upsilon^*(\rho)\times\mathfrak{G}^*(\rho)+\Upsilon^*(\zeta)\times\mathfrak{G}^*(\zeta)]=\frac{11}{30}\cdot 8(2-\theta)^2,$$

$$\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_*(\rho,\zeta)=[\Upsilon_*(\rho)\times\mathfrak{G}_*(\zeta)+\Upsilon_*(\zeta)\times\mathfrak{G}_*(\rho)]=\frac{2}{15}(0),$$

$$\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_*(\rho,\zeta)=[\Upsilon^*(\rho)\times\mathfrak{G}^*(\zeta)+\Upsilon^*(\zeta)\times\mathfrak{G}^*(\rho)]=\frac{2}{15}(0).$$

Therefore, we have

$$\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}_\theta((\rho,\zeta),\,\theta)+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_\theta((\rho,\zeta),\,\theta)$$

$$=\frac{11}{30}[8\theta^2, 8(2-\theta)^2]+\frac{2}{15}[0,0]\approx\left[2.9332\theta^2, 2.9332(2-\theta)^2\right].$$

It follows that

$$\left[2.9332\theta^2, 2.9332(2-\theta)^2\right]\leq_I\left[2.9332\theta^2, 2.9332(2-\theta)^2\right],$$

and Theorem 8 has been demonstrated.

Theorem 9. *Let $\Upsilon, \mathfrak{G} : [\rho, \zeta] \to \mathbb{E}_C$ be two p-convex f-i-v-ms, as well as θ-levels define the family of i-v-ms Υ_θ, $\mathfrak{G}_\theta : [\rho, \zeta] \subset \mathbb{R} \to \mathfrak{X}_C^+$, satisfying that $\Upsilon_\theta(\varkappa) = [\Upsilon_*(\varkappa, \theta), \Upsilon^*(\varkappa, \theta)]$ and $\mathfrak{G}_\theta(\varkappa) = [\mathfrak{G}_*(\varkappa, \theta), \mathfrak{G}^*(\varkappa, \theta)]$ for every $\varkappa \in [\rho, \zeta]$ and for every $\theta \in [0, 1]$. If $\Upsilon\widetilde{\times}\mathfrak{G} \in L([\rho, \zeta], \mathbb{E}_C)$, then*

$$\frac{1}{\beta}\,\Upsilon\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right)\widetilde{\times}\mathfrak{G}\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq$$

$$\frac{p^\beta\Gamma(\beta+1)}{4(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}^{p,\beta}_{\rho^+}\,\Upsilon(\zeta)\widetilde{\times}\mathfrak{G}(\zeta)\widetilde{+}\mathcal{I}^{p,\beta}_{\zeta^-}\,\Upsilon(\rho)\widetilde{\times}\mathfrak{G}(\rho)\right]$$

$$+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}(\rho,\zeta)\widetilde{+}\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}(\rho,\zeta),$$

where $\mathcal{M}(\rho,\zeta) = \Upsilon(\rho)\widetilde{\times}\mathfrak{G}(\rho) \widetilde{+} \Upsilon(\zeta)\widetilde{\times}\mathfrak{G}(\zeta)$, $\mathcal{N}(\rho,\zeta) = \Upsilon(\rho)\widetilde{\times}\mathfrak{G}(\zeta) \widetilde{+} \Upsilon(\zeta)\widetilde{\times}\mathfrak{G}(\rho)$, *and* $\mathcal{M}_\theta(\rho,\zeta) = [\mathcal{M}_*((\rho,\zeta),\,\theta),\ \mathcal{M}^*((\rho,\zeta),\,\theta)]$ *and* $\mathcal{N}_\theta(\rho,\zeta) = [\mathcal{N}_*((\rho,\zeta),\,\theta),\ \mathcal{N}^*((\rho,\zeta),\,\theta)]$.

Proof. Since Υ, $\mathfrak{G}$ both are p-convex f-i-v-ms, then from (1), (2), and (3), and by hypothesis, for every $\theta \in [0,\ 1]$, we have

$$\begin{aligned}
&\Upsilon_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\theta\right) \times \mathfrak{G}_*\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\theta\right) \\
&\le \frac{1}{4}\begin{bmatrix} \Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \\ +\Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \end{bmatrix} \\
&+\frac{1}{4}\begin{bmatrix} \Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \\ +\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \end{bmatrix}, \\
&\le \frac{1}{4}\begin{bmatrix} \Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \\ +\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \end{bmatrix} \\
&+\frac{1}{4}\begin{bmatrix} (\nu\Upsilon_*(\rho,\,\theta)+(1-\nu)\Upsilon_*(\zeta,\,\theta)) \\ \times((1-\nu)\mathfrak{G}_*(\rho,\,\theta)+\nu\mathfrak{G}_*(\zeta,\,\theta)) \\ +((1-\nu)\Upsilon_*(\rho,\,\theta)+\nu\Upsilon_*(\zeta,\,\theta)) \\ \times(\nu\mathfrak{G}_*(\rho,\,\theta)+(1-\nu)\mathfrak{G}_*(\zeta,\,\theta)) \end{bmatrix}, \\
&= \frac{1}{4}\begin{bmatrix} \Upsilon_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([\nu\rho^p+(1-\nu)\zeta^p]^{\frac{1}{p}},\,\theta\right) \\ +\Upsilon_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \times \mathfrak{G}_*\left([(1-\nu)\rho^p+\nu\zeta^p]^{\frac{1}{p}},\,\theta\right) \end{bmatrix} \\
&+\frac{1}{4}\begin{bmatrix} \left\{\nu^2+(1-\nu)^2\right\}\mathcal{N}_*((\rho,\zeta),\,\theta) \\ +\{\nu(1-\nu)+(1-\nu)\nu\}\mathcal{M}_*((\rho,\zeta),\,\theta) \end{bmatrix}.
\end{aligned} \tag{43}$$

Taking multiplication of (43) with $\nu^{\beta-1}$ and integrating over $(0,\ 1)$, we get

$$\begin{aligned}
&\frac{1}{\beta}\,\Upsilon_*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\theta\right) \times \mathfrak{G}_*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\theta\right) \\
&\le \tfrac{p}{4(\zeta^p-\rho^p)^\beta}\left[\int_\rho^\zeta(\zeta^p-\varkappa^p)^{\beta-1}\Upsilon_*(\varkappa,\theta)\times\mathfrak{G}_*(\varkappa,\theta)d\varkappa+\int_\rho^\zeta(y^p-\rho^p)^{\beta-1}\Upsilon_*(y,\theta)\times\mathfrak{G}_*(y,\theta)dy\right] \\
&+\tfrac{1}{2\beta}\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_*((\rho,\zeta),\,\theta)+\tfrac{1}{2\beta}\left(\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}_*((\rho,\zeta),\,\theta), \\
&= \tfrac{p^\beta\Gamma(\beta+1)}{4(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\,\Upsilon_*(\zeta)\times\mathfrak{G}_*(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,\Upsilon_*(\rho)\times\mathfrak{G}_*(\rho)\right] \\
&+\tfrac{1}{2\beta}\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_*((\rho,\zeta),\,\theta)+\tfrac{1}{2\beta}\left(\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}_*((\rho,\zeta),\,\theta).
\end{aligned}$$

In a similar way as above, for $\Upsilon^*(\varkappa,\ \theta)$ and $\mathfrak{G}^*(\varkappa,\ \theta)$ we have

$$\begin{aligned}
&\frac{1}{\beta}\,\Upsilon^*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\theta\right) \times \mathfrak{G}^*\left(\left[\tfrac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}},\theta\right) \\
&= \tfrac{p^\beta\Gamma(\beta+1)}{4(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\,\Upsilon^*(\zeta)\times\mathfrak{G}^*(\zeta)+\mathcal{I}_{\zeta^-}^{p,\beta}\,\Upsilon^*(\rho)\times\mathfrak{G}^*(\rho)\right] \\
&+\tfrac{1}{2\beta}\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}^*((\rho,\zeta),\,\theta) \\
&+\tfrac{1}{2\beta}\left(\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}^*((\rho,\zeta),\,\theta).
\end{aligned}$$

That is,

$$\frac{1}{\beta}\Upsilon\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right)\tilde{\times}\mathfrak{G}\left(\left[\frac{\rho^p+\zeta^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq$$
$$\frac{p^\beta\Gamma(\beta+1)}{4(\zeta^p-\rho^p)^\beta}\left[\mathcal{I}_{\rho^+}^{p,\beta}\,\Upsilon(\zeta)\tilde{\times}\mathfrak{G}(\zeta)\tilde{+}\mathcal{I}_{\zeta^-}^{p,\beta}\,\Upsilon(\rho)\tilde{\times}\mathfrak{G}(\rho)\right]$$
$$+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}(\rho,\zeta)\tilde{+}\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}(\rho,\zeta).$$

Hence, the required result. □

4. Conclusions and Future Plans

The p-convex (concave) class of f-i-v-ms and various related topics were explored in this paper. We also used fuzzy order relations and fuzzy generalized fractional integrals to establish certain $\mathcal{H}$–$\mathcal{H}$ inequalities for p-convex f-i-v-ms. We demonstrated that our conclusions cover a broad range of new and well-known inequalities for p-convex f-i-v-ms and their variant forms as special cases. In the near future, we will try to analyze Jensen and $\mathcal{H}$–$\mathcal{H}$ inequalities for i-v-m and f-i-v-ms on a temporal scale. Moreover, we will extend these concepts for (p,h)-convex f-i-v-ms. We hope that the concepts and methodologies presented in this study will serve as a springboard for future research in this field.

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Abbreviations

fuzzy-interval-valued functions	f-i-v-ms
interval-valued functions	i-v-ms
Hermite–Hadamard inequality	$\mathcal{H}$–$\mathcal{H}$ inequality
Hermite–Hadamard–Fejér inequality	$\mathcal{H}$–$\mathcal{H}$–Fejér inequality
Aumann integrable	IA-integrable
$\mathbb{I}$	interval
$\mathfrak{X}_C$	set of intervals
$\leq_I$	order relation defined on $\mathfrak{X}_C$
$\mathbb{E}$	set of fuzzy sets
$\mathbb{E}_C$	set of fuzzy numbers or intervals
$\succcurlyeq$	order relation defined on $\mathbb{E}_C$

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