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On a System of Riemann–Liouville Fractional Boundary Value Problems with ϱ-Laplacian Operators and Positive Parameters

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Abstract: In this paper, we study the existence and nonexistence of positive solutions of a system of Riemann–Liouville fractional differential equations with ϱ-Laplacian operators, supplemented with coupled nonlocal boundary conditions containing Riemann–Stieltjes integrals, fractional derivatives of various orders, and positive parameters. We apply the Schauder fixed point theorem in the proof of the existence result.

Keywords: Riemann–Liouville fractional differential equations; nonlocal coupled boundary conditions; positive solutions; existence; nonexistence; positive parameters

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1. Introduction

We consider the system of fractional differential equations with ϱ_1-Laplacian and ϱ_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}\mathrm{u}(t))) + \mathfrak{a}(t)\mathfrak{f}(\mathrm{v}(t)) = 0, & t \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}\mathrm{v}(t))) + \mathfrak{b}(t)\mathfrak{g}(\mathrm{u}(t)) = 0, & t \in (0,1), \end{cases} \tag{1}$$

subject to the coupled nonlocal boundary conditions

$$\begin{cases} \mathrm{u}^{(j)}(0) = 0,\ j = 0,\dots,p-2;\ D_{0+}^{\delta_1}\mathrm{u}(0) = 0,\ D_{0+}^{\alpha_0}\mathrm{u}(1) = \sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}\mathrm{v}(\tau)\,d\mathfrak{H}_j(\tau) + \mathfrak{c}_0, \\ \mathrm{v}^{(j)}(0) = 0,\ j = 0,\dots,q-2;\ D_{0+}^{\delta_2}\mathrm{v}(0) = 0,\ D_{0+}^{\beta_0}\mathrm{v}(1) = \sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}\mathrm{u}(\tau)\,d\mathfrak{K}_j(\tau) + \mathfrak{d}_0, \end{cases} \tag{2}$$

where $\gamma_1, \gamma_2 \in (0,1]$, $\delta_1 \in (p-1,p]$, $\delta_2 \in (q-1,q]$, $p, q \in \mathbb{N}$, $p, q \geq 3$, $n, m \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ for all $j = 0,1,\dots,n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\beta_j \in \mathbb{R}$ for all $j = 0,1,\dots,m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, the functions $\mathfrak{f}, \mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mathfrak{a}, \mathfrak{b} : [0,1] \to \mathbb{R}_+$ are continuous, ($\mathbb{R}_+ = [0,\infty)$), $\mathfrak{c}_0$ and $\mathfrak{d}_0$ are positive parameters, $\varrho_1, \varrho_2 > 1$, $\varphi_{\varrho_i}(\zeta) = |\zeta|^{\varrho_i - 2}\zeta$, $\varphi_{\varrho_i}^{-1} = \varphi_{\rho_i}$, $\rho_i = \frac{\varrho_i}{\varrho_i - 1}$, and $i = 1,2$. The integrals from the conditions (2) are Riemann–Stieltjes integrals with $\mathfrak{H}_j$, $j = 1,\dots,n$ and $\mathfrak{K}_i$, $i = 1,\dots,m$ functions of bounded variation, and D_{0+}^{k} denotes the Riemann–Liouville derivative of order k (for $k = \gamma_1, \delta_1, \gamma_2, \delta_2, \alpha_j$; for $j = 0,1,\dots,n$, β_i; and for $i = 0,1,\dots,m$).

We present in this paper sufficient conditions for the functions $\mathfrak{f}$ and $\mathfrak{g}$, and intervals for the parameters $\mathfrak{c}_0$ and $\mathfrak{d}_0$ such that problem (1) and (2) has at least one positive solution, or it has no positive solutions. We apply the Schauder fixed point theorem in the proof of the main existence result. A positive solution of (1) and (2) is a pair of functions $(\mathrm{u},\mathrm{v}) \in (C([0,1];\mathbb{R}_+))^2$ that satisfy the system (1) and the boundary conditions (2), with

$u(t) > 0$ and $v(t) > 0$ for all $t \in (0,1]$. Now, we present some recent results related to our problem. By using the Guo–Krasnosel'skii fixed point theorem, in [1], the authors investigated the system of fractional differential equations

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}u(t))) + \lambda f(t, u(t), v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}v(t))) + \mu g(t, u(t), v(t)) = 0, & t \in (0,1), \end{cases} \tag{3}$$

supplemented with the boundary conditions (2) with $\mathfrak{c}_0 = \mathfrak{d}_0 = 0$, where λ and μ are positive parameters, and $f, g \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. They presented various intervals for λ and μ such that problem (2) and (3) with $\mathfrak{c}_0 = \mathfrak{d}_0 = 0$ has at least one positive solution ($u(t) > 0$ for all $t \in (0,1]$, or $v(t) > 0$ for all $t \in (0,1]$). They also studied the nonexistence of positive solutions. In [2], the author investigated the existence and nonexistence of positive solutions for the system (3) with the uncoupled boundary conditions

$$\begin{cases} u^{(j)}(0) = 0,\ j = 0, \ldots, p-2;\ D_{0+}^{\delta_1}u(0) = 0,\ D_{0+}^{\alpha_0}u(1) = \sum_{j=1}^{n} \int_0^1 D_{0+}^{\alpha_j}u(\tau)\, d\mathfrak{H}_j(\tau), \\ v^{(j)}(0) = 0,\ j = 0, \ldots, q-2;\ D_{0+}^{\delta_2}v(0) = 0,\ D_{0+}^{\beta_0}v(1) = \sum_{j=1}^{m} \int_0^1 D_{0+}^{\beta_j}v(\tau)\, d\mathfrak{K}_j(\tau), \end{cases}$$

where $\alpha_j \in \mathbb{R}$ for all $j = 0, 1, \ldots, n$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le \alpha_0 < \delta_1 - 1$, $\alpha_0 \ge 1$, $\beta_j \in \mathbb{R}$ for all $j = 0, 1, \ldots, m$, $0 \le \beta_1 < \beta_2 < \cdots < \beta_m \le \beta_0 < \delta_2 - 1$, $\beta_0 \ge 1$, $\mathfrak{H}_i$, $i = 1, \ldots, n$, and $\mathfrak{K}_j$, $j = 1, \cdots, m$ are functions of bounded variation. In [3], the authors studied the positive solutions for the system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + \mathfrak{a}(t)\mathfrak{f}(v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\beta}v(t) + \mathfrak{b}(t)\mathfrak{g}(u(t)) = 0, & t \in (0,1), \end{cases}$$

subject to the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ u(1) = \int_0^1 v(\tau) d\mathfrak{H}(\tau) + \mathfrak{c}_0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0,\ v(1) = \int_0^1 u(\tau) d\mathfrak{K}(\tau) + \mathfrak{d}_0, \end{cases}$$

where $n - 1 < \alpha \le n$, $m - 1 < \beta \le m$, $n, m \in \mathbb{N}$, $n, m \ge 3$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{f}, \mathfrak{g}$ are nonnegative continuous functions, $\mathfrak{c}_0$ and $\mathfrak{d}_0$ are positive parameters, and $\mathfrak{H}$ and $\mathfrak{K}$ are bounded variation functions. In [4], the authors investigated the existence and nonexistence of positive solutions for the system (1) with the nonlocal uncoupled boundary conditions with positive parameters

$$\begin{cases} u^{(j)}(0) = 0,\ j = 0, \ldots, p-2;\ D_{0+}^{\delta_1}u(0) = 0,\ D_{0+}^{\alpha_0}u(1) = \sum_{j=1}^{n} \int_0^1 D_{0+}^{\alpha_j}u(\tau)\, d\mathfrak{H}_j(\tau) + \mathfrak{c}_0, \\ v^{(j)}(0) = 0,\ j = 0, \ldots, q-2;\ D_{0+}^{\delta_2}v(0) = 0,\ D_{0+}^{\beta_0}v(1) = \sum_{j=1}^{m} \int_0^1 D_{0+}^{\beta_j}v(\tau)\, d\mathfrak{K}_j(\tau) + \mathfrak{d}_0. \end{cases}$$

We note that our problem (1) and (2) is different than the problem studied in [4], because of the boundary conditions, which are coupled in (2) and uncoupled in [4]. Based on this difference, here, we will use, for problem (1) and (2), other Green functions, different systems of integral equations, and different operators than those in [4]. We would also like to mention the papers [5–10], and the monographs [11–13], which contain other recent results for fractional differential equations and systems of fractional differential equations with or without Laplacian operators, and for various applications. The novelties of our problem (1) and (2) with respect to the above papers consist in the consideration of positive parameters $\mathfrak{c}_0$ and $\mathfrak{d}_0$ in the coupled nonlocal boundary conditions (2) containing fractional derivatives of various orders and Riemann–Stieltjes integrals, combined with the system of fractional differential Equation (1), which has ϱ-Laplacian operators.

The paper is structured as follows. In Section 2, we present some auxiliary results, which include the Green functions associated with our problem (1) and (2) and their properties. In Section 3, we give the main theorems for the existence and nonexistence of positive solutions for (1) and (2), and Section 4 contains an example illustrating our results. Finally, in Section 5, we present the conclusions of this work.

2. Auxiliary Results

In this section, we present some results from [1], which will be used in our main theorems in the next section.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}\mathrm{u}(t))) + \widetilde{\mathrm{h}}(t) = 0, \ \ t \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}\mathrm{v}(t))) + \widetilde{\mathrm{k}}(t) = 0, \ \ t \in (0,1), \end{cases} \tag{4}$$

with the coupled boundary conditions

$$\begin{cases} \mathrm{u}^{(j)}(0) = 0, \ j = 0,\dots,p-2; \ D_{0+}^{\delta_1}\mathrm{u}(0) = 0, \ D_{0+}^{\alpha_0}\mathrm{u}(1) = \sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}\mathrm{v}(\tau)\,d\mathfrak{H}_j(\tau), \\ \mathrm{v}^{(j)}(0) = 0, \ j = 0,\dots,q-2; \ D_{0+}^{\delta_2}\mathrm{v}(0) = 0, \ D_{0+}^{\beta_0}\mathrm{v}(1) = \sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}\mathrm{u}(\tau)\,d\mathfrak{K}_j(\tau), \end{cases} \tag{5}$$

where $\widetilde{\mathrm{h}}$, $\widetilde{\mathrm{k}} \in C[0,1]$. We denote this by

$$\Delta_1 = \sum_{i=1}^{n}\frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\alpha_i)}\int_0^1 \tau^{\delta_2-\alpha_i-1}\,d\mathfrak{H}_i(\tau), \ \ \Delta_2 = \sum_{i=1}^{m}\frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\beta_i)}\int_0^1 \tau^{\delta_1-\beta_i-1}\,d\mathfrak{K}_i(\tau),$$
$$\Delta = \frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1-\alpha_0)\Gamma(\delta_2-\beta_0)} - \Delta_1\Delta_2.$$

Lemma 1 ([1])**.** *If $\Delta \neq 0$, then the unique solution $(\mathrm{u},\mathrm{v}) \in (C[0,1])^2$ of problem (4) and (5) is given by*

$$\begin{cases} \mathrm{u}(t) = \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{\mathrm{h}}(\zeta))\,d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{\mathrm{k}}(\zeta))\,d\zeta, \ \ \forall\, t \in [0,1], \\ \mathrm{v}(t) = \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{\mathrm{h}}(\zeta))\,d\zeta + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{\mathrm{k}}(\zeta))\,d\zeta, \ \ \forall\, t \in [0,1], \end{cases} \tag{6}$$

where

$$\begin{aligned} \mathfrak{G}_1(t,\zeta) &= \mathfrak{g}_1(t,\zeta) + \frac{t^{\delta_1-1}\Delta_1}{\Delta}\left(\sum_{j=1}^{m}\int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)\,d\mathfrak{K}_j(\tau)\right), \\ \mathfrak{G}_2(t,\zeta) &= \frac{t^{\delta_1-1}\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\sum_{j=1}^{n}\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)\,d\mathfrak{H}_j(\tau), \\ \mathfrak{G}_3(t,\zeta) &= \frac{t^{\delta_2-1}\Gamma(\delta_1)}{\Delta\Gamma(\delta_1-\alpha_0)}\sum_{j=1}^{m}\int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)\,d\mathfrak{K}_j(\tau), \\ \mathfrak{G}_4(t,\zeta) &= \mathfrak{g}_2(t,\zeta) + \frac{t^{\delta_2-1}\Delta_2}{\Delta}\left(\sum_{j=1}^{n}\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)\,d\mathfrak{H}_j(\tau)\right), \end{aligned} \tag{7}$$

for all $(t,\zeta) \in [0,1]\times[0,1]$ and

$$\mathfrak{g}_1(t,\zeta) = \frac{1}{\Gamma(\delta_1)}\begin{cases} t^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1} - (t-\zeta)^{\delta_1-1}, \ \ 0 \le \zeta \le t \le 1, \\ t^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1}, \ \ 0 \le t \le \zeta \le 1, \end{cases}$$
$$\mathfrak{g}_{1j}(\tau,\zeta) = \frac{1}{\Gamma(\delta_1-\beta_j)}\begin{cases} \tau^{\delta_1-\beta_j-1}(1-\zeta)^{\delta_1-\alpha_0-1} - (\tau-\zeta)^{\delta_1-\beta_j-1}, \ \ 0 \le \zeta \le \tau \le 1, \\ \tau^{\delta_1-\beta_j-1}(1-\zeta)^{\delta_1-\alpha_0-1}, \ \ 0 \le \tau \le \zeta \le 1, \end{cases}$$

$$\mathfrak{g}_2(t,\zeta) = \frac{1}{\Gamma(\delta_2)} \begin{cases} t^{\delta_2-1}(1-\zeta)^{\delta_2-\beta_0-1} - (t-\zeta)^{\delta_2-1}, & 0 \le \zeta \le t \le 1, \\ t^{\delta_2-1}(1-\zeta)^{\delta_2-\beta_0-1}, & 0 \le t \le \zeta \le 1, \end{cases}$$
$$\mathfrak{g}_{2k}(\tau,\zeta) = \frac{1}{\Gamma(\delta_2-\alpha_k)} \begin{cases} \tau^{\delta_2-\alpha_k-1}(1-\zeta)^{\delta_2-\beta_0-1} - (\tau-\zeta)^{\delta_2-\alpha_k-1}, & 0 \le \zeta \le \tau \le 1, \\ \tau^{\delta_2-\alpha_k-1}(1-\zeta)^{\delta_2-\beta_0-1}, & 0 \le \tau \le \zeta \le 1, \end{cases}$$

for all $j = 1, \dots, m$ *and* $k = 1, \dots, n$.

Lemma 2 ([1]). *We suppose that* $\Delta > 0$, $\mathfrak{H}_j$, $j = 1, \dots, n$, $\mathfrak{K}_j$, $j = 1, \dots, m$ *are nondecreasing functions. Therefore, the functions* $\mathfrak{G}_i$, $i = 1, \dots, 4$ *(given by (7)) have the following properties:*

(1) $\mathfrak{G}_i : [0,1] \times [0,1] \to \mathbb{R}_+$, $i = 1, \dots, 4$ *are continuous functions;*

(2) $\mathfrak{G}_1(t,\zeta) \le \mathfrak{J}_1(\zeta)$ *for all* $(t,\zeta) \in [0,1] \times [0,1]$, *where*
$$\mathfrak{J}_1(\zeta) = \mathfrak{h}_1(\zeta) + \frac{\Delta_1}{\Delta}\left(\sum_{j=1}^{m} \int_0^1 g_{1j}(\tau,\zeta)\,d\mathfrak{K}_j(\tau)\right), \ \forall \zeta \in [0,1],$$
and $\mathfrak{h}_1(\zeta) = \frac{1}{\Gamma(\delta_1)}(1-\zeta)^{\delta_1-\alpha_0-1}(1-(1-\zeta)^{\alpha_0})$, *for all* $\zeta \in [0,1]$.

(3) $\mathfrak{G}_1(t,\zeta) \ge t^{\delta_1-1}\mathfrak{J}_1(\zeta)$ *for all* $(t,\zeta) \in [0,1] \times [0,1]$;

(4) $\mathfrak{G}_2(t,\zeta) \le \mathfrak{J}_2(\zeta)$, *for all* $(t,\zeta) \in [0,1] \times [0,1]$, *where*
$$\mathfrak{J}_2(\zeta) = \frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)} \sum_{j=1}^{n} \int_0^1 g_{2j}(\tau,\zeta)\,d\mathfrak{H}_j(\tau), \ \forall \zeta \in [0,1];$$

(5) $\mathfrak{G}_2(t,\zeta) = t^{\delta_1-1}\mathfrak{J}_2(\zeta)$ *for all* $(t,\zeta) \in [0,1] \times [0,1]$;

(6) $\mathfrak{G}_3(t,\zeta) \le \mathfrak{J}_3(\zeta)$ *for all* $(t,\zeta) \in [0,1] \times [0,1]$, *where*
$$\mathfrak{J}_3(\zeta) = \frac{\Gamma(\delta_1)}{\Delta\Gamma(\delta_1-\alpha_0)} \sum_{j=1}^{m} \int_0^1 g_{1j}(\tau,\zeta)\,d\mathfrak{K}_j(\tau), \ \forall \zeta \in [0,1];$$

(7) $\mathcal{G}_3(t,\zeta) = t^{\delta_2-1}\mathfrak{J}_3(\zeta)$ *for all* $(t,\zeta) \in [0,1] \times [0,1]$;

(8) $\mathfrak{G}_4(t,\zeta) \le \mathfrak{J}_4(\zeta)$ *for all* $(t,\zeta) \in [0,1] \times [0,1]$, *where*
$$\mathfrak{J}_4(\zeta) = \mathfrak{h}_2(\zeta) + \frac{\Delta_2}{\Delta}\left(\sum_{j=1}^{n} \int_0^1 g_{2j}(\tau,\zeta)\,dH_j(\tau)\right), \ \forall \zeta \in [0,1],$$
and $\mathfrak{h}_2(\zeta) = \frac{1}{\Gamma(\delta_2)}(1-\zeta)^{\delta_2-\beta_0-1}(1-(1-\zeta)^{\beta_0})$, *for all* $\zeta \in [0,1]$.

(9) $\mathfrak{G}_4(t,\zeta) \ge t^{\delta_2-1}\mathfrak{J}_4(\zeta)$, *for all* $(t,\zeta) \in [0,1] \times [0,1]$.

Lemma 3. *We suppose that* $\Delta > 0$, $\mathfrak{H}_i$, $i = 1, \dots, n$, $\mathfrak{K}_j$, $j = 1, \dots, m$ *are nondecreasing functions, and* $\widetilde{\mathrm{h}}, \widetilde{\mathrm{k}} \in C([0,1];\mathbb{R}_+)$. *Therefore, the solution* $(\mathrm{u}(t), \mathrm{v}(t))$, $t \in [0,1]$ *of problem (4) and (5) (given by (6)) satisfies the inequalities* $\mathrm{u}(t) \ge 0$, $\mathrm{v}(t) \ge 0$, $\mathrm{u}(t) \ge t^{\delta_1-1}\mathrm{u}(\nu)$, $\mathrm{v}(t) \ge t^{\delta_2-1}\mathrm{v}(\nu)$ *for all* $t, \nu \in [0,1]$.

Proof. Under the assumptions of this lemma, by using relations (6) and Lemma 2, we find that $\mathrm{u}(t) \ge 0$ and $\mathrm{v}(t) \ge 0$ for all $t \in [0,1]$. In addition, for all t, $\nu \in [0,1]$, we obtain the following inequalities:

$$\begin{aligned}
\mathrm{u}(t) &\ge t^{\delta_1-1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{\mathrm{h}}(\zeta))\,d\zeta + \int_0^1 \mathfrak{J}_2(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{\mathrm{k}}(\zeta))\,d\zeta\right) \\
&\ge t^{\delta_1-1}\left(\int_0^1 \mathfrak{G}_1(\nu,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{\mathrm{h}}(\zeta))\,d\zeta + \int_0^1 \mathfrak{G}_2(\nu,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{\mathrm{k}}(\zeta))\,d\zeta\right) \\
&= t^{\delta_1-1}\mathrm{u}(\nu), \\
\mathrm{v}(t) &\ge t^{\delta_2-1}\left(\int_0^1 \mathfrak{J}_3(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{\mathrm{h}}(\zeta))\,d\zeta + \int_0^1 \mathcal{J}_4(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{\mathrm{k}}(\zeta))\,d\zeta\right) \\
&\ge t^{\delta_2-1}\left(\int_0^1 \mathfrak{G}_3(\nu,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{\mathrm{h}}(\zeta))\,d\zeta + \int_0^1 \mathfrak{G}_4(\nu,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{\mathrm{k}}(\zeta))\,d\zeta\right) \\
&= t^{\delta_2-1}\mathrm{v}(\nu).
\end{aligned}$$

□

3. Main Results

In this section, we study the existence and nonexistence of positive solutions for problem (1) and (2) under some conditions on $\mathfrak{a}, \mathfrak{b}, \mathfrak{f}$, and $\mathfrak{g}$, when the positive parameters $\mathfrak{c}_0$ and $\mathfrak{d}_0$ belong to some intervals.

We now give the assumptions that we will use in the next part.

$(K1)$ $\gamma_1,\ \gamma_2 \in (0,1]$, $\delta_1 \in (p-1,p]$, $\delta_2 \in (q-1,q]$, $p,\ q \in \mathbb{N}$, $p,\ q \geq 3$, $n,\ m \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ for all $j = 0,1,\ldots,n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\beta_j \in \mathbb{R}$ for all $j = 0,1,\ldots,m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, $\mathfrak{c}_0 > 0$ and $\mathfrak{d}_0 > 0$, $\mathfrak{H}_i$, $i = 1,\ldots,n$ and $\mathfrak{K}_j$, $j = 1,\ldots,m$ are nondecreasing functions, and $\Delta > 0$.

$(K2)$ The functions $\mathfrak{a},\ \mathfrak{b} : [0,1] \to \mathbb{R}_+$ are continuous, and there exist $\tau_1, \tau_2 \in (0,1)$ such that $\mathfrak{a}(\tau_1) > 0$, $\mathfrak{b}(\tau_2) > 0$.

$(K3)$ The functions $\mathfrak{f},\ \mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, and there exists $\mathfrak{e}_0 > 0$ such that $\mathfrak{f}(z) < \frac{\mathfrak{e}_0^{\varrho_1-1}}{L}$, $\mathfrak{g}(z) < \frac{\mathfrak{e}_0^{\varrho_2-1}}{L}$ for all $z \in [0,\mathfrak{e}_0]$, where

$$L = \max\left\{ \frac{2^{\varrho_1-1}\Xi_1}{\Gamma(\gamma_1+1)} \left(\int_0^1 \mathfrak{I}_i(\zeta)\zeta^{\gamma_1(\rho_1-1)}\,d\zeta \right)^{\varrho_1-1},\ i \in \{1,3\}; \right.$$
$$\left. \frac{2^{\varrho_2-1}\Xi_2}{\Gamma(\gamma_2+1)} \left(\int_0^1 \mathfrak{I}_j(\zeta)\zeta^{\gamma_2(\rho_2-1)}\,d\zeta \right)^{\varrho_2-1},\ j \in \{2,4\} \right\},$$

with $\Xi_1 = \sup_{\tau\in[0,1]} \mathfrak{a}(\tau)$, $\Xi_2 = \sup_{\tau\in[0,1]} \mathfrak{b}(\tau)$.

$(K4)$ The functions $\mathfrak{f},\ \mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy the conditions $\lim_{w\to\infty} \frac{\mathfrak{f}(w)}{w^{\varrho_1-1}} = \infty$ and $\lim_{w\to\infty} \frac{\mathfrak{g}(w)}{w^{\varrho_2-1}} = \infty$.

By assumptions $(K1)$ and $(K2)$ and Lemma 2, we obtain that the constant L from assumption $(K3)$ is positive.

Now, we consider the following system of fractional differential equations:

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}x(t))) = 0, & t \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}y(t))) = 0, & t \in (0,1), \end{cases} \tag{8}$$

subject to the coupled boundary conditions

$$\begin{cases} x^{(j)}(0) = 0,\ j = 0,\ldots,p-2;\ D_{0+}^{\delta_1}x(0) = 0,\ D_{0+}^{\alpha_0}x(1) = \sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}y(\tau)\,d\mathfrak{H}_j(\tau) + \mathfrak{c}_0, \\ y^{(j)}(0) = 0,\ j = 0,\ldots,q-2;\ D_{0+}^{\delta_2}y(0) = 0,\ D_{0+}^{\beta_0}y(1) = \sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}x(\tau)\,d\mathfrak{K}_j(\tau) + \mathfrak{d}_0. \end{cases} \tag{9}$$

Lemma 4. *Under assumption $(K1)$, the unique solution $(x,y) \in (C[0,1])^2$ of problem (8) and (9) is*

$$x(t) = \frac{t^{\delta_1-1}}{\Delta}\left(\mathfrak{c}_0\frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} + \mathfrak{d}_0\Delta_1\right),\ y(t) = \frac{t^{\delta_2-1}}{\Delta}\left(\mathfrak{c}_0\Delta_2 + \mathfrak{d}_0\frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)}\right),\ t \in [0,1], \tag{10}$$

which satisfies the conditions $x(t) > 0$ and $y(t) > 0$ for all $t \in (0,1]$.

Proof. We note that $\varphi_{\varrho_1}(D_{0+}^{\delta_1}x(t)) = \phi(t)$, $\varphi_{\varrho_2}(D_{0+}^{\delta_2}y(t)) = \psi(t)$. Therefore, the problem (8) and (9) is equivalent to the following three problems:

$$(I)\begin{cases} D_{0+}^{\gamma_1}\phi(t) = 0, \\ \phi(0) = 0, \end{cases} \qquad (II)\begin{cases} D_{0+}^{\gamma_2}\psi(t) = 0, \\ \psi(0) = 0, \end{cases}$$

and

$$(III)\begin{cases} \begin{cases} D_{0+}^{\delta_1}x(t) = \varphi_{\rho_1}(\phi(t)), & t \in (0,1), \\ D_{0+}^{\delta_2}y(t) = \varphi_{\rho_2}(\psi(t)), & t \in (0,1), \end{cases} \quad (III)_1 \\ \text{with} \\ \begin{cases} x^{(j)}(0) = 0, \ j = 0,\dots,p-2, \ D_{0+}^{\alpha_0}x(1) = \sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}y(\tau)\,d\mathfrak{H}_j(\tau) + \mathfrak{c}_0, \\ y^{(j)}(0) = 0, \ j = 0,\dots,q-2, \ D_{0+}^{\beta_0}y(1) = \sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}x(\tau)\,d\mathfrak{K}_j(\tau) + \mathfrak{d}_0. \end{cases} \quad (III)_2 \end{cases}$$

Problem (I) has the solution $\phi(t) = 0$ for all $t \in [0,1]$, and problem (II) has the solution $\psi(t) = 0$ for all $t \in [0,1]$. Therefore, problem (III) can be written as

$$\begin{cases} D_{0+}^{\delta_1}x(t) = 0, & t \in (0,1), \\ D_{0+}^{\delta_2}y(t) = 0, & t \in (0,1), \end{cases} \tag{11}$$

supplemented with the boundary conditions $(III)_2$. The solutions of system (11) are

$$\begin{array}{l} x(t) = a_1 t^{\delta_1-1} + a_2 t^{\delta_1-2} + \cdots + a_p t^{\delta_1-p}, \ t \in [0,1], \\ y(t) = b_1 t^{\delta_2-1} + b_2 t^{\delta_2-2} + \cdots + b_q t^{\delta_2-q}, \ t \in [0,1], \end{array} \tag{12}$$

with $a_1,\dots,a_p,b_1,\dots,b_q \in \mathbb{R}$. By using the boundary conditions $x^{(j)}(0) = 0, \ j = 0,\dots,p-2$, $y^{(j)}(0) = 0, \ j = 0,\dots,q-2$ (from $(III)_2$), we obtain $a_2 = \cdots = a_p = 0$ and $b_2 = \cdots = b_q = 0$. Then, the functions in Equation (12) become $x(t) = a_1 t^{\delta_1-1}$, $t \in [0,1]$, $y(t) = b_1 t^{\delta_2-1}$, $t \in [0,1]$. For these functions, we find

$$\begin{array}{l} D_{0+}^{\alpha_0}x(t) = a_1 \dfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)} t^{\delta_1-\alpha_0-1}, \ D_{0+}^{\beta_0}y(t) = b_1 \dfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} t^{\delta_2-\beta_0-1}, \\ D_{0+}^{\alpha_j}y(t) = b_1 \dfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\alpha_j)} t^{\delta_2-\alpha_j-1}, \ D_{0+}^{\beta_j}x(t) = a_1 \dfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\beta_j)} t^{\delta_1-\beta_j-1}. \end{array}$$

Therefore, by now using the above fractional derivatives and the conditions $D_{0+}^{\alpha_0}x(1) = \sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}y(\tau)\,d\mathfrak{H}_j(\tau) + \mathfrak{c}_0$ and $D_{0+}^{\beta_0}y(1) = \sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}x(\tau)\,d\mathfrak{K}_j(\tau) + \mathfrak{d}_0$ (from $(III)_2$), we deduce the following system for a_1 and b_1:

$$\begin{cases} a_1 \dfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)} = \displaystyle\sum_{j=1}^{n}\int_0^1 b_1 \frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\alpha_j)} \tau^{\delta_2-\alpha_j-1}\,d\mathfrak{H}_j(\tau) + \mathfrak{c}_0, \\ b_1 \dfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} = \displaystyle\sum_{j=1}^{m}\int_0^1 a_1 \frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\beta_j)} \tau^{\delta_1-\beta_j-1}\,d\mathfrak{K}_j(\tau) + \mathfrak{d}_0, \end{cases}$$

or equivalently

$$\begin{cases} a_1 \dfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)} = b_1\Delta_1 + \mathfrak{c}_0, \\ b_1 \dfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} = a_1\Delta_2 + \mathfrak{d}_0. \end{cases}$$

The determinant of the above system in the unknown a_1 and b_1 is

$$\begin{vmatrix} \dfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)} & -\Delta_1 \\ -\Delta_2 & \dfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} \end{vmatrix} = \frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1-\alpha_0)\Gamma(\delta_2-\beta_0)} - \Delta_1\Delta_2 = \Delta.$$

Then, we obtain

$$a_1 = \frac{1}{\Delta}\left(\mathfrak{c}_0 \frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} + \mathfrak{d}_0\Delta_1\right), \ b_1 = \frac{1}{\Delta}\left(\mathfrak{d}_0 \frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)} + \mathfrak{c}_0\Delta_2\right).$$

Therefore, we deduce the solution $(x(t), y(t))$ of problem (8) and (9) presented in (10). By assumption $(K1)$, we find that $x(t) > 0$ and $y(t) > 0$ for all $t \in (0,1]$. $\square$

We use the functions $x(t)$ and $y(t)$, $t \in [0,1]$ (given by (10)), and we make a change of unknown functions for our boundary value problem (1) and (2) such that the new boundary conditions have no positive parameters. For a solution (u,v) of problem (1) and (2), we define the functions $h(t)$ and $k(t)$, $t \in [0,1]$ by

$$\begin{aligned} h(t) &= \mathrm{u}(t) - x(t) = \mathrm{u}(t) - \frac{t^{\delta_1-1}}{\Delta}\left(\mathfrak{c}_0 \frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} + \mathfrak{d}_0\Delta_1\right), \ t \in [0,1], \\ k(t) &= \mathrm{v}(t) - y(t) = \mathrm{v}(t) - \frac{t^{\delta_2-1}}{\Delta}\left(\mathfrak{c}_0\Delta_2 + \mathfrak{d}_0 \frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)}\right), \ t \in [0,1]. \end{aligned}$$

Then, problem (1) and (2) can be equivalently written as the system of fractional differential equations

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}h(t))) + \mathfrak{a}(t)\mathfrak{f}(k(t)+y(t)) = 0, \ t \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}k(t))) + \mathfrak{b}(t)\mathfrak{g}(h(t)+x(t)) = 0, \ t \in (0,1), \end{cases} \tag{13}$$

with the boundary conditions without parameters

$$\begin{cases} h^{(j)}(0) = 0, \ j = 0,\ldots,p-2; \ D_{0+}^{\delta_1}h(0) = 0, \ D_{0+}^{\alpha_0}h(1) = \displaystyle\sum_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}k(\tau)\,d\mathfrak{H}_j(\tau), \\ k^{(j)}(0) = 0, \ j = 0,\ldots,q-2; \ D_{0+}^{\delta_2}k(0) = 0, \ D_{0+}^{\beta_0}k(1) = \displaystyle\sum_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}h(\tau)\,d\mathfrak{K}_j(\tau). \end{cases} \tag{14}$$

Using the Green functions $\mathfrak{G}_i$, $i = 1,\ldots,4$ and Lemma 1, a pair of functions (h,k) is a solution of problem (13) and (14) if and only if (h,k) is a solution of the system of integral equations

$$\begin{aligned} h(t) &= \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))))\,d\zeta \\ &\quad + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))))\,d\zeta, \ t \in [0,1], \\ k(t) &= \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))))\,d\zeta \\ &\quad + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))))\,d\zeta, \ t \in [0,1]. \end{aligned} \tag{15}$$

We consider the Banach space $\mathcal{X} = C[0,1]$ with the supremum norm $\|z\| = \sup_{\tau\in[0,1]}|z(\tau)|$ for $z \in \mathcal{X}$, and the Banach space $\mathcal{Y} = \mathcal{X}\times\mathcal{X}$ with the norm $\|(h,k)\|_{\mathcal{Y}} = \max\{\|h\|,\|k\|\}$ for $(h,k) \in \mathcal{Y}$. We define the set $\mathcal{V} = \{(h,k) \in \mathcal{Y}, \ 0 \le h(t) \le \mathfrak{e}_0, \ 0 \le k(t) \le \mathfrak{e}_0, \ \forall t \in [0,1]\}$. We also define the operator $\mathcal{S} : \mathcal{V} \to \mathcal{Y}$, $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$,

$$\begin{aligned} \mathcal{S}_1(h,k)(t) &= \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))))\,d\zeta \\ &\quad + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))))\,d\zeta, \ t \in [0,1], \\ \mathcal{S}_2(h,k)(t) &= \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))))\,d\zeta \\ &\quad + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))))\,d\zeta, \ t \in [0,1], \end{aligned}$$

for $(h,k) \in \mathcal{V}$. We easily see that (h,k) is a solution of system (15) if and only if (h,k) is a fixed point of operator $\mathcal{S}$. Therefore, our next task is the detection of the fixed points of operator $\mathcal{S}$. The first result is the following existence theorem for problem (1) and (2):

Theorem 1. *We assume that assumptions $(K1)-(K3)$ are satisfied. Therefore, there exist $\mathfrak{c}_1 > 0$ and $\mathfrak{d}_1 > 0$ such that for any $\mathfrak{c}_0 \in (0,\mathfrak{c}_1]$ and $\mathfrak{d}_0 \in (0,\mathfrak{d}_1]$, the problem (1) and (2) has at least one positive solution.*

Proof. By assumption $(K3)$ we deduce that there exist $\mathfrak{s}_0 > 0$ and $\mathfrak{t}_0 > 0$ such that $\mathfrak{f}(w) \le \frac{\mathfrak{e}_0^{\varrho_1-1}}{L}$ for all $w \in [0,\mathfrak{e}_0+\mathfrak{s}_0]$, and $\mathfrak{g}(w) \le \frac{\mathfrak{e}_0^{\varrho_2-1}}{L}$ for all $w \in [0,\mathfrak{e}_0+\mathfrak{t}_0]$. We define now $\mathfrak{c}_1$ and $\mathfrak{d}_1$ as follows:

- If $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$, then
$$\mathfrak{c}_1 = \min\left\{\frac{\mathfrak{s}_0\Delta}{2\Delta_2}, \frac{\mathfrak{t}_0\Delta\Gamma(\delta_2-\beta_0)}{2\Gamma(\delta_2)}\right\},\ \ \mathfrak{d}_1 = \min\left\{\frac{\mathfrak{s}_0\Delta\Gamma(\delta_1-\alpha_0)}{2\Gamma(\delta_1)}, \frac{\mathfrak{t}_0\Delta}{2\Delta_1}\right\}.$$
- If $\Delta_1 = 0$ and $\Delta_2 \neq 0$, then
$$\mathfrak{c}_1 = \min\left\{\frac{\mathfrak{s}_0\Delta}{2\Delta_2}, \frac{\mathfrak{t}_0\Delta\Gamma(\delta_2-\beta_0)}{\Gamma(\delta_2)}\right\},\ \ \mathfrak{d}_1 = \frac{\mathfrak{s}_0\Delta\Gamma(\delta_1-\alpha_0)}{2\Gamma(\delta_1)}.$$
- If $\Delta_1 \neq 0$ and $\Delta_2 = 0$, then
$$\mathfrak{c}_1 = \frac{\mathfrak{t}_0\Delta\Gamma(\delta_2-\beta_0)}{2\Gamma(\delta_2)},\ \ \mathfrak{d}_1 = \min\left\{\frac{\mathfrak{s}_0\Delta\Gamma(\delta_1-\alpha_0)}{\Gamma(\delta_1)}, \frac{\mathfrak{t}_0\Delta}{2\Delta_1}\right\}.$$
- If $\Delta_1 = 0$ and $\Delta_2 = 0$, then
$$\mathfrak{c}_1 = \frac{\mathfrak{t}_0\Delta\Gamma(\delta_2-\beta_0)}{\Gamma(\delta_2)},\ \ \mathfrak{d}_1 = \frac{\mathfrak{s}_0\Delta\Gamma(\delta_1-\alpha_0)}{\Gamma(\delta_1)}.$$

Let $\mathfrak{c}_0 \in (0,\mathfrak{c}_1]$ and $\mathfrak{d}_0 \in (0,\mathfrak{d}_1]$. Then, for $(h,k) \in \mathcal{V}$ and $\zeta \in [0,1]$, we have

$$k(\zeta)+y(\zeta) \le \mathfrak{e}_0 + \tfrac{1}{\Delta}\left(\mathfrak{c}_0\Delta_2 + \mathfrak{d}_0\tfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)}\right) \le \mathfrak{e}_0 + \tfrac{1}{\Delta}\left(\mathfrak{c}_1\Delta_2 + \mathfrak{d}_1\tfrac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)}\right) \le \mathfrak{e}_0+\mathfrak{s}_0,$$
$$h(\zeta)+x(\zeta) \le \mathfrak{e}_0 + \tfrac{1}{\Delta}\left(\mathfrak{c}_0\tfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} + \mathfrak{d}_0\Delta_1\right) \le \mathfrak{e}_0 + \tfrac{1}{\Delta}\left(\mathfrak{c}_1\tfrac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)} + \mathfrak{d}_1\Delta_1\right) \le \mathfrak{e}_0+\mathfrak{t}_0,$$

and so
$$\mathfrak{f}(k(\zeta)+y(\zeta)) \le \frac{\mathfrak{e}_0^{\varrho_1-1}}{L},\ \ \mathfrak{g}(h(\zeta)+x(\zeta)) \le \frac{\mathfrak{e}_0^{\varrho_2-1}}{L}. \tag{16}$$

By using Lemma 3, we deduce that $\mathcal{S}_i(h,k)(t) \ge 0$, $i=1,2$ for all $t \in [0,1]$ and $(h,k) \in \mathcal{V}$. By inequalities (16), for all $(h,k) \in \mathcal{V}$, we obtain

$$\begin{aligned}
& I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))) = \frac{1}{\Gamma(\gamma_1)}\int_0^\zeta (\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\mathfrak{f}(k(\tau)+y(\tau))\,d\tau \\
& \le \frac{\mathfrak{e}_0^{\varrho_1-1}}{L\Gamma(\gamma_1)}\int_0^\zeta (\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\,d\tau \le \frac{\Xi_1\mathfrak{e}_0^{\varrho_1-1}}{L\Gamma(\gamma_1)}\int_0^\zeta (\zeta-\tau)^{\gamma_1-1}\,d\tau \\
& = \frac{\Xi_1\mathfrak{e}_0^{\varrho_1-1}\zeta^{\gamma_1}}{L\Gamma(\gamma_1+1)},\ \ \forall\,\zeta \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
& I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))) = \frac{1}{\Gamma(\gamma_2)}\int_0^\zeta (\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\mathfrak{g}(h(\tau)+x(\tau))\,d\tau \\
& \le \frac{\mathfrak{e}_0^{\varrho_2-1}}{L\Gamma(\gamma_2)}\int_0^\zeta (\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\,d\tau \le \frac{\Xi_2\mathfrak{e}_0^{\varrho_2-1}}{L\Gamma(\gamma_2)}\int_0^\zeta (\zeta-\tau)^{\gamma_2-1}\,d\tau \\
& = \frac{\Xi_2\mathfrak{e}_0^{\varrho_2-1}\zeta^{\gamma_2}}{L\Gamma(\gamma_2+1)},\ \ \forall\,\zeta \in [0,1].
\end{aligned}$$

Then, by Lemma 2 and the definition of L from $(K3)$, we find

$$\begin{aligned}\mathcal{S}_1(h,k)(t) &\le \int_0^1 \mathfrak{J}_1(\zeta)\varphi_{\rho_1}\left(\frac{\Xi_1\mathfrak{e}_0^{\varrho_1-1}\zeta^{\gamma_1}}{L\Gamma(\gamma_1+1)}\right)d\zeta + \int_0^1 \mathfrak{J}_2(\zeta)\varphi_{\rho_2}\left(\frac{\Xi_2\mathfrak{e}_0^{\varrho_2-1}\zeta^{\gamma_2}}{L\Gamma(\gamma_2+1)}\right)d\zeta \\ &= \left(\frac{\Xi_1\mathfrak{e}_0^{\varrho_1-1}}{L\Gamma(\gamma_1+1)}\right)^{\rho_1-1}\int_0^1 \mathfrak{J}_1(\zeta)\zeta^{\gamma_1(\rho_1-1)}\,d\zeta \\ &\quad+\left(\frac{\Xi_2\mathfrak{e}_0^{\varrho_2-1}}{L\Gamma(\gamma_2+1)}\right)^{\rho_2-1}\int_0^1 \mathfrak{J}_2(\zeta)\zeta^{\gamma_2(\rho_2-1)}\,d\zeta \le \frac{\mathfrak{e}_0}{2}+\frac{\mathfrak{e}_0}{2}=\mathfrak{e}_0,\ \ \forall\, t\in[0,1],\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_2(h,k)(t) &\le \int_0^1 \mathfrak{J}_3(\zeta)\varphi_{\rho_1}\left(\frac{\Xi_1\mathfrak{e}_0^{\varrho_1-1}\zeta^{\gamma_1}}{L\Gamma(\gamma_1+1)}\right)d\zeta + \int_0^1 \mathfrak{J}_4(\zeta)\varphi_{\rho_2}\left(\frac{\Xi_2\mathfrak{e}_0^{\varrho_2-1}\zeta^{\gamma_2}}{L\Gamma(\gamma_2+1)}\right)d\zeta \\ &= \left(\frac{\Xi_1\mathfrak{e}_0^{\varrho_1-1}}{L\Gamma(\gamma_1+1)}\right)^{\rho_1-1}\int_0^1 \mathfrak{J}_3(\zeta)\zeta^{\gamma_1(\rho_1-1)}\,d\zeta \\ &\quad+\left(\frac{\Xi_2\mathfrak{e}_0^{\varrho_2-1}}{L\Gamma(\gamma_2+1)}\right)^{\rho_2-1}\int_0^1 \mathfrak{J}_4(\zeta)\zeta^{\gamma_2(\rho_2-1)}\,d\zeta \le \frac{\mathfrak{e}_0}{2}+\frac{\mathfrak{e}_0}{2}=\mathfrak{e}_0,\ \ \forall\, t\in[0,1].\end{aligned}$$

Therefore, we find that $\mathcal{S}(\mathcal{V})\subset\mathcal{V}$. By using a standard method, we conclude that $\mathcal{S}$ is a completely continuous operator. Therefore, by the Schauder fixed point theorem, we deduce that $\mathcal{S}$ has a fixed point $(h,k)\in\mathcal{V}$, which is a non-negative solution for problem (15), or equivalently, for problem (13) and (14). Hence, (u,v), where $\mathrm{u}(t)=h(t)+x(t)$ and $\mathrm{v}(t)=k(t)+y(t)$ for all $t\in[0,1]$, is a positive solution of problem (1) and (2). This solution (u,v) satisfies the conditions $\frac{t^{\delta_1-1}}{\Delta}(\mathfrak{c}_0\frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)}+\mathfrak{d}_0\Delta_1)\le \mathrm{u}(t)\le \frac{t^{\delta_1-1}}{\Delta}(\mathfrak{c}_0\frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\beta_0)}+\mathfrak{d}_0\Delta_1)+\mathfrak{e}_0$ and $\frac{t^{\delta_2-1}}{\Delta}(\mathfrak{c}_0\Delta_2+\mathfrak{d}_0\frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)})\le \mathrm{v}(t)\le \frac{t^{\delta_2-1}}{\Delta}(\mathfrak{c}_0\Delta_2+\mathfrak{d}_0\frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)})+\mathfrak{e}_0$ for all $t\in[0,1]$. □

The second result is the following nonexistence theorem for the boundary value problem (1) and (2).

Theorem 2. *We assume that assumptions $(K1)$, $(K2)$, and $(K4)$ are satisfied. Then, there exist $\mathfrak{c}_2>0$ and $\mathfrak{d}_2>0$ such that for any $\mathfrak{c}_0\ge\mathfrak{c}_2$ and $\mathfrak{d}_0\ge\mathfrak{d}_2$, the problem (1) and (2) has no positive solution.*

Proof. By assumption $(K2)$, there exist $[\eta_1,\eta_2]\subset(0,1)$, $\eta_1<\eta_2$ such that $\tau_1,\tau_2\in(\eta_1,\eta_2)$, and then

$$\Lambda_1=\int_{\eta_1}^{\eta_2}\mathfrak{J}_1(\zeta)\left(\int_{\eta_1}^{\zeta}\mathfrak{a}(\tau)(\zeta-\tau)^{\gamma_1-1}\,d\tau\right)^{\rho_1-1}d\zeta>0,$$
$$\Lambda_4=\int_{\eta_1}^{\eta_2}\mathfrak{J}_4(\zeta)\left(\int_{\eta_1}^{\zeta}\mathfrak{b}(\tau)(\zeta-\tau)^{\gamma_2-1}\,d\tau\right)^{\rho_2-1}d\zeta>0.$$

We define the number

$$R_0=\max\left\{\frac{2^{\varrho_1-1}\Gamma(\gamma_1)}{\eta_1^{(\delta_1+\delta_2-2)(\varrho_1-1)}\Lambda_1^{\varrho_1-1}},\ \frac{2^{\varrho_2-1}\Gamma(\gamma_2)}{\eta_1^{(\delta_1+\delta_2-2)(\varrho_2-1)}\Lambda_4^{\varrho_2-1}}\right\}.$$

By using $(K4)$, for R_0 defined above, we obtain that there exists $L_0>0$ such that $\mathfrak{f}(w)\ge R_0 w^{\varrho_1-1}$ and $\mathfrak{g}(w)\ge R_0 w^{\varrho_2-1}$ for all $w\ge L_0$. We define now $\mathfrak{c}_2$ and $\mathfrak{d}_2$ as follows:

- If $\Delta_1\neq0$ and $\Delta_2\neq0$, then

$$\mathfrak{c}_2=\max\left\{\frac{L_0\Delta\Gamma(\delta_2-\beta_0)}{2\eta_1^{\delta_1-1}\Gamma(\delta_2)},\ \frac{L_0\Delta}{2\eta_1^{\delta_2-1}\Delta_2}\right\},\ \ \mathfrak{d}_2=\max\left\{\frac{L_0\Delta}{2\eta_1^{\delta_1-1}\Delta_1},\ \frac{L_0\Delta\Gamma(\delta_1-\alpha_0)}{2\eta_1^{\delta_2-1}\Gamma(\delta_1)}\right\}.$$

- If $\Delta_1 = 0$ and $\Delta_2 \neq 0$, then

$$\mathfrak{c}_2 = \max\left\{ \frac{L_0 \Delta \Gamma(\delta_2 - \beta_0)}{\eta_1^{\delta_1 - 1}\Gamma(\delta_2)}, \frac{L_0 \Delta}{2\eta_1^{\delta_2 - 1}\Delta_2} \right\}, \quad \mathfrak{d}_2 = \frac{L_0 \Delta \Gamma(\delta_1 - \alpha_0)}{2\eta_1^{\delta_2 - 1}\Gamma(\delta_1)}.$$

- If $\Delta_1 \neq 0$ and $\Delta_2 = 0$, then

$$\mathfrak{c}_2 = \frac{L_0 \Delta \Gamma(\delta_2 - \beta_0)}{2\eta_1^{\delta_1 - 1}\Gamma(\delta_2)}, \quad \mathfrak{d}_2 = \max\left\{ \frac{L_0 \Delta}{2\eta_1^{\delta_1 - 1}\Delta_1}, \frac{L_0 \Delta \Gamma(\delta_1 - \alpha_0)}{\eta_1^{\delta_2 - 1}\Gamma(\delta_1)} \right\}.$$

- If $\Delta_1 = 0$ and $\Delta_2 = 0$, then

$$\mathfrak{c}_2 = \frac{L_0 \Delta \Gamma(\delta_2 - \beta_0)}{\eta_1^{\delta_1 - 1}\Gamma(\delta_2)}, \quad \mathfrak{d}_2 = \frac{L_0 \Delta \Gamma(\delta_1 - \alpha_0)}{\eta_1^{\delta_2 - 1}\Gamma(\delta_1)}.$$

Let $\mathfrak{c}_0 \geq \mathfrak{c}_2$ and $\mathfrak{d}_0 \geq \mathfrak{d}_2$. We assume that (u, v) is a positive solution of (1) and (2). Then, the pair (h, k), where $h(t) = \mathrm{u}(t) - x(t)$, $k(t) = \mathrm{v}(t) - y(t)$, $t \in [0, 1]$, with x and y given by (10), is a solution of problem (13) and (14), or equivalently, of system (15). By using Lemma 3, we find that $h(t) \geq t^{\delta_1 - 1}\|h\|$, $k(t) \geq t^{\delta_2 - 1}\|k\|$ for all $t \in [0, 1]$. Then, $\inf_{s \in [\eta_1, \eta_2]} h(s) \geq \eta_1^{\delta_1 - 1}\|h\|$, $\inf_{s \in [\eta_1, \eta_2]} k(s) \geq \eta_1^{\delta_2 - 1}\|k\|$. By the definition of the functions x and y, we obtain

$$\begin{aligned}
\inf_{s \in [\eta_1, \eta_2]} x(s) &= \frac{\eta_1^{\delta_1 - 1}}{\Delta}\left(\mathfrak{c}_0 \frac{\Gamma(\delta_2)}{\Gamma(\delta_2 - \beta_0)} + \mathfrak{d}_0 \Delta_1 \right) = \eta_1^{\delta_1 - 1}\|x\|, \\
\inf_{s \in [\eta_1, \eta_2]} y(s) &= \frac{\eta_1^{\delta_2 - 1}}{\Delta}\left(\mathfrak{c}_0 \Delta_2 + \mathfrak{d}_0 \frac{\Gamma(\delta_1)}{\Gamma(\delta_1 - \alpha_0)} \right) = \eta_1^{\delta_2 - 1}\|y\|.
\end{aligned}$$

Hence, we deduce

$$\begin{aligned}
&\inf_{s \in [\eta_1, \eta_2]} (h(s) + x(s)) \geq \inf_{s \in [\eta_1, \eta_2]} h(s) + \inf_{s \in [\eta_1, \eta_2]} x(s) \geq \eta_1^{\delta_1 - 1}\|h\| + \eta_1^{\delta_1 - 1}\|x\| \\
&\quad = \eta_1^{\delta_1 - 1}(\|h\| + \|x\|) \geq \eta_1^{\delta_1 - 1}\|h + x\|, \\
&\inf_{s \in [\eta_1, \eta_2]} (k(s) + y(s)) \geq \inf_{s \in [\eta_1, \eta_2]} k(s) + \inf_{s \in [\eta_1, \eta_2]} y(s) \geq \eta_1^{\delta_2 - 1}\|k\| + \eta_1^{\delta_2 - 1}\|y\| \\
&\quad = \eta_1^{\delta_2 - 1}(\|k\| + \|y\|) \geq \eta_1^{\delta_2 - 1}\|k + y\|.
\end{aligned}$$

In addition we have

$$\begin{aligned}
&\inf_{s \in [\eta_1, \eta_2]} (h(s) + x(s)) \geq \eta_1^{\delta_1 - 1}\|x\| = \eta_1^{\delta_1 - 1}\frac{1}{\Delta}\left(\mathfrak{c}_0 \frac{\Gamma(\delta_2)}{\Gamma(\delta_2 - \beta_0)} + \mathfrak{d}_0 \Delta_1 \right) \\
&\geq \eta_1^{\delta_1 - 1}\frac{1}{\Delta}\left(\mathfrak{c}_2 \frac{\Gamma(\delta_2)}{\Gamma(\delta_2 - \beta_0)} + \mathfrak{d}_2 \Delta_1 \right) \geq L_0,
\end{aligned}$$

$$\begin{aligned}
&\inf_{s \in [\eta_1, \eta_2]} (k(s) + y(s)) \geq \eta_1^{\delta_2 - 1}\|y\| = \eta_1^{\delta_2 - 1}\frac{1}{\Delta}\left(\mathfrak{c}_0 \Delta_2 + \mathfrak{d}_0 \frac{\Gamma(\delta_1)}{\Gamma(\delta_1 - \alpha_0)} \right) \\
&\geq \eta_1^{\delta_2 - 1}\frac{1}{\Delta}\left(\mathfrak{c}_2 \Delta_2 + \mathfrak{d}_2 \frac{\Gamma(\delta_1)}{\Gamma(\delta_1 - \alpha_0)} \right) \geq L_0.
\end{aligned}$$

By using Lemma 3 and the above inequalities we find

$$
\begin{aligned}
& I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))) \\
& \geq \frac{1}{\Gamma(\gamma_1)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\mathfrak{f}(k(\tau)+y(\tau))\,d\tau \\
& \geq \frac{R_0}{\Gamma(\gamma_1)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)(k(\tau)+y(\tau))^{\varrho_1-1}\,d\tau \\
& \geq \frac{R_0}{\Gamma(\gamma_1)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\left(\inf_{\tau\in[\eta_1,\eta_2]}(k(\tau)+y(\tau))\right)^{\varrho_1-1}\,d\tau \\
& \geq \frac{R_0L_0^{\varrho_1-1}}{\Gamma(\gamma_1)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\,d\tau,\ \ \forall\,\zeta\in[\eta_1,\eta_2],
\end{aligned}
$$

and then

$$
\begin{aligned}
h(\eta_1) & \geq \int_0^1 \eta_1^{\delta_1-1}\mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))))\,d\zeta \\
& \geq \int_{\eta_1}^{\eta_2}\eta_1^{\delta_1-1}\mathfrak{J}_1(\zeta)\left(\frac{R_0L_0^{\varrho_1-1}}{\Gamma(\gamma_1)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\,d\tau\right)^{\rho_1-1}d\zeta \\
& = \frac{R_0^{\rho_1-1}L_0\eta_1^{\delta_1-1}\Lambda_1}{(\Gamma(\gamma_1))^{\rho_1-1}} > 0.
\end{aligned}
$$

We deduce that $\|h\| \geq h(\eta_1) > 0$. In a similar manner, we obtain

$$
\begin{aligned}
& I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))) \\
& \geq \frac{R_0}{\Gamma(\gamma_2)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\left(\inf_{\tau\in[\eta_1,\eta_2]}(h(\tau)+x(\tau))\right)^{\varrho_2-1}\,d\tau \\
& \geq \frac{R_0L_0^{\varrho_2-1}}{\Gamma(\gamma_2)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\,d\tau,\ \ \forall\,\zeta\in[\eta_1,\eta_2],
\end{aligned}
$$

and so

$$
\begin{aligned}
k(\eta_1) & \geq \int_0^1 \eta_1^{\delta_2-1}\mathfrak{J}_4(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))))\,d\zeta \\
& \geq \int_{\eta_1}^{\eta_2}\eta_1^{\delta_2-1}\mathfrak{J}_4(\zeta)\left(\frac{R_0L_0^{\varrho_2-1}}{\Gamma(\gamma_2)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\,d\tau\right)^{\rho_2-1}d\zeta \\
& = \frac{R_0^{\rho_2-1}L_0\eta_1^{\delta_2-1}\Lambda_4}{(\Gamma(\gamma_2))^{\rho_2-1}} > 0.
\end{aligned}
$$

We deduce that $\|k\| \geq k(\eta_1) > 0$.
In addition, from the above inequalities we have

$$
\begin{aligned}
& I_{0+}^{\gamma_1}(\mathfrak{a}(\zeta)\mathfrak{f}(k(\zeta)+y(\zeta))) \\
& \geq \frac{R_0}{\Gamma(\gamma_1)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\left(\inf_{\tau\in[\eta_1,\eta_2]}(k(\tau)+y(\tau))\right)^{\varrho_1-1}\,d\tau \\
& \geq \frac{R_0\eta_1^{(\delta_2-1)(\varrho_1-1)}}{\Gamma(\gamma_1)}\|k+y\|^{\varrho_1-1}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\,d\tau,\ \ \forall\,\zeta\in[\eta_1,\eta_2],
\end{aligned}
$$

and so

$$
\begin{aligned}
& h(\eta_1) \geq \int_{\eta_1}^{\eta_2}\eta_1^{\delta_1-1}\mathfrak{J}_1(\zeta)\left(\frac{R_0\eta_1^{(\delta_2-1)(\varrho_1-1)}}{\Gamma(\gamma_1)}\right)^{\rho_1-1}\|k+y\|\left(\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_1-1}\mathfrak{a}(\tau)\,d\tau\right)^{\rho_1-1}d\zeta \\
& = \frac{\eta_1^{\delta_1+\delta_2-2}R_0^{\rho_1-1}}{(\Gamma(\gamma_1))^{\rho_1-1}}\Lambda_1\|k+y\| \geq 2\|k+y\| \geq 2\|k\|.
\end{aligned}
$$

Hence,

$$\|k\| \leq \frac{1}{2}h(\eta_1) \leq \frac{1}{2}\|h\|. \tag{17}$$

In a similar manner, we deduce

$$\begin{aligned}
&I_{0+}^{\gamma_2}(\mathfrak{b}(\zeta)\mathfrak{g}(h(\zeta)+x(\zeta))) \\
&\geq \frac{R_0}{\Gamma(\gamma_2)}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\left(\inf_{\tau\in[\eta_1,\eta_2]}(h(\tau)+x(\tau))\right)^{\varrho_2-1}d\tau \\
&\geq \frac{R_0\eta_1^{(\delta_1-1)(\varrho_2-1)}}{\Gamma(\gamma_2)}\|h+x\|^{\varrho_2-1}\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\,d\tau,\ \ \forall\,\zeta\in[\eta_1,\eta_2],
\end{aligned}$$

and then

$$\begin{aligned}
k(\eta_1) &\geq \int_{\eta_1}^{\eta_2}\eta_1^{\delta_2-1}\mathfrak{J}_4(\zeta)\left(\frac{R_0\eta_1^{(\delta_1-1)(\varrho_2-1)}}{\Gamma(\gamma_2)}\right)^{\rho_2-1}\|h+x\|\left(\int_{\eta_1}^{\zeta}(\zeta-\tau)^{\gamma_2-1}\mathfrak{b}(\tau)\,d\tau\right)^{\rho_2-1}d\zeta \\
&= \frac{\eta_1^{\delta_1+\delta_2-2}R_0^{\rho_2-1}}{(\Gamma(\gamma_2))^{\rho_2-1}}\Lambda_4\|h+x\| \geq 2\|h+x\| \geq 2\|h\|.
\end{aligned}$$

Therefore,

$$\|h\| \leq \frac{1}{2}k(\eta_1) \leq \frac{1}{2}\|k\|. \tag{18}$$

Hence, by (17) and (18), we conclude that $\|h\| \leq \frac{1}{2}\|k\| \leq \frac{1}{4}\|h\|$, which is a contradiction (we saw before that $\|h\| > 0$). Therefore, problem (1) and (2) has no positive solution. □

4. An Example

We consider $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{2}{5}$, $\delta_1 = \frac{14}{3}$, $(p=5)$, $\delta_2 = \frac{11}{2}$, $(q=6)$, $n=2$, $m=1$, $\alpha_0 = \frac{17}{8}$, $\beta_0 = \frac{19}{6}$, $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{16}{7}$, $\beta_1 = \frac{3}{7}$, $\varrho_1 = \frac{73}{12}$, $\varrho_2 = \frac{59}{8}$, $\rho_1 = \frac{73}{61}$, $\rho_2 = \frac{59}{51}$, $\mathfrak{a}(t) = 1$, $\mathfrak{b}(t) = 1$ for all $t \in [0,1]$, $\mathfrak{H}_1(t) = \frac{91}{6}t$ for all $t \in [0,1]$, $\mathfrak{H}_2(t) = \left\{\frac{1}{3},\ t \in \left[0,\frac{2}{3}\right);\ \frac{17}{15},\ t \in \left[\frac{2}{3},1\right]\right\}$, $\mathfrak{K}_1(t) = \left\{\frac{1}{2},\ t \in \left[0,\frac{8}{11}\right);\ \frac{33}{26},\ t \in \left[\frac{8}{11},1\right]\right\}$. We introduce the functions $\mathfrak{f}, \mathfrak{g}; [0,\infty) \to [0,\infty)$, $\mathfrak{f}(z) = \omega_1 z^{\sigma_1}$, $\mathfrak{g}(z) = \omega_2 z^{\sigma_2}$ for all $z \in [0,\infty)$ with $\omega_1, \omega_2 > 0$, $\sigma_1, \sigma_2 > 0$, $\sigma_1 > \frac{61}{12}$, $\sigma_2 > \frac{51}{8}$. We have $\lim_{z\to\infty}\frac{\mathfrak{f}(z)}{z^{\varrho_1-1}} = \infty$ and $\lim_{z\to\infty}\frac{\mathfrak{g}(z)}{z^{\varrho_2-1}} = \infty$.

We consider the system of Riemann–Liouville fractional differential equations

$$\begin{cases} D_{0+}^{3/4}\left(\varphi_{73/12}\left(D_{0+}^{14/3}\mathrm{u}(t)\right)\right) + \omega_1(\mathrm{v}(t))^{\sigma_1} = 0,\ \ t \in (0,1), \\ D_{0+}^{2/5}\left(\varphi_{59/8}\left(D_{0+}^{11/2}\mathrm{v}(t)\right)\right) + \omega_2(\mathrm{u}(t))^{\sigma_2} = 0,\ \ t \in (0,1), \end{cases} \tag{19}$$

subject to the coupled boundary conditions

$$\begin{cases} \mathrm{u}^{(i)}(0) = 0,\ i = 0,\ldots,3,\ \ D_{0+}^{14/3}\mathrm{u}(0) = 0, \\ D_{0+}^{17/8}\mathrm{u}(1) = \dfrac{91}{6}\displaystyle\int_0^1 D_{0+}^{3/2}\mathrm{v}(t)\,dt + \frac{4}{5}D_{0+}^{16/7}\mathrm{v}\left(\frac{2}{3}\right) + \mathfrak{c}_0, \\ \mathrm{v}^{(i)}(0) = 0,\ i = 0,\ldots,4,\ \ D_{0+}^{11/2}\mathrm{v}(0) = 0,\ \ D_{0+}^{19/6}\mathrm{v}(1) = \dfrac{10}{13}D_{0+}^{3/7}\mathrm{u}\left(\frac{8}{11}\right) + \mathfrak{d}_0. \end{cases} \tag{20}$$

We obtain here $\Delta_1 \approx 40.01662964$, $\Delta_2 \approx 0.49478575$, and $\Delta \approx 452.46647281 > 0$. Therefore, assumptions $(K1)$, $(K2)$, and $(K4)$ are satisfied. In addition, we deduce

$$\begin{aligned}
\mathfrak{g}_1(t,\zeta) &= \frac{1}{\Gamma(14/3)}\begin{cases} t^{11/3}(1-\zeta)^{37/24} - (t-\zeta)^{11/3}, & 0 \le \zeta \le t \le 1, \\ t^{11/3}(1-\zeta)^{37/24}, & 0 \le t \le \zeta \le 1, \end{cases} \\
\mathfrak{g}_{11}(\tau,\zeta) &= \frac{1}{\Gamma(89/21)}\begin{cases} \tau^{68/21}(1-\zeta)^{37/24} - (\tau-\zeta)^{68/21}, & 0 \le \zeta \le \tau \le 1, \\ \tau^{68/21}(1-\zeta)^{37/24}, & 0 \le \tau \le \zeta \le 1, \end{cases} \\
\mathfrak{g}_2(t,\zeta) &= \frac{1}{\Gamma(11/2)}\begin{cases} t^{9/2}(1-\zeta)^{4/3} - (t-\zeta)^{9/2}, & 0 \le \zeta \le t \le 1, \\ t^{9/2}(1-\zeta)^{4/3}, & 0 \le t \le \zeta \le 1, \end{cases} \\
\mathfrak{g}_{21}(\tau,\zeta) &= \frac{1}{6}\begin{cases} \tau^{3}(1-\zeta)^{4/3} - (\tau-\zeta)^{3}, & 0 \le \zeta \le \tau \le 1, \\ \tau^{3}(1-\zeta)^{4/3}, & 0 \le \tau \le \zeta \le 1, \end{cases} \\
\mathfrak{g}_{22}(\tau,\zeta) &= \frac{1}{\Gamma(45/14)}\begin{cases} \tau^{31/14}(1-\zeta)^{4/3} - (\tau-\zeta)^{31/14}, & 0 \le \zeta \le \tau \le 1, \\ \tau^{31/14}(1-\zeta)^{4/3}, & 0 \le \tau \le \zeta \le 1, \end{cases} \\
\mathfrak{G}_1(t,\zeta) &= \mathfrak{g}_1(t,\zeta) + \frac{10\Delta_1 t^{11/3}}{13\Delta}\mathfrak{g}_{11}\left(\frac{8}{11},\zeta\right), \\
\mathfrak{G}_2(t,\zeta) &= \frac{t^{11/3}\Gamma(11/2)}{\Delta\Gamma(7/3)}\left(\frac{91}{6}\int_0^1 \mathfrak{g}_{21}(\tau,\zeta)\,d\tau + \frac{4}{5}\mathfrak{g}_{22}\left(\frac{2}{3},\zeta\right)\right), \\
\mathfrak{G}_3(t,\zeta) &= \frac{10t^{9/2}\Gamma(14/3)}{13\Delta\Gamma(61/24)}\mathfrak{g}_{11}\left(\frac{8}{11},\zeta\right), \\
\mathfrak{G}_4(t,\zeta) &= \mathfrak{g}_2(t,\zeta) + \frac{t^{9/2}\Delta_2}{\Delta}\left(\frac{91}{6}\int_0^1 \mathfrak{g}_{21}(\tau,\zeta)\,d\tau + \frac{4}{5}\mathfrak{g}_{22}\left(\frac{2}{3},\zeta\right)\right), \\
\mathfrak{h}_1(\zeta) &= \frac{1}{\Gamma(14/3)}(1-\zeta)^{37/24}\left(1-(1-\zeta)^{17/8}\right), \\
\mathfrak{h}_2(\zeta) &= \frac{1}{\Gamma(11/2)}(1-\zeta)^{4/3}\left(1-(1-\zeta)^{19/6}\right),
\end{aligned}$$

for all $t,\tau,\zeta \in [0,1]$. In addition, we find

$$\mathfrak{J}_1(\zeta) = \begin{cases} \mathfrak{h}_1(\zeta) + \frac{10\Delta_1}{13\Delta\Gamma(89/21)}\left[\left(\frac{8}{11}\right)^{68/21}(1-\zeta)^{37/24} - \left(\frac{8}{11}-\zeta\right)^{68/21}\right], & 0 \le \zeta < \frac{8}{11}, \\ \mathfrak{h}_1(\zeta) + \frac{10\Delta_1}{13\Delta\Gamma(89/21)}\left(\frac{8}{11}\right)^{68/21}(1-\zeta)^{37/24}, & \frac{8}{11} \le \zeta \le 1, \end{cases}$$

$$\mathfrak{J}_2(\zeta) = \begin{cases} \frac{\Gamma(11/2)}{\Delta\Gamma(7/3)}\left\{\frac{91}{144}(1-\zeta)^{4/3} - \frac{91}{144}(1-\zeta)^4 + \frac{4}{5\Gamma(45/14)} \right. \\ \qquad \left. \times\left[\left(\frac{2}{3}\right)^{31/14}(1-\zeta)^{4/3} - \left(\frac{2}{3}-\zeta\right)^{31/14}\right]\right\}, & 0 \le \zeta < \frac{2}{3}, \\ \frac{\Gamma(11/2)}{\Delta\Gamma(7/3)}\left[\frac{91}{144}(1-\zeta)^{4/3} - \frac{91}{144}(1-\zeta)^4 + \frac{4}{5\Gamma(45/14)} \right. \\ \qquad \left. \times\left(\frac{2}{3}\right)^{31/14}(1-\zeta)^{4/3}\right], & \frac{2}{3} \le \zeta \le 1, \end{cases}$$

$$\mathfrak{J}_3(\zeta) = \begin{cases} \frac{10\Gamma(14/3)}{13\Delta\Gamma(61/24)\Gamma(89/21)}\left[\left(\frac{8}{11}\right)^{68/21}(1-\zeta)^{37/24} - \left(\frac{8}{11}-\zeta\right)^{68/21}\right], & 0 \le \zeta < \frac{8}{11}, \\ \frac{10\Gamma(14/3)}{13\Delta\Gamma(61/24)\Gamma(89/21)}\left(\frac{8}{11}\right)^{68/21}(1-\zeta)^{37/24}, & \frac{8}{11} \le \zeta \le 1, \end{cases}$$

$$\mathfrak{J}_4(\zeta) = \begin{cases} \mathfrak{h}_2(\zeta) + \frac{\Delta_2}{\Delta}\left\{\frac{91}{144}(1-\zeta)^{4/3} - \frac{91}{144}(1-\zeta)^4 + \frac{4}{5\Gamma(45/14)} \right. \\ \qquad \left. \times\left[\left(\frac{2}{3}\right)^{31/14}(1-\zeta)^{4/3} - \left(\frac{2}{3}-\zeta\right)^{31/14}\right]\right\}, & 0 \le \zeta < \frac{2}{3}, \\ \mathfrak{h}_2(\zeta) + \frac{\Delta_2}{\Delta}\left[\frac{91}{144}(1-\zeta)^{4/3} - \frac{91}{144}(1-\zeta)^4 + \frac{4}{5\Gamma(45/14)} \right. \\ \qquad \left. \times\left(\frac{2}{3}\right)^{31/14}(1-\zeta)^{4/3}\right], & \frac{2}{3} \le \zeta \le 1. \end{cases}$$

We also obtain $\Xi_1 = 1$ and $\Xi_2 = 1$. After some computations, we find

$$\begin{aligned}
P_1 &:= \frac{2^{61/12}}{\Gamma(7/4)}\left(\int_0^1 \mathfrak{J}_1(\zeta)\zeta^{9/61}\,d\zeta\right)^{61/12} \approx 4.11609161 \times 10^{-9}, \\
P_2 &:= \frac{2^{51/8}}{\Gamma(7/5)}\left(\int_0^1 \mathfrak{J}_2(\zeta)\zeta^{16/255}\,d\zeta\right)^{51/8} \approx 3.11233481 \times 10^{-10},
\end{aligned}$$

$$P_3 := \frac{2^{61/12}}{\Gamma(7/4)}\left(\int_0^1 \mathfrak{J}_3(\zeta)\zeta^{9/61}\,d\zeta\right)^{61/12} \approx 1.39796164 \times 10^{-18},$$
$$P_4 := \frac{2^{51/8}}{\Gamma(7/5)}\left(\int_0^1 \mathfrak{J}_4(\zeta)\zeta^{16/255}\,d\zeta\right)^{51/8} \approx 1.16007238 \times 10^{-13},$$

and so $L = \max\{P_i,\ i = 1,\ldots,4\} = P_1$. We choose $\mathfrak{c}_0 = 10$, $\sigma_1 = \frac{31}{6}$, $\sigma_2 = \frac{13}{2}$, and if we select $\omega_1 < \frac{1}{L}10^{-1/12}$ and $\omega_2 < \frac{1}{L}10^{-1/8}$, then we deduce that $\mathfrak{f}(z) < \frac{10^{61/12}}{L}$ and $\mathfrak{g}(z) < \frac{10^{51/8}}{L}$ for all $z \in [0,10]$. For example, if $\omega_1 \leq 2.0053 \times 10^8$ and $\omega_2 \leq 1.8218 \times 10^8$, then the above conditions for $\mathfrak{f}$ and $\mathfrak{g}$ are satisfied. Therefore, assumption $(K3)$ is also satisfied. By Theorem 1, we conclude that there exist positive constants $\mathfrak{c}_1$ and $\mathfrak{d}_1$ such that for any $\mathfrak{c}_0 \in (0,\mathfrak{c}_1]$ and $\mathfrak{d}_0 \in (0,\mathfrak{d}_1]$, problem (19) and (20) has at least one positive solution $(\mathrm{u}(t),\mathrm{v}(t))$, $t \in [0,1]$. By Theorem 2, we deduce that there exist positive constants $\mathfrak{c}_2$ and $\mathfrak{d}_2$ such that for any $\mathfrak{c}_0 \geq \mathfrak{c}_2$ and $\mathfrak{d}_0 \geq \mathfrak{d}_2$, problem (19) and (20) has no positive solution.

5. Conclusions

In this paper, we studied the system of coupled Riemann–Liouville fractional differential Equation (1) with ϱ_1-Laplacian and ϱ_2-Laplacian operators, subject to the nonlocal coupled boundary conditions (2), which contain fractional derivatives of various orders, Riemann–Stieltjes integrals, and two positive parameters $\mathfrak{c}_0$ and $\mathfrak{d}_0$. Under some assumptions for the nonlinearities $\mathfrak{f}$ and $\mathfrak{g}$ of system (1), we established intervals for the parameters $\mathfrak{c}_0$ and $\mathfrak{d}_0$ such that our problem (1) and (2) has at least one positive solution. First, we made a change of unknown functions such that the new boundary conditions have no positive parameters. By using the corresponding Green functions, the new boundary value problem was then written equivalently as a system of integral equations (namely the system (15)). We associated to this integral system an operator $(\mathcal{S})$, and we proved the existence of at least one fixed point for it by applying the Schauder fixed point theorem. Intervals for parameters $\mathfrak{c}_0$ and $\mathfrak{d}_0$ were also given such that problem (1) and (2) has no positive solution. Finally, we presented an example to illustrate our main results.

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