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Existence and Stability Results for a Tripled System of the Caputo Type with Multi-Point and Integral Boundary Conditions

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Abstract: In this paper, we introduce and investigate the existence and stability of a tripled system of sequential fractional differential equations (SFDEs) with multi-point and integral boundary conditions. The existence and uniqueness of the solutions are established by the principle of Banach's contraction and the alternative of Leray–Schauder. The stability of the Hyer–Ulam solutions are investigated. A few examples are provided to identify the major results.

Keywords: fractional differential equations; tripled system; existence; fixed point theorems; stability

MSC: 26A33; 34B18; 34B15

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1. Introduction

For Leibniz in 1695, L'Hospital summarized his discoveries for him and, for the first time, utilized fractional calculus. Fractional calculus (FC) is founded on the idea of substituting rational numbers in the derivative order for natural numbers. Fractional calculus has been studied by several great authors, including Liouville, Grunwald, Letnikov, and Riemann. Numerous phenomena in various disciplines of engineering and science are described by fractional differential equations, ranging from theoretical physics to astrophysics and astronomy, from dynamical chaos to signal processing, and from signal processing to networking. For current and wide-ranging analyses of fractional derivatives and their applications, we recommend certain monographs [1–4] and recently mentioned papers [5–14]. The majority of research on FDEs is based on fractional derivatives of the Riemann–Liouville and Caputo types (see [5,8,11,12,15]).

Fractional differential equations can be used to represent a wide range of phenomena, such as chaotic synchronization, anomalous diffusion, ecological consequences, and biological models. Some important results have been published in this field using a variety of boundary conditions (see [16–21] and the references therein for more information). Another area of study that has gained considerable attention from researchers in the field of fractional differential equations is the concept of Ulam stability. Ulam [22] proposed Ulam-type stability in 1940, and it has since been studied and generalized using a variety of methods. Recent research in the stability analysis of fractional-order systems is in [23–25] and the references cited therein. Recently, Caputo-Hadamard-type fractional differential equations have been studied by Subramanian et al. [11]. The authors explored existence and uniqueness, which can be handled through the use of well-known fixed point theorems and Hyer–Ulam stability, which are also described in the article. Tripled fractional boundary conditions are regulated by three related differential equations with three initial or boundary conditions. Despite popular belief, researchers have paid less attention to studies of tripled fractional systems. According to the author's observations, there is no analytical research on the existence of tripled systems of sequential fractional differential equations. This is true to the best of the author's knowledge. A tripled fractional boundary value problem is being investigated by a few researchers [26–28]. Nonlinear mappings in partially ordered complete metric spaces were only studied by Berinde and Borcut [26], who developed the idea of tripled fixed points. The authors in [29] studied tripled fixed points for a class of condensing operators in Banach spaces. The authors investigated tripled fractional systems with cyclic boundary conditions in [28]:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha_k} x_k(\tau) = f_k(\tau, x(\tau)), & 1 < \alpha_k \le 2, \\ x_k^{(j)}(0) = a_{k,j} x_{\varepsilon(k)}^{(j)}(\mathcal{T}), & k = 1,2,3; \quad j = 0,1, \end{cases} \tag{1}$$

where ${}^{c}\mathcal{D}_0^{\alpha_k}$ denotes the Caputo fractional derivatives of the order α_k, $\tau \in J = [0, \mathcal{T}]$, $f_k : J \times \mathcal{R}_e^3 \to \mathcal{R}_e$ are continuous functions, $x = (x_1, x_2, x_3) \in \mathcal{R}_e^3, \varepsilon = (1,2,3)$ is a cyclic permutation, and $a_{k,j} \in k = 1,2,3, j = 0,1$. The existence and uniqueness of solutions are investigated. The authors investigated a nonlinear coupled system of three fractional differential equations with non-local coupled boundary conditions in [30]:

$$\begin{cases} {}^{c}\mathcal{D}_{a^+}^{\eta} u(\varrho) = \rho(\varrho, u(\varrho), x(\varrho), y(\varrho)), & 1 < \eta \le 2, \varrho \in [a,b], \\ {}^{c}\mathcal{D}_{a^+}^{\xi} x(\varrho) = \varphi(\varrho, u(\varrho), x(\varrho), y(\varrho)), & 1 < \xi \le 2, \varrho \in [a,b], \\ {}^{c}D_{a^+}^{\zeta} y(\varrho) = \psi(\varrho, u(\varrho), x(\varrho), y(\varrho)), & 2 < \zeta \le 3, \varrho \in [a,b], \\ u(a) = u_0, \qquad u(b) = \sum_{i=1}^{m} p_i x(\alpha_i), \\ x(a) = 0, \qquad x(b) = \sum_{j=1}^{n} q_j y(\beta_j), \\ y(\xi_1) = 0, \qquad y(\xi_2) = 0, \qquad y(b) = \sum_{k=1}^{l} r_k u(\gamma_k), \\ a < \xi_1 < \xi_2 < \alpha_1 < \cdots < \alpha_m < \beta_1 < \cdots < \beta_n < \gamma_1 < \cdots < \gamma_l < b, \end{cases} \tag{2}$$

where ${}^{c}D^{\chi}$ is a Caputo fractional derivative of the order $\chi \in \{\eta, \xi, \zeta\}, \rho, \lambda, \psi : [a,b] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are given functions, and $p_i, q_j, r_k \in \mathcal{R}_e, i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, l$. The existence and uniqueness results for the problem are proven via the Leray–Schauder alternative and Banach's contraction mapping principle. Recently, Tavazoei and Asemani [31] investigated the robust stability of fractional-order systems described in a pseudo-state space model with incommensurate fractional orders. The generalized Nyquist theorem is used to extend an existing non-conservative robust stability criterion for integer-order systems in incommensurate-order fractional systems. In [32], the author discussed a fractional-order generalization of the susceptible, infected, and recovered (SIR) epidemic model for predicting the spread of an infectious disease. Additionally, a system of incommensurate fractional-order differential equations is used, which involves the Caputo fractional derivative. The definition of the Caputo derivative is mainly practical. The Riemann–Liouville approach requires the initial conditions for differential equations in terms of non-integer derivatives which are hardly physically interpreted, whereas the Caputo approach uses the integer-order initial conditions. Moreover, we sometimes also need the fractional derivatives of the constants to be zero. The Riemann–Liouville derivative with a finite lower bound does not satisfy this, while the Caputo derivative does. Mathematical biology is concerned with modeling, studying, analyzing, and interpreting biological phenomena such as species interactions, coexistence, and evolution. These interactions may occur between individuals of the same species, between individuals of other species, or between individuals of different species, as well as interactions with the environment, disease, and food supplies. The Lotka–Volterra model, which was developed independently by Lotka [33] and Volterra [34], was the first stone in this subject. Later developments included density-dependent prey growth and a functional response to the model [35]. Initially, the authors suggested a system of coupled differential equations in two dimensions [36]. In the last two decades, fractional-order differential equations appeared, and the study of predator-prey models in the fractional-order form began. The scenario then shifted to discrete mathematical models with a variety of sophisticated dynamical behaviors [37]. In [38], the authors studied the predator-prey model of Holling type II with harvesting and predators in disease:

$$\begin{cases} \frac{dx}{dt} = rx(1-\frac{x}{k}) - \frac{ayx}{m+x} - azx - h_1 x, \\ \frac{dy}{dt} = bxy + \alpha yz + \frac{\gamma yx}{m+x} - h_2 y, \\ \frac{dz}{dt} = bzx - \alpha yz - dz, \end{cases}$$

where x, y, and z are the prey, infected predator, and susceptible predator, respectively, and r, k, a, b,γ, α, h_1, h_2, and d are assumed to be positive constants. They have studied the existence of a positive biological equilibrium and the uniform boundedness of the system, and the local stability conditions are also defined based on the Routh-Hurwitz criterion. In [39], the authors discussed the fractional order model of a two-prey-one-predator system:

$$\begin{cases} {}^cD_*^{\alpha} x_1(t) = f_1(x_1,x_2,x_3) = ax_1(t)(1-x_1(t)) - x_1(t)x_3(t) + x_1(t)x_2(t)x_3(t), \ t \in [0,T], \\ {}^cD_*^{\alpha} x_2(t) = f_2(x_1,x_2,x_3) = bx_2(t)(1-x_2(t)) - x_2(t)x_3(t) + x_1(t)x_2(t)x_3(t), \ t \in [0,T], \\ {}^cD_*^{\alpha} x_3(t) = f_3(x_1,x_2,x_3) = -cx_3^2(t) + dx_1(t)x_3(t) + ex_2(t)x_3(t), \ t \in [0,T], \end{cases}$$

where c is the death rate of the predator, $0 \le \alpha \le 1$, and $x_1(t) \ge 0$, $x_2(t) \ge 0$, $x_3(t) \ge 0$. Meanwhile, a, b, c, d, and e are all positive constants. Thelocal asymptotic stability of the equilibrium solutions of the proposed model were studied. One of the most important disciplines in the study of fractional-order differential equations is the theory of existence, uniqueness, and stability of the solutions. Many researchers have recently been interested in this idea, and we refer to [40–42] and the references therein for some of the recent growth. In [30], the authors investigated the existence of positive solutions for three fractional differential equations complemented with non-local multi-point discrete boundary conditions. In the present paper, inspired by the above-mentioned works, we introduce and investigate the existence and stability of solutions for a coupled system of nonlinear Caputo-type sequential three-fractional differential equations enhanced with boundary conditions:

$$\begin{cases} ({}^cD^{\alpha} + \lambda {}^cD^{\alpha-1})x(\tau) = f(\tau, x(\tau), y(\tau), z(\tau)), \quad 2 < \alpha \le 3, \\ ({}^cD^{\beta} + \lambda {}^cD^{\beta-1})y(\tau) = g(\tau, x(\tau), y(\tau), z(\tau)), \quad 2 < \beta \le 3, \\ ({}^cD^{\gamma} + \lambda {}^cD^{\gamma-1})z(\tau) = h(\tau, x(\tau), y(\tau), z(\tau)), \quad 3 < \gamma \le 4, \\ x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = \mathcal{A}_1 \sum_{j=1}^{k-2} p_j y(\theta_j) + \mathcal{B}_1 \mathcal{I}^{\omega} y(\varpi), \\ y(0) = 0, \qquad y'(0) = 0, \qquad y(1) = \mathcal{A}_2 \sum_{j=1}^{k-2} q_j z(\delta_j) + \mathcal{B}_2 \mathcal{I}^{\psi} z(\varrho), \\ z(0) = 0, \qquad z'(0) = 0, \qquad z''(0) = 0, \qquad z(1) = \mathcal{A}_3 \sum_{j=1}^{k-2} r_j x(\eta_j) + \mathcal{B}_3 \mathcal{I}^{\varsigma} x(\vartheta), \end{cases} \tag{3}$$

where ${}^cD^{\chi}$ is a Caputo fractional derivative (CFD) of the order $\chi \in \{\alpha, \beta, \gamma\}$, f, g, $h : [0,1] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are given functions, $p_j, q_j, r_j \in \mathcal{R}_e, j = 1, \ldots, k-2$ are non-negative real constants, and $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ are real constants. Notably, the multi-point strip boundary conditions in Equation (3) are novel and can be thought of as the value of unknown functions with the right endpoint 1, which is proportional to the sum of the Riemann–Liouville fractional integrals of unknown functions with different strip lengths $(0,\omega)$, $(0,\varrho)$, and $(0,\vartheta)$ and with different multi-point values of unknown functions with θ_j, δ_j, η_j $j = 1, 2, \ldots, k-2$. Using fixed point theory, we obtain their existence and unique results. Additionally, we investigate Hyers–Ulam stability.

Section 2 discusses several fundamental definitions and lemmas of fractional calculus. Additionally, an auxiliary lemma involving a linear function of Equation (3) is proven, which is necessary for obtaining the main results. Section 3 summarizes the main results. This section introduces the existence of a solution to the problem at hand using the Leray–Schauder alternative while also verifying the existence of a unique solution using Banach's contraction mapping principle. In Section 4, we prove that the proposed problem in Equation (3) is Ulam–Hyers stable under certain conditions. In Section 5, examples are provided to illustrate the theoretical results.

2. Preliminary Assertions

This portion introduces basic fractional calculus (FC) concepts, definitions, and tentative results [4,43,44]:

Definition 1. *The fractional integral of the order α with the lower limit zero for a function k is defined as* [2]

$$I^{\alpha}k(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{k(s)}{(\tau - s)^{1-\alpha}} ds, \tau > 0, \alpha > 0, \tag{4}$$

provided the right-hand side is point-wise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^{\infty} \tau^{\alpha-1} e^{-\tau} d\tau$.

Definition 2. *The Riemann–Liouville fractional derivative of the order $\alpha > 0, n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as* [1]

$$D_{0+}^{\alpha}k(\tau) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\tau}\right)^n \int_0^{\tau} (\tau - s)^{n-\alpha-1} k(s) ds, \tau > 0, \tag{5}$$

where the function k has an absolutely continuous derivative up to the order $(n-1)$.

Definition 3. *The Caputo derivative of the order $\alpha \in [n-1, n)$ for a function $k : [0, \infty) \to (\mathbb{R})$ can be written as* [1]

$$^{c}D_{0+}^{\alpha}k(\tau) = D_{0+}^{\alpha}\left(k(\tau) - \sum_{m=0}^{n-1} \frac{\tau^m}{m!} f^{(m)}(0)\right), \tau > 0, n-1 < r < n. \tag{6}$$

Note that the Caputo fractional derivative of the order $\alpha \in [n-1, n)$ is almost everywhere on $[0, \infty)$ if $k \in \mathcal{AC}^n([0, \infty), (\mathbb{R}))$.

Remark 1. *If $k \in \mathcal{C}^n[0, \infty)$, then*

$$^{c}D_{0+}^{\alpha}k(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\tau} \frac{k^n(s)}{(\tau - s)^{\alpha+1-n}} ds = I^{n-\alpha}k(n)(\tau), \tau > 0, n-1 < \alpha < n.$$

We are now ready to introduce an important lemma that we use to solve the problem:

Lemma 1. *Let $x, y, z \in \mathcal{C}[0,1]$ and $\Delta \neq 0$. Then, the solution of the linear fractional differential system*

$$\begin{cases} ({}^cD^{\alpha} + \lambda {}^cD^{\alpha-1})x(\tau) = f(\tau, x(\tau), y(\tau), z(\tau)), \quad 2 < \alpha \leq 3, \\ ({}^cD^{\beta} + \lambda {}^cD^{\beta-1})y(\tau) = g(\tau, x(\tau), y(\tau), z(\tau)), \quad 2 < \beta \leq 3, \\ ({}^cD^{\gamma} + \lambda {}^cD^{\gamma-1})z(\tau) = h(\tau, x(\tau), y(\tau), z(\tau)), \quad 3 < \gamma \leq 4, \\ x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = \mathcal{A}_1 \sum_{j=1}^{k-2} p_j y(\theta_j) + \mathcal{B}_1 \mathcal{I}^{\omega} y(\varpi), \\ y(0) = 0, \qquad y'(0) = 0, \qquad y(1) = \mathcal{A}_2 \sum_{j=1}^{k-2} q_j z(\delta_j) + \mathcal{B}_2 \mathcal{I}^{\psi} z(\varrho), \\ z(0) = 0, \qquad z'(0) = 0, \qquad z''(0) = 0, \qquad z(1) = \mathcal{A}_3 \sum_{j=1}^{k-2} r_j x(\eta_j) + \mathcal{B}_3 \mathcal{I}^{\varsigma} x(\vartheta), \end{cases} \tag{7}$$

is given by

$$\begin{aligned} x(\tau) =& \frac{(\lambda\tau - 1 + e^{-\lambda\tau})}{\lambda^2 \mathscr{G}_1} \Bigg[\mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} H_2(u) du \right) ds \\ &+ \mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} H_2(m) dm \right) du \right) ds \\ &- \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} H_1(u) du \right) ds \\ &+ \frac{1}{\Delta} \Bigg(\mathscr{G}_2 \mathscr{G}_4 \mathscr{G}_6 \Bigg\{ \mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} H_2(u) du \right) ds \\ &+ \mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} H_2(m) dm \right) du \right) ds \\ &- \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} H_1(u) du \right) ds \Bigg\} \\ &+ \mathscr{G}_1 \mathscr{G}_2 \mathscr{G}_4 \Bigg\{ \mathcal{A}_3 \sum_{j=1}^{k-2} r_j \int_0^{\eta_j} e^{-\lambda(\eta_j - s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} H_1(u) du \right) ds \\ &+ \mathcal{B}_3 \int_0^{\vartheta} \frac{(\vartheta - s)^{\varsigma-1}}{\Gamma(\varsigma)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)} H_1(m) dm \right) du \right) ds \\ &- \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} H_3(u) du \right) ds \Bigg\} \\ &+ \mathscr{G}_1 \mathscr{G}_2 \mathscr{G}_5 \Bigg\{ \mathcal{A}_2 \sum_{j=1}^{k-2} q_j \int_0^{\delta_j} e^{-\lambda(\delta_j - s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} H_3(u) du \right) ds \\ &+ \mathcal{B}_2 \int_0^{\varrho} \frac{(\varrho - s)^{\psi-1}}{\Gamma(\psi)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)} H_3(m) dm \right) du \right) ds \\ &- \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} H_2(u) du \right) ds \Bigg\} \Bigg) \Bigg] \\ &+ \int_0^{\tau} e^{-\lambda(\tau - s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} H_1(u) du \right) ds, \end{aligned} \tag{8}$$

$$
\begin{aligned}
y(\tau) = & \frac{(\lambda\tau - 1 + e^{-\lambda\tau})}{\lambda^2\Delta}\Bigg[\Bigg(\mathscr{G}_4\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds \\
& +\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}H_2(m)dm\bigg)du\bigg)ds \\
& -\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds\Bigg\} \\
& +\mathscr{G}_1\mathscr{G}_4\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds \\
& +\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(m)dm\bigg)du\bigg)ds \\
& -\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\Bigg\} \\
& +\mathscr{G}_1\mathscr{G}_5\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds \\
& +\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(m)dm\bigg)du\bigg)ds \\
& -\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\Bigg\}\Bigg)\Bigg] \\
& +\int_0^{\tau}e^{-\lambda(\tau-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds,
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
& z(\tau) \\
= & \frac{(\lambda^2\tau^2 - 2\lambda\tau + 2 - 2e^{-\lambda\tau})}{\lambda^3\Delta}\Bigg[\Bigg(\mathscr{G}_3\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds \\
& +\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}H_2(m)dm\bigg)du\bigg)ds \\
& -\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds\Bigg\} \\
& +\mathscr{G}_1\mathscr{G}_3\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds \\
& +\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(m)dm\bigg)du\bigg)ds \\
& -\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\Bigg\} \\
& +\mathscr{G}_2\mathscr{G}_6\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds \\
& +\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(m)dm\bigg)du\bigg)ds \\
& -\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\Bigg\}\Bigg)\Bigg] \\
& +\int_0^{\tau}e^{-\lambda(\tau-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds,
\end{aligned} \tag{10}
$$

where

$$\begin{aligned}
\mathscr{G}_1 =&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2},\\
\mathscr{G}_2 =&\frac{1}{\lambda^2}\left[\mathcal{A}_1\sum_{j=1}^{k-2}p_j(\lambda\theta_j-1+e^{-\lambda\theta_j})+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}(\lambda s-1+e^{-\lambda s})ds\right],\\
\mathscr{G}_3 =&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2},\\
\mathscr{G}_4 =&\frac{1}{\lambda^3}\left[\mathcal{A}_2\sum_{j=1}^{k-2}q_j(\lambda^2\delta_j{}^2-2\lambda\delta+2-e^{-\lambda\delta_j})+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}(\lambda^2s^2-2\lambda s+2-2e^{-\lambda s})ds\right],\\
\mathscr{G}_5 =&\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3},\\
\mathscr{G}_6 =&\frac{1}{\lambda^2}\left[\mathcal{A}_3\sum_{j=1}^{k-2}r_j(\lambda\eta_j-1+e^{-\lambda\eta_j})+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}(\lambda s-1+e^{-\lambda s})ds\right],\\
&\Delta=(\mathscr{G}_1\mathscr{G}_3\mathscr{G}_5-\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6).
\end{aligned}\tag{11}$$

Proof. The solution of the system in Equation (7) can be written as

$$\begin{aligned}
x(\tau) =&b_0e^{-\lambda\tau}+\frac{b_1}{\lambda}(1-e^{-\lambda\tau})+\frac{b_2}{\lambda^2}(\lambda\tau-1+e^{-\lambda\tau})+\left(\int_0^\tau e^{-\lambda(\tau-s)}\left(I^{\alpha-1}H_1(s)\right)ds\right),\\
y(\tau) =&c_0e^{-\lambda\tau}+\frac{c_1}{\lambda}(1-e^{-\lambda\tau})+\frac{c_2}{\lambda^2}(\lambda\tau-1+e^{-\lambda\tau})+\left(\int_0^\tau e^{-\lambda(\tau-s)}\left(I^{\beta-1}H_2(s)\right)ds\right),\\
z(\tau) =&d_0e^{-\lambda\tau}+\frac{d_1}{\lambda}(1-e^{-\lambda\tau})+\frac{d_2}{\lambda^2}(\lambda\tau-1+e^{-\lambda\tau})+\frac{d_3}{\lambda^3}(\lambda^2\tau^2-2\lambda\tau+2-2e^{-\lambda\tau})\\
&+\left(\int_0^\tau e^{-\lambda(\tau-s)}\left(I^{\gamma-1}H_3(s)\right)ds\right),
\end{aligned}\tag{12}$$

where $b_0, b_1, b_2, c_0, c_1, c_2, d_0, d_1, d_2$, and d_3 are unknown arbitrary constants. With conditions $x(0)=0$, $x'(0)=0$, $y(0)=0$, $y'(0)=0$, $z(0)=0$, $z'(0)=0$, and $z''(0)=0$ applied in Equation (12), we find that $b_0=0, b_1=0, c_0=0, c_1=0, d_0=0, d_1=0$, and $d_2=0$. Using these conditions,we get

$$\begin{aligned}
b_2 =&\frac{1}{\mathscr{G}_1}\left[\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\right)ds\right.\\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}H_2(m)dm\right)du\right)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\right)ds\\
&+\frac{1}{\Delta}\left(\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6\left\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\right)ds\right.\right.\\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}H_2(m)dm\right)du\right)ds\\
&\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\right)ds\right\}\\
&+\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4\left\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\right)ds\right.\\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(m)dm\right)du\right)ds\\
&\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\right)ds\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\Bigg\}\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
c_2=&\frac{1}{\Delta}\Bigg[\Bigg(\mathscr{G}_4\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}H_2(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds\Bigg\}\\
&+\mathscr{G}_1\mathscr{G}_4\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds\\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\Bigg\}\\
&+\mathscr{G}_1\mathscr{G}_5\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\Bigg\}\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
d_3=&\frac{1}{\Delta}\Bigg[\Bigg(\mathscr{G}_3\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}H_2(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds\Bigg\}\\
&+\mathscr{G}_1\mathscr{G}_3\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(u)du\bigg)ds\\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}H_1(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\Bigg\}\\
&+\mathscr{G}_2\mathscr{G}_6\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(u)du\bigg)ds\\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\bigg(\int_0^s e^{-\lambda(s-u)}\bigg(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}H_3(m)dm\bigg)du\bigg)ds\\
&-\int_0^1 e^{-\lambda(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}H_2(u)du\bigg)ds\Bigg\}\Bigg)\Bigg]
\end{aligned}
$$

Finally, the values of b_2, c_2, and d_3 being substituted into Equation (12) yields the desired solutions for Equations (8)–(10). □

3. Main Results

We define the Banach space $\hat{\mathcal{J}} = \mathcal{C}([0,1], \mathcal{R}_e)$ of continuous real functions as being defined on $[0,1]$ with the usual maximum norm. Then, $(\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}, \|(x,y,z)\|_{\hat{\mathcal{J}}})$ is also a Banach space equipped with the norm $\|(x,y,z)\|_{\hat{\mathcal{J}}} = \|x\| + \|y\| + \|z\|, x, y, z \in \hat{\mathcal{J}}$. In light of Lemma 1, we define an operator $\mathcal{E} : \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} \to \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$ by

$$\mathcal{E}\big(x(\tau), y(\tau), z(\tau)\big) = \big(\mathcal{E}_1\big(x(\tau), y(\tau), z(\tau)\big), \mathcal{E}_2\big(x(\tau), y(\tau), z(\tau)\big), \mathcal{E}_3\big(x(\tau), y(\tau), z(\tau)\big)\big),$$

where

$$\begin{aligned}
&\mathcal{E}_1\big(x(\tau), y(\tau), z(\tau)\big)\\
=&\frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \mathscr{G}_1}\Bigg[\mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u, x(u), y(u), z(u)) du\right) ds\\
&+\mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} g(m, x(m), y(m), z(m)) dm\right) du\right) ds\\
&-\int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u, x(u), y(u), z(u)) du\right) ds\\
&+\frac{1}{\Delta}\Bigg(\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6\Bigg\{\mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u, x(u), y(u), z(u)) du\right) ds\\
&+\mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} g(m, x(m), y(m), z(m)) dm\right) du\right) ds\\
&-\int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} h(u, x(u), y(u), z(u)) du\right) ds\Bigg\}\\
&+\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4\Bigg\{\mathcal{A}_3 \sum_{j=1}^{k-2} r_j \int_0^{\eta_j} e^{-\lambda(\eta_j - s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u, x(u), y(u), z(u)) du\right) ds\\
&+\mathcal{B}_3 \int_0^{\vartheta} \frac{(\vartheta - s)^{\varsigma-1}}{\Gamma(\varsigma)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)} f(m, x(m), y(m), z(m)) dm\right) du\right) ds\\
&-\int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u, x(u), y(u), z(u)) du\right) ds\Bigg\}\\
&+\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5\Bigg\{\mathcal{A}_2 \sum_{j=1}^{k-2} q_j \int_0^{\delta_j} e^{-\lambda(\delta_j - s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u, x(u), y(u), z(u)) du\right) ds\\
&+\mathcal{B}_2 \int_0^{\varrho} \frac{(\varrho - s)^{\psi-1}}{\Gamma(\psi)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)} h(m, x(m), y(m), z(m)) dm\right) du\right) ds\\
&-\int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u, x(u), y(u), z(u)) du\right) ds\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\tau} e^{-\lambda(\tau-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u, x(u), y(u), z(u)) du\right) ds,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}_2\big(x(\tau), y(\tau), z(\tau)\big)\\
=&\frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \Delta}\Bigg[\Bigg(\mathscr{G}_4\mathscr{G}_6\Bigg\{\mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u, x(u), y(u), z(u)) du\right) ds\\
&+\mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} g(m, x(m), y(m), z(m)) dm\right) du\right) ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du\right)ds\Bigg\} \\
&+\mathscr{G}_1\mathscr{G}_4\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2} r_j\int_0^{\eta_j} e^{-\lambda(\eta_j-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du\right)ds \\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)} f(m,x(m),y(m),z(m))dm\right)du\right)ds \\
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du\right)ds\Bigg\} \\
&+\mathscr{G}_1\mathscr{G}_5\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2} q_j\int_0^{\delta_j} e^{-\lambda(\delta_j-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du\right)ds \\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)} h(m,x(m),y(m),z(m))dm\right)du\right)ds \\
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du\right)ds\Bigg\}\Bigg)\Bigg] \\
&+\int_0^{\tau} e^{-\lambda(\tau-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du\right)ds,
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}_3\big(x(\tau),y(\tau),z(\tau)\big) \\
=&\frac{(\lambda^2\tau^2-2\lambda\tau+2-2e^{\lambda\tau})}{\lambda^3\Delta} \\
&\times\Bigg[\Bigg(\mathscr{G}_3\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2} p_j\int_0^{\theta_j} e^{-\lambda(\theta_j-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du\right)ds \\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} g(m,x(m),y(m),z(m))dm\right)du\right)ds \\
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du\right)ds\Bigg\} \\
&+\mathscr{G}_1\mathscr{G}_3\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2} r_j\int_0^{\eta_j} e^{-\lambda(\eta_j-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du\right)ds \\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)} f(m,x(m),y(m),z(m))dm\right)du\right)ds \\
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du\right)ds\Bigg\} \\
&+\mathscr{G}_2\mathscr{G}_6\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2} q_j\int_0^{\delta_j} e^{-\lambda(\delta_j-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du\right)ds \\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)} h(m,x(m),y(m),z(m))dm\right)du\right)ds \\
&-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du\right)ds\Bigg\}\Bigg)\Bigg] \\
&+\int_0^{\tau} e^{-\lambda(\tau-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du\right)ds.
\end{aligned}
$$

For the sake of computational simplicity, we define

$$
\mathcal{U}_1=\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathscr{G}_1}\Bigg[\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}+\frac{\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}+\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4}{\Delta}\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\left(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\right)
$$

$$
\begin{aligned}
&+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\Big\}\Big]+\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)},\\
\mathcal{V}_1=&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathcal{G}_1}\Bigg[\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\Bigg(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\Bigg)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\
&+\frac{\mathcal{G}_2\mathcal{G}_4\mathcal{G}_6}{\Delta}\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\Bigg(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\Bigg)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\
&+\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\Bigg],\\
\mathcal{W}_1=&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathcal{G}_1}\Bigg[\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4}{\Delta}\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\Bigg\}+\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5}{\Delta}\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\Bigg(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\Bigg)\\
&+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\Bigg\}\Bigg],\\
\mathcal{U}_2=&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\Delta}\Bigg[\mathcal{G}_4\mathcal{G}_6\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}\Bigg\}+\mathcal{G}_1\mathcal{G}_4\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\Bigg(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\Bigg)\\
&+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\Bigg\}\Bigg],\\
\mathcal{V}_2=&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\Delta}\Bigg[\mathcal{G}_4\mathcal{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\Bigg(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\Bigg)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\
&+\mathcal{G}_1\mathcal{G}_5\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\Bigg\}\Bigg]+\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)},\\
\mathcal{W}_2=&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\Delta}\Bigg[\mathcal{G}_1\mathcal{G}_4\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\Bigg\}+\mathcal{G}_1\mathcal{G}_5\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\Bigg(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\Bigg)\\
&+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\Bigg\}\Bigg],\\
\mathcal{U}_3=&\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\Bigg[\mathcal{G}_3\mathcal{G}_6\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}\Bigg\}+\mathcal{G}_1\mathcal{G}_3\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\Bigg(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\Bigg)\\
&+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\Bigg\}\Bigg]\\
\mathcal{V}_3=&\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\Bigg[\mathcal{G}_3\mathcal{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\Bigg(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\Bigg)\\
&+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}+\mathcal{G}_2\mathcal{G}_6\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\Bigg\}\Bigg],\\
\mathcal{W}_3=&\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\Bigg[\mathcal{G}_1\mathcal{G}_3\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\Bigg\}+\mathcal{G}_1\mathcal{G}_6\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\Bigg(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\Bigg)\\
&+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\Bigg\}\Bigg]+\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}.
\end{aligned}
$$

We established the existence of solutions to the system in Equation (3) in our first result by implementing the Leray–Schauder alternative.

Lemma 2. *(Leray–Schauder alternative) Let $J:\varpi\to\varpi$ be a completely continuous operator. Let $\mathcal{Q}(J)=\{y\in\varpi: y=\alpha J(y)$ for some $0<\alpha<1\}$. Then, either the set $\mathcal{Q}(J)$ is unbounded or J has at least one fixed point.*

Theorem 1. *Let $\Upsilon \neq 0$, where Υ is defined by Equation (11).*

We assume that $(\mathcal{H}_2) : f, g, h : [0,1] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are continuous functions and there exist real constants $\sigma_i, \varepsilon_i, \kappa_i \geq 0, (i = 1,2,3)$ and $\sigma_0 > 0, \varepsilon_0 > 0, \kappa_0 > 0$ such that, for all $\tau \in [0,1]$ and $x, y, z \in \mathcal{R}_e$, the following is true:

$$
\begin{aligned}
\big|f(\tau, x, y, z)\big| &\leq \sigma_0 + \sigma_1|x| + \sigma_2|y| + \sigma_3|z|, \\
\big|g(\tau, x, y, z)\big| &\leq \varepsilon_0 + \varepsilon_1|x| + \varepsilon_2|y| + \varepsilon_3|z|, \\
\big|h(\tau, x, y, z)\big| &\leq \kappa_0 + \kappa_1|x| + \kappa_2|y| + \kappa_3|z|.
\end{aligned}
$$

Then, the system in Equation (3) has at least one solution on $[0,1]$, provided that

$$
\begin{aligned}
(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_1 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_1 < 1, \\
(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_2 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_2 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_2 < 1, \\
(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_3 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_3 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_3 < 1,
\end{aligned}
\tag{13}
$$

where $\mathcal{U}_i, \mathcal{V}_i$, and $\mathcal{W}_i, i = 1,2,3$, are given in Section 3.

Proof. Observe that the continuity of the operator $\mathcal{E} : \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} \to \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$ follows from that of the functions f, g, and h. Next, let $\Omega \subset \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$ be bounded such that

$$
\begin{aligned}
\big|f\big(\tau, x(\tau), y(\tau), z(\tau)\big)\big| &\leq \wp_1, \\
\big|g\big(\tau, x(\tau), y(\tau), z(\tau)\big)\big| &\leq \wp_2, \\
\big|h\big(\tau, x(\tau), y(\tau), z(\tau)\big)\big| &\leq \wp_3, \quad \forall (x, y, z) \in \Omega,
\end{aligned}
$$

for non-negative constants $\wp_1, \wp_2$, and $\wp_3$. Then, for any $(x, y, z) \in \Omega$, we have

$$
\begin{aligned}
&\big|\mathcal{E}_1\big(x(\tau), y(\tau), z(\tau)\big)\big| \\
=&\frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \mathscr{G}_1}\Bigg[\mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}|g(u, x(u), y(u), z(u))|du\right)ds \\
&+\mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}|g(m, x(m), y(m), z(m))|dm\right)du\right)ds \\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}|f(u, x(u), y(u), z(u))|du\right)ds \\
&+\frac{1}{\Delta}\Bigg(\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6\Bigg\{\mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}|g(u, x(u), y(u), z(u))|du\right)ds \\
&+\mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}|g(m, x(m), y(m), z(m))|dm\right)du\right)ds \\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}|h(u, x(u), y(u), z(u))|du\right)ds\Bigg\} \\
&+\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4\Bigg\{\mathcal{A}_3 \sum_{j=1}^{k-2} r_j \int_0^{\eta_j} e^{-\lambda(\eta_j - s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}|f(u, x(u), y(u), z(u))|du\right)ds \\
&+\mathcal{B}_3 \int_0^{\vartheta} \frac{(\vartheta - s)^{\varsigma-1}}{\Gamma(\varsigma)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}|f(m, x(m), y(m), z(m))|dm\right)du\right)ds \\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}|h(u, x(u), y(u), z(u))|du\right)ds\Bigg\} \\
&+\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5\Bigg\{\mathcal{A}_2 \sum_{j=1}^{k-2} q_j \int_0^{\delta_j} e^{-\lambda(\delta_j - s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}|h(u, x(u), y(u), z(u))|du\right)ds \\
&+\mathcal{B}_2 \int_0^{\varrho} \frac{(\varrho - s)^{\psi-1}}{\Gamma(\psi)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}|h(m, x(m), y(m), z(m))|dm\right)du\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}|g(u,x(u),y(u),z(u))|du\right)ds\Big\}\Big)\Big]\\
&+\int_0^\tau e^{-\lambda(\tau-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}|f(u,x(u),y(u),z(u))|du\right)ds,\\
\leq&\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathscr{G}_1}\left\{\left[\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}+\frac{\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}+\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4}{\Delta}\left\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\left(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\right)\right.\right.\right.\\
&\left.\left.+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\right\}\right]\\
&+\left[\left\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\right\}\right.\\
&+\frac{\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6}{\Delta}\left\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\right\}\\
&\left.+\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\right]+\left[\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4}{\Delta}\left\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\right\}+\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5}{\Delta}\left\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\left(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\right)\right.\right.\\
&\left.\left.\left.+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\right\}\right]\right\}+\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)},\\
\leq&\left\{\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathscr{G}_1}\right\}(\mathcal{U}_1\wp_1+\mathcal{V}_1\wp_2+\mathcal{W}_1\wp_3),
\end{aligned}
$$

This implies

$$||\mathcal{E}_1(x(\tau),y(\tau),z(\tau))||_{\hat{\mathcal{J}}}\leq\left\{\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathscr{G}_1}\right\}(\mathcal{U}_1\wp_1+\mathcal{V}_1\wp_2+\mathcal{W}_1\wp_3).$$

Similarly, it implies that

$$||\mathcal{E}_2(x(\tau),y(\tau),z(\tau))||_{\hat{\mathcal{J}}}\leq\left\{\frac{(\lambda-1+e^{\lambda})}{\lambda^2\Delta}\right\}(\mathcal{U}_2\wp_1+\mathcal{V}_2\wp_2+\mathcal{W}_2\wp_3),$$

and

$$
\begin{aligned}
&\left|\mathcal{E}_3(x(\tau),y(\tau),z(\tau))\right|\\
\leq&\frac{(\lambda^2-2\lambda+2-2e^{\lambda})}{\lambda^3\Delta}\\
&\times\left[\left(\mathscr{G}_3\mathscr{G}_6\left\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}|g(u,x(u),y(u),z(u))|du\right)ds\right.\right.\right.\\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}|g(m,x(m),y(m),z(m))|dm\right)du\right)ds\\
&\left.+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}|f(u,x(u),y(u),z(u))|du\right)ds\right\}\\
&+\mathscr{G}_1\mathscr{G}_3\left\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}|f(u,x(u),y(u),z(u))|du\right)ds\right.\\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}|f(m,x(m),y(m),z(m))|dm\right)du\right)ds\\
&\left.+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}|h(u,x(u),y(u),z(u))|du\right)ds\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathscr{G}_2\mathscr{G}_6\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\left(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}|h(u,x(u),y(u),z(u))|du\right)ds\\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}|h(m,x(m),y(m),z(m))|dm\right)du\right)ds\\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}|g(u,x(u),y(u),z(u))|du\right)ds\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\tau}e^{-\lambda(\tau-s)}\left(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}|h(u,x(u),y(u),z(u))|du\right)ds,\\
\leq&\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\Bigg\{\Bigg[\mathscr{G}_3\mathscr{G}_6\left\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}\right\}+\mathscr{G}_1\mathscr{G}_3\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\left(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\right)\\
&+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\Bigg\}\Bigg]+\Bigg[\mathscr{G}_3\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)\\
&+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}+\mathscr{G}_2\mathscr{G}_6\left\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\right\}\Bigg]\\
&+\Bigg[\mathscr{G}_1\mathscr{G}_3\left\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\right\}+\mathscr{G}_1\mathscr{G}_6\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\left(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\right)\\
&+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\Bigg\}+\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\Bigg]\Bigg\},\\
&\leq\left[\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\right](\mathcal{U}_3\wp_1+\mathcal{V}_3\wp_2+\mathcal{W}_3\wp_3),
\end{aligned}
$$

This implies that

$$
||\mathcal{E}_3(x(\tau),y(\tau),z(\tau))||_{\hat{\mathcal{J}}}\leq\left[\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\right](\mathcal{U}_3\wp_1+\mathcal{V}_3\wp_2+\mathcal{W}_3\wp_3),
$$

We can conclude that the operator $\mathcal{E}$ is bounded uniformly from the above argument:

$$
\begin{aligned}
\|\mathcal{E}(x(\tau),y(\tau),z(\tau))\|\leq&\left[\frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathscr{G}_1}+\frac{(\lambda-1+e^{\lambda})}{\lambda^2\Delta}+\frac{(\lambda^2-2\lambda+2-2e^{-\lambda})}{\lambda^3\Delta}\right]\\
&+(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\wp_1+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)\wp_2+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)\wp_3.
\end{aligned}
$$

Following that, we demonstrate that $\mathcal{E}$ is equi-continuous. By letting $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$, we get

$$
\begin{aligned}
&|\mathcal{E}_1(x(\tau_2),y(\tau_2),z(\tau_2))-\mathcal{E}_1(x(\tau_1),y(\tau_1),z(\tau_1))|\\
\leq&\frac{(\lambda\tau_2-\lambda\tau_1+e^{\lambda\tau_2}-e^{\lambda\tau_1})}{\lambda^2\mathscr{G}_1}\\
&\times\Bigg[\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}g(u,x(u),y(u),z(u))du\right)ds\\
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}g(m,x(m),y(m),z(m))dm\right)du\right)ds\\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}f(u,x(u),y(u),z(u))du\right)ds\\
&+\frac{1}{\Delta}\Bigg(\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}p_j\int_0^{\theta_j}e^{-\lambda(\theta_j-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}g(u,x(u),y(u),z(u))du\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{B}_1\int_0^{\varpi}\frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)}g(m,x(m),y(m),z(m))dm\right)du\right)ds\\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}h(u,x(u),y(u),z(u))du\right)ds\Bigg\}\\
&+\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4\Bigg\{\mathcal{A}_3\sum_{j=1}^{k-2}r_j\int_0^{\eta_j}e^{-\lambda(\eta_j-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}f(u,x(u),y(u),z(u))du\right)ds\\
&+\mathcal{B}_3\int_0^{\vartheta}\frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)}f(m,x(m),y(m),z(m))dm\right)du\right)ds\\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}h(u,x(u),y(u),z(u))du\right)ds\Bigg\}\\
&+\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}q_j\int_0^{\delta_j}e^{-\lambda(\delta_j-s)}\left(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}h(u,x(u),y(u),z(u))du\right)ds\\
&+\mathcal{B}_2\int_0^{\varrho}\frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)}\left(\int_0^s e^{-\lambda(s-u)}\left(\int_0^u\frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)}h(m,x(m),y(m),z(m))dm\right)du\right)ds\\
&+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}g(u,x(u),y(u),z(u))du\right)ds\Bigg\}\Bigg)\Bigg]\\
&+\Bigg|\int_0^{\tau_1}(e^{-\varphi(\tau_2-s)}-e^{-\varphi(\tau_1-s)})\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}f(u,x(u),y(u),z(u))du\right)ds\\
&+\int_{\tau_2}^{\tau_1}e^{-\varphi(\tau_2-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}f(u,x(u),y(u),z(u))du\right)ds\Bigg|,
\end{aligned}
$$

$$
\begin{aligned}
\le&\frac{(\lambda\tau_2-\lambda\tau_1+e^{\lambda\tau_2}-e^{\lambda\tau_1})}{\lambda^2\mathcal{G}_1}\\
&\times\Bigg\{\Bigg[\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}+\frac{\mathcal{G}_2\mathcal{G}_4\mathcal{G}_6}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}+\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4}{\Delta}\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\left(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\right)\\
&+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\Bigg\}\Bigg]\wp_1\\
&+\Bigg[\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\
&+\frac{\mathcal{G}_2\mathcal{G}_4\mathcal{G}_6}{\Delta}\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\
&+\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\Bigg]\wp_2+\Bigg[\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4}{\Delta}\left\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\right\}+\frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5}{\Delta}\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\left(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\right)\\
&+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\Bigg\}\Bigg]\wp_3\Bigg\}\\
&+\Bigg|\int_0^{\tau_1}(e^{-\varphi(\tau_2-s)}-e^{-\varphi(\tau_1-s)})\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}du\right)ds\\
&+\int_{\tau_2}^{\tau_1}e^{-\varphi(\tau_2-s)}\left(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}du\right)ds\Bigg|\wp_1.
\end{aligned}
$$

Similarly, it holds that

$$
\begin{aligned}
&|\mathcal{E}_2(x(\tau_2),y(\tau_2),z(\tau_2))-\mathcal{E}_2(x(\tau_1),y(\tau_1),z(\tau_1))|\\
=&\frac{(\lambda\tau_2-\lambda\tau_1+e^{\lambda\tau_2}-e^{\lambda\tau_1})}{\lambda^2\Delta}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg[\Bigg(\mathscr{G}_4\mathscr{G}_6 \Bigg\{ \mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du \bigg) ds \\
&+ \mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi - s)^{\omega-1}}{\Gamma(\omega)} \bigg(\int_0^s e^{-\lambda(s-u)} \bigg(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} g(m,x(m),y(m),z(m))dm \bigg) du \bigg) ds \\
&+ \int_0^1 e^{-\lambda(1-s)} \bigg(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du \bigg) ds \Bigg\} \\
&+ \mathscr{G}_1\mathscr{G}_4 \Bigg\{ \mathcal{A}_3 \sum_{j=1}^{k-2} r_j \int_0^{\eta_j} e^{-\lambda(\eta_j - s)} \bigg(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du \bigg) ds \\
&+ \mathcal{B}_3 \int_0^{\vartheta} \frac{(\vartheta - s)^{\varsigma-1}}{\Gamma(\varsigma)} \bigg(\int_0^s e^{-\lambda(s-u)} \bigg(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)} f(m,x(m),y(m),z(m))dm \bigg) du \bigg) ds \\
&+ \int_0^1 e^{-\lambda(1-s)} \bigg(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du \bigg) ds \Bigg\} \\
&+ \mathscr{G}_1\mathscr{G}_5 \Bigg\{ \mathcal{A}_2 \sum_{j=1}^{k-2} q_j \int_0^{\delta_j} e^{-\lambda(\delta_j - s)} \bigg(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du \bigg) ds \\
&+ \mathcal{B}_2 \int_0^{\varrho} \frac{(\varrho - s)^{\psi-1}}{\Gamma(\psi)} \bigg(\int_0^s e^{-\lambda(s-u)} \bigg(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)} h(m,x(m),y(m),z(m))dm \bigg) du \bigg) ds \\
&+ - \int_0^1 e^{-\lambda(1-s)} \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du \bigg) ds \Bigg\} \Bigg) \Bigg] + \\
&+ \bigg| \int_0^{\tau_1} (e^{-\varphi(\tau_2 - s)} - e^{-\varphi(\tau_1 - s)}) \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du \bigg) ds \\
&+ \int_{\tau_1}^{\tau_2} (e^{-\varphi(\tau_2 - s)}) \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du \bigg) ds \bigg|, \\
\leq& \frac{(\lambda\tau_2 - \lambda\tau_1 + e^{\lambda\tau_2} - e^{\lambda\tau_1})}{\lambda^2\Delta} \Bigg\{ \Bigg[\mathscr{G}_4\mathscr{G}_6 \bigg\{ \frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)} \bigg\} + \mathscr{G}_1\mathscr{G}_4 \Bigg\{ \mathcal{A}_1 \sum_{j=1}^{k-2} |r_j| \bigg(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)} \bigg) \\
&+ \mathcal{B}_3 \frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)} (\vartheta\lambda + e^{-\lambda\vartheta} - 1) \Bigg\} \Bigg] \wp_1 \\
&+ \Bigg[\mathscr{G}_4\mathscr{G}_6 \Bigg\{ \mathcal{A}_1 \sum_{j=1}^{k-2} |p_j| \bigg(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)} \bigg) + \mathcal{B}_1 \frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)} (\varpi\lambda + e^{-\lambda\varpi} - 1) \Bigg\} \\
&+ \mathscr{G}_1\mathscr{G}_5 \bigg\{ \frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)} \bigg\} \Bigg] \wp_2 + \Bigg[\mathscr{G}_1\mathscr{G}_4 \bigg\{ \frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)} \bigg\} + \mathscr{G}_1\mathscr{G}_5 \Bigg\{ \mathcal{A}_2 \sum_{j=1}^{k-2} |q_j| \bigg(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)} \bigg) \\
&+ \mathcal{B}_2 \frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)} (\varrho\lambda + e^{-\lambda\varrho} - 1) \Bigg\} \Bigg] \wp_3 \Bigg\} \\
&+ \bigg| \int_0^{\tau_1} (e^{-\varphi(\tau_2 - s)} - e^{-\varphi(\tau_1 - s)}) \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} du \bigg) ds \\
&+ \int_{\tau_1}^{\tau_2} (e^{-\varphi(\tau_2 - s)}) \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} du \bigg) ds \bigg| \wp_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{E}_3(x(\tau_2), y(\tau_2), z(\tau_2)) - \mathcal{E}_3(x(\tau_1), y(\tau_1), z(\tau_1))| \\
\leq& \frac{(\lambda^2\tau^2 - 2\lambda\tau + 2 - 2e^{\lambda\tau})}{\lambda^3\Delta} \\
&\times \Bigg[\Bigg(\mathscr{G}_3\mathscr{G}_6 \Bigg\{ \mathcal{A}_1 \sum_{j=1}^{k-2} p_j \int_0^{\theta_j} e^{-\lambda(\theta_j - s)} \bigg(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du \bigg) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{B}_1 \int_0^{\varpi} \frac{(\varpi-s)^{\omega-1}}{\Gamma(\omega)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\beta-2}}{\Gamma(\beta-1)} g(m,x(m),y(m),z(m))dm \right) du \right) ds \\
&+ \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du \right) ds \Bigg\} \\
&+\mathcal{G}_1\mathcal{G}_3 \Bigg\{ \mathcal{A}_3 \sum_{j=1}^{k-2} r_j \int_0^{\eta_j} e^{-\lambda(\eta_j-s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} f(u,x(u),y(u),z(u))du \right) ds \\
&+\mathcal{B}_3 \int_0^{\vartheta} \frac{(\vartheta-s)^{\varsigma-1}}{\Gamma(\varsigma)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\alpha-2}}{\Gamma(\alpha-1)} f(m,x(m),y(m),z(m))dm \right) du \right) ds \\
&+ \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du \right) ds \Bigg\} \\
&+\mathcal{G}_2\mathcal{G}_6 \Bigg\{ \mathcal{A}_2 \sum_{j=1}^{k-2} q_j \int_0^{\delta_j} e^{-\lambda(\delta_j-s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} h(u,x(u),y(u),z(u))du \right) ds \\
&+\mathcal{B}_2 \int_0^{\varrho} \frac{(\varrho-s)^{\psi-1}}{\Gamma(\psi)} \left(\int_0^s e^{-\lambda(s-u)} \left(\int_0^u \frac{(u-m)^{\gamma-2}}{\Gamma(\gamma-1)} h(m,x(m),y(m),z(m))dm \right) du \right) ds \\
&+ \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} g(u,x(u),y(u),z(u))du \right) ds \Bigg\} \Bigg) \Bigg] \\
&+ \left| \int_0^{\tau_1} (e^{-\varphi(\tau_2-s)} - e^{-\varphi(\tau_1-s)}) \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} du \right) ds \right. \\
&\left. + \int_{\tau_1}^{\tau_2} e^{-\varphi(\tau_2-s)} \left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} du \right) ds \right| \wp_3.
\end{aligned}
$$

The preceding inequalities are unrelated to x, y, and z and tend toward zero as $\tau_1 \to \tau_2$. This demonstrates that the operator $\mathcal{E}(x,y,z)$ is equi-continuous. As a result, we can conclude that the operator $\mathcal{E}(x,y,z)$ is completely continuous.

Finally, the set $\mathcal{P} = \{(x,y,z) \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} : (x,y,z) = \nu\mathcal{E}(x,y,z), 0 \leq \nu \leq 1\}$, showing that it is bounded.

Let $(x,y,z) \in \mathcal{P}$ with $(x,y,z) = \nu\mathcal{E}(x,y,z)$. For any $\tau \in [0,1]$, we have $x(\tau) = \nu\mathcal{E}_1(x,y,z)(\tau)$, $y(\tau) = \nu\mathcal{E}_2(x,y,z)(\tau)$, $z(\tau) = \nu\mathcal{E}_3(x,y,z)(\tau)$. Then, by $(\mathcal{H}_2)$, we have

$$
\begin{aligned}
|x(\tau)| \leq & \left\{ \frac{(\lambda\tau - 1 + e^{-\lambda\tau})}{\lambda^2 \mathcal{G}_1} \right\} + \mathcal{U}_1(\sigma_0 + \sigma_1|x| + \sigma_2|y| + \sigma_3|z|) \\
& + \mathcal{V}_1(\varepsilon_0 + \varepsilon_1|x| + \varepsilon_2|y| + \varepsilon_3|z|) + \mathcal{W}_1(\kappa_0 + \kappa_1|x| + \kappa_2|y| + \kappa_3|z|), \\
\leq & \left\{ \frac{(\lambda\tau - 1 + e^{-\lambda\tau})}{\lambda^2 \mathcal{G}_1} \right\} + \mathcal{U}_1\sigma_0 + \mathcal{V}_1\varepsilon_0 + \mathcal{W}_1\kappa_0 \\
& + (\mathcal{U}_1\sigma_1 + \mathcal{V}_1\varepsilon_1 + \mathcal{W}_1\kappa_1)|x| + (\mathcal{U}_1\sigma_2 + \mathcal{V}_1\varepsilon_2 + \mathcal{W}_1\kappa_2)|y| + (\mathcal{U}_1\sigma_3 + \mathcal{V}_1\varepsilon_3 + \mathcal{W}_1\kappa_3)|z|,
\end{aligned}
$$

$$
\begin{aligned}
|y(\tau)| \leq & \left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \Delta} \right\} + \mathcal{U}_2\sigma_0 + \mathcal{V}_2\varepsilon_0 + \mathcal{W}_2\kappa_0 \\
& + (\mathcal{U}_2\sigma_1 + \mathcal{V}_2\varepsilon_1 + \mathcal{W}_2\kappa_1)|x| + (\mathcal{U}_2\sigma_2 + \mathcal{V}_2\varepsilon_2 + \mathcal{W}_2\kappa_2)|y| + (\mathcal{U}_2\sigma_3 + \mathcal{V}_2\varepsilon_3 + \mathcal{W}_2\kappa_3)|z|,
\end{aligned}
$$

and

$$
\begin{aligned}
|z(\tau)| \leq & \left[\frac{(\lambda^2\tau^2 - 2\lambda\tau + 2 - 2e^{-\lambda\tau})}{\lambda^3 \Delta} \right] + \mathcal{U}_3\sigma_0 + \mathcal{V}_3\varepsilon_0 + \mathcal{W}_3\kappa_0 \\
& + (\mathcal{U}_3\sigma_1 + \mathcal{V}_3\varepsilon_1 + \mathcal{W}_3\kappa_1)|x| + (\mathcal{U}_3\sigma_2 + \mathcal{V}_3\varepsilon_2 + \mathcal{W}_3\kappa_2)|y| + (\mathcal{U}_3\sigma_3 + \mathcal{V}_3\varepsilon_3 + \mathcal{W}_3\kappa_3)|z|.
\end{aligned}
$$

From the previous arguments, it follows that

$$||x(\tau)|| \leq \left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \mathscr{G}_1} \right\} + \mathcal{U}_1\sigma_0 + \mathcal{V}_1\varepsilon_0 + \mathcal{W}_1\kappa_0 + (\mathcal{U}_1\sigma_1 + \mathcal{V}_1\varepsilon_1 + \mathcal{W}_1\kappa_1)||x|| \\ + (\mathcal{U}_1\sigma_2 + \mathcal{V}_1\varepsilon + \mathcal{W}_1\kappa_2)||y|| + (\mathcal{U}_1\sigma_3 + \mathcal{V}_1\varepsilon_3 + \mathcal{W}_1\kappa_3)||z||,$$

$$||y(\tau)|| \leq \left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \Delta} \right\} + \mathcal{U}_2\sigma_0 + \mathcal{V}_2\varepsilon_0 + \mathcal{W}_2\kappa_0 + (\mathcal{U}_2\sigma_1 + \mathcal{V}_2\varepsilon_1 + \mathcal{W}_2\kappa_1)||x|| \\ + (\mathcal{U}_2\sigma_2 + \mathcal{V}_2\varepsilon_2 + \mathcal{W}_2\kappa_2)||y|| + (\mathcal{U}_2\sigma_3 + \mathcal{V}_2\varepsilon_3 + \mathcal{W}_2\kappa_3)||z||,$$

$$||z(\tau)|| \leq \left[\frac{(\lambda^2\tau^2 - 2\lambda\tau + 2 - 2e^{-\lambda\tau})}{\lambda^3 \Delta} \right] + \mathcal{U}_3\sigma_0 + \mathcal{V}_3\varepsilon_0 + \mathcal{W}_3\kappa_0 + (\mathcal{U}_3\sigma_1 + \mathcal{V}_3\varepsilon_1 + \mathcal{W}_3\kappa_1)||x|| \\ + (\mathcal{U}_3\sigma_2 + \mathcal{V}_3\varepsilon_2 + \mathcal{W}_3\kappa_2)||y|| + (\mathcal{U}_3\sigma_3 + \mathcal{V}_3\varepsilon_3 + \mathcal{W}_3\kappa_3)||z||.$$

By combining the three previously mentioned inequalities, we arrive at

$$\begin{aligned} ||x|| + ||y|| + ||z|| \leq & \left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \mathscr{G}_1} \right\} + \left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \Delta} \right\} + \left[\frac{(\lambda^2\tau^2 - 2\lambda\tau + 2 - 2e^{-\lambda\tau})}{\lambda^3 \Delta} \right] \\ & + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_0 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_0 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_0 \\ & + [(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_1 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_1]||x|| \\ & + [(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_2 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_2 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_2]||y|| \\ & + [(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_3 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_3 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_3]||z||, \end{aligned}$$

This implies that

$$||(x,y,z)||_{\hat{\mathcal{J}}} \leq \frac{1}{\Phi} \left[\left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \mathscr{G}_1} \right\} + \left\{ \frac{(\lambda\tau - 1 + e^{\lambda\tau})}{\lambda^2 \Delta} \right\} + \left[\frac{(\lambda^2\tau^2 - 2\lambda\tau + 2 - 2e^{-\lambda\tau})}{\lambda^3 \Delta} \right] \\ + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_0 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_0 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_0 \right],$$

where $\Phi = \min\{1 - [(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_i + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_i + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_i], i = 1,2,3\}$.

As a result, the set $\mathcal{P}$ is bounded. Thus, using the Leray–Schauder alternative, we deduced that the operator $\mathcal{E}$ has at least one fixed point, implying that the problem in Equation (3) has at least one solution in the interval $[0,1]$. The proof is now complete.

Our next existence and uniqueness results are based on Banach's principle of contraction mapping. □

Theorem 2. *Let $\Upsilon \neq 0$, where Υ is defined by Equation(11). Assuming that $f, g, h : [0,1] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are continuous functions (c.f) and there exist non-negative constants Θ_1, Θ_2, and Θ_3 such that $\forall \tau \in [0,1]$ and $x_i, y_i, z_i \in \mathcal{R}_e, i = 1,2,3$, we have*

$$\begin{aligned} \left| f(\tau, y_1, y_2, y_3) - f(\tau, z_1, z_2, z_3) \right| &\leq \Theta_1 \left(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| \right), \\ \left| g(\tau, y_1, y_2, y_3) - g(\tau, z_1, z_2, z_3) \right| &\leq \Theta_2 \left(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| \right), \\ \left| h(\tau, y_1, y_2, y_3) - h(\tau, z_1, z_2, z_3) \right| &\leq \Theta_3 \left(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| \right). \end{aligned}$$

if

$$(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\Theta_2 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\Theta_3 < 1, \tag{14}$$

where $\mathcal{U}_i, \mathcal{V}_i$, and $\mathcal{W}_i$ are specified. Then, the system in Equation (3) has a unique solution on the interval $[0,1]$.

Proof. Define $\sup_{\tau\in[0,1]} f(\tau,0,0,0) = Q_1 < \infty, \sup_{\tau\in[0,1]} g(\tau,0,0,0) = Q_2 < \infty$, $\sup_{\tau\in[0,1]} h(\tau,0,0,0) = Q_3 < \infty$ and $\Psi > 0$ such that

$$\Psi > \frac{\left\{\frac{(\lambda\tau-1+e^{\lambda\tau})}{\lambda^2\mathscr{G}_1}\right\} + \left\{\frac{(\lambda\tau-1+e^{\lambda\tau})}{\lambda^2\Delta}\right\} + \left[\frac{(\lambda^2\tau^2-2\lambda\tau+2-2e^{-\lambda\tau})}{\lambda^3\Delta}\right] + \mathcal{L}}{1-(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\Theta_1-(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)\Theta_2-(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)\Theta_3},$$

where $\mathcal{L} = (\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)Q_1 + (\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)Q_2 + (\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)Q_3$.

In the first step, we show that $\mathcal{E}\mathcal{B}_\Psi \subset \mathcal{B}_\Psi$, where $\mathcal{B}_\Psi = \{(x,y,z) \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} : \|(x,y,z)\| \leq \Psi\}$.

By assumption $(\mathcal{H}_2)$, for $(x,y,z) \subset \mathcal{B}_\Psi, \tau \in [0,1]$, we have

$$\begin{aligned}\left|x\big(\tau,x(\tau),y(\tau),z(\tau)\big)\right| &\leq \left|x\big(\tau,x(\tau),y(\tau),y(\tau)\big) - x(\tau,0,0,0)\right|,\\ &\leq \Theta_1\big(|x(\tau)|+|y(\tau)|+|z(\tau)|\big)+Q_1,\\ &\leq \Theta_1\big(\|x\|+\|y\|+\|z\|\big)+Q_1, \leq \Theta_1\Psi+Q_1,\end{aligned} \tag{15}$$

$$\begin{aligned}\left|y\big(\tau,x(\tau),y(\tau),z(\tau)\big)\right| &\leq \left|y\big(\tau,x(\tau),y(\tau),y(\tau)\big) - y(\tau,0,0,0)\right|,\\ &\leq \Theta_2\big(|x(\tau)|+|y(\tau)|+|z(\tau)|\big)+Q_2,\\ &\leq \Theta_2\big(\|x\|+\|y\|+\|z\|\big)+Q_2, \leq \Theta_2\Psi+Q_2,\end{aligned} \tag{16}$$

$$\begin{aligned}\left|z\big(\tau,x(\tau),y(\tau),z(\tau)\big)\right| &\leq \left|z\big(\tau,x(\tau),y(\tau),y(\tau)\big) - z(\tau,0,0,0)\right|,\\ &\leq \Theta_3\big(|x(\tau)|+|y(\tau)|+|z(\tau)|\big)+Q_3,\\ &\leq \Theta_3\big(\|x\|+\|y\|+\|z\|\big)+Q_3, \leq \Theta_3\Psi+Q_3,\end{aligned} \tag{17}$$

Using Equations (15–17), we obtain

$$\begin{aligned}&|\mathcal{E}_1((x,y,z)(\tau))|\\ &\leq \frac{(\lambda\tau-1+e^{-\lambda\tau})}{\lambda^2\mathscr{G}_1}\Bigg\{\Bigg[\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)} + \frac{\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)} + \frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4}{\Delta}\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\left(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\right)\\ &\quad+\mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\Bigg\}\Bigg]||f||\\ &\quad+\Bigg[\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\ &\quad+\frac{\mathscr{G}_2\mathscr{G}_4\mathscr{G}_6}{\Delta}\Bigg\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right)+\mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\Bigg\}\\ &\quad+\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)}\Bigg]||g||+\Bigg[\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_4}{\Delta}\Bigg\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\Bigg\}+\frac{\mathscr{G}_1\mathscr{G}_2\mathscr{G}_5}{\Delta}\Bigg\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\left(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\right)\\ &\quad+\mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\Bigg\}\Bigg]||h||\Bigg\}+\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)},\\ &\leq\left(\frac{(\lambda\tau-1+e^{-\lambda\tau})}{\lambda^2\mathscr{G}_1}\right)+\mathcal{U}_1(\Theta_1\Psi+Q_1)+\mathcal{V}_1(\Theta_2\Psi+Q_2)+\mathcal{W}_1(\Theta_3\Psi+Q_3),\\ &\leq\left(\frac{(\lambda\tau-1+e^{-\lambda\tau})}{\lambda^2\mathscr{G}_1}\right)+(\mathcal{U}_1\Theta_1+\mathcal{V}_1\Theta_2+\mathcal{W}_1\Theta_3)\Psi+\mathcal{U}_1Q_1+\mathcal{V}_1Q_2+\mathcal{W}_1Q_3,\end{aligned}$$

Which, when the norm for $\tau \in [0,1]$ is taken, yields

$$\|\mathcal{E}_1(x,y,z)\|_{\hat{\mathcal{J}}} \leq \left(\frac{(\lambda\tau-1+e^{-\lambda\tau})}{\lambda^2\mathscr{G}_1}\right)+(\mathcal{U}_1\Theta_1+\mathcal{V}_1\Theta_2+\mathcal{W}_1\Theta_3)\Psi+\mathcal{U}_1Q_1+\mathcal{V}_1Q_2+\mathcal{W}_1Q_3.$$

Similarly, we can obtain

$$\left\|\mathcal{E}_2(x,y,z)\right\|_{\hat{\mathcal{J}}} \leq \left(\frac{(\lambda\tau-1+e^{-\lambda\tau})}{\lambda^2\Delta}\right) + (\mathcal{U}_2\Theta_1+\mathcal{V}_2\Theta_2+\mathcal{W}_2\Theta_3)\Psi + \mathcal{U}_2Q_1+\mathcal{V}_2Q_2+\mathcal{W}_2Q_3.$$

and

$$\begin{aligned}\left\|\mathcal{E}_3(x,y,z)\right\|_{\hat{\mathcal{J}}} \leq & \left[\frac{(\lambda^2\tau^2-2\lambda\tau+2-2e^{-\lambda\tau})}{\lambda^3\Delta}\right] \\ & + (\mathcal{U}_3\Theta_1+\mathcal{V}_3\Theta_2+\mathcal{W}_3\Theta_3)\Psi + \mathcal{U}_3Q_1+\mathcal{V}_3Q_2+\mathcal{W}_3Q_3.\end{aligned}$$

Consequently, it holds that

$$\begin{aligned}\left\|\mathcal{E}(x,y,z)\right\|_{\hat{\mathcal{J}}} \leq & \left\{\frac{(\lambda\tau-1+e^{\lambda\tau})}{\lambda^2\mathcal{G}_1}\right\} + \left\{\frac{(\lambda\tau-1+e^{\lambda\tau})}{\lambda^2\Delta}\right\} + \left[\frac{(\lambda^2\tau^2-2\lambda\tau+2-2e^{-\lambda\tau})}{\lambda^3\Delta}\right] \\ & + \left[(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\Theta_1 + (\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)\Theta_2 + (\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)\Theta_3\right]\Psi \\ & + (\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)Q_1 + (\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)Q_2 + (\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)Q_3 \\ \leq & \Psi.\end{aligned}$$

Now, for $(x_1,y_1,z_1),(x_2,y_2,z_2) \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$ and for any $\tau \in [0,1]$, we get

$$\begin{aligned}& \left|\mathcal{E}_1\big((x_2,y_2,z_2)(\tau)\big) - \mathcal{E}_1\big((x_1,y_1,z_1)(\tau)\big)\right| \\ \leq & \frac{(\lambda-1+e^{-\lambda})}{\lambda^2\mathcal{G}_1}\left[\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)} + \frac{\mathcal{G}_2\mathcal{G}_4\mathcal{G}_6}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)} + \frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4}{\Delta}\left\{\mathcal{A}_1\sum_{j=1}^{k-2}|r_j|\left(\frac{\eta_j(1-e^{-\lambda\eta_j})}{\lambda\Gamma(\alpha)}\right)\right.\right. \\ & \left. + \mathcal{B}_3\frac{\vartheta^{\alpha+\varsigma-1}}{\lambda^2\Gamma(\alpha)\Gamma(\varsigma)}(\vartheta\lambda+e^{-\lambda\vartheta}-1)\right\} \times \Theta_1(||x_2-x_1||+||y_2-y_1||+||z_2-z_1||) \\ & + \left\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right) + \mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\right\} \\ & + \frac{\mathcal{G}_2\mathcal{G}_4\mathcal{G}_6}{\Delta}\left\{\mathcal{A}_1\sum_{j=1}^{k-2}|p_j|\left(\frac{\theta_j(1-e^{-\lambda\theta_j})}{\lambda\Gamma(\beta)}\right) + \mathcal{B}_1\frac{\varpi^{\beta+\omega-1}}{\lambda^2\Gamma(\beta)\Gamma(\omega)}(\varpi\lambda+e^{-\lambda\varpi}-1)\right\} \\ & + \frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5}{\Delta}\frac{(1-e^{-\lambda})}{\lambda\Gamma(\beta)} \times \Theta_2(||x_2-x_1||+||y_2-y_1||+||z_2-z_1||) \\ & + \frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4}{\Delta}\left\{\frac{(1-e^{-\lambda})}{\lambda\Gamma(\gamma)}\right\} + \frac{\mathcal{G}_1\mathcal{G}_2\mathcal{G}_5}{\Delta}\left\{\mathcal{A}_2\sum_{j=1}^{k-2}|q_j|\left(\frac{\delta_j(1-e^{-\lambda\delta_j})}{\lambda\Gamma(\gamma)}\right)\right. \\ & \left.\left. + \mathcal{B}_2\frac{\varrho^{\gamma+\psi-1}}{\lambda^2\Gamma(\gamma)\Gamma(\psi)}(\varrho\lambda+e^{-\lambda\varrho}-1)\right\}\right] \times \Theta_3(||x_2-x_1||+||y_2-y_1||+||z_2-z_1||) \\ & + \frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}, \\ \leq & (\mathcal{U}_1\Theta_1+\mathcal{V}_1\Theta_2+\mathcal{W}_1\Theta_3)(||x_2-x_1||+||y_2-y_1||+||z_2-z_1||).\end{aligned}$$

This implies that

$$\begin{aligned}& \left\|\mathcal{E}_1\big((x_2,y_2,z_2)(\tau)\big) - \mathcal{E}_1\big((x_1,y_1,z_1)(\tau)\big)\right\|_{\hat{\mathcal{J}}} \\ & \quad \leq (\mathcal{U}_1\Theta_1+\mathcal{V}_1\Theta_2+\mathcal{W}_1\Theta_3)(||x_2-x_1||+||y_2-y_1||+||z_2-z_1||).\end{aligned}$$

Similarly, we get

$$\begin{aligned}& ||\mathcal{E}_2\big((x_2,y_2,z_2)(\tau)\big) - \mathcal{E}_2\big((x_1,y_1,z_1)(\tau)\big)||_{\hat{\mathcal{J}}} \\ & \leq (\mathcal{U}_2\Theta_1+\mathcal{V}_2\Theta_2+\mathcal{W}_2\Theta_3)(||x_2-x_1||+||y_2-y_1||+||z_2-z_1||).\end{aligned}$$

and

$$
\begin{aligned}
&||\mathcal{E}_3((x_2,y_2,z_2)(\tau)) - \mathcal{E}_3((x_1,y_1,z_1)(\tau))||_{\hat{\mathcal{J}}} \\
&\leq (\mathcal{U}_3\Theta_1 + \mathcal{V}_3\Theta_2 + \mathcal{W}_3\Theta_3)(||x_2 - x_1|| + ||y_2 - y_1|| + ||z_2 - z_1||).
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\mathcal{E}(x_2,y_2,z_2) - \mathcal{E}(x_1,y_1,z_1)\right\|_{\hat{\mathcal{J}}} \\
&\quad \leq \left[(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\Theta_2 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\Theta_3\right] \\
&\qquad \times (||x_2 - x_1|| + ||y_2 - y_1|| + ||z_2 - z_1||).
\end{aligned}
$$

When combined with Equation (14), the previous inequality implies that $\mathcal{E}$ is a contraction. As a result of Banach's fixed point theorem, there exists a unique fixed point for the operator $\mathcal{E}$ which corresponds to a unique solution for the problem in Equation (3) in the range $[0,1]$. The proof has been completed. □

4. Hyers–Ulam Stability

Let us define operators $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3 \in \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e) \to \mathcal{C}([0,1],\mathcal{R}_e)$, where $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ are defined by

$$
\begin{cases}
(\mathcal{D}^{\alpha} + \varphi\mathcal{D}^{\alpha-1})x(\tau) - f(\tau, x(\tau), y(\tau), z(\tau)) = \mathcal{Z}_1(x,y,z)(\tau), \\
(\mathcal{D}^{\beta} + \varphi\mathcal{D}^{\beta-1})y(\tau) - g(\tau, x(\tau), y(\tau), z(\tau)) = \mathcal{Z}_2(x,y,z)(\tau), \\
(\mathcal{D}^{\gamma} + \varphi\mathcal{D}^{\gamma-1})z(\tau) - h(\tau, x(\tau), y(\tau), z(\tau)) = \mathcal{Z}_3(x,y,z)(\tau),
\end{cases} \tag{18}
$$

for $\tau \in [0,1]$. For some $\pi_1, \pi_2, \pi_3 > 0$, we consider the following inequalities:

$$
\left\|\mathcal{Z}_1(x,y,z)\right\| \leq \pi_1, \qquad \left\|\mathcal{Z}_2(x,y,z)\right\| \leq \pi_2 \qquad \left\|\mathcal{Z}_3(x,y,z)\right\| \leq \pi_3. \tag{19}
$$

Definition 4. *The coupled system in Equation (3) is said to be stable in the Hyers–Ulam sense if $\wp_1, \wp_2, \wp_3 > 0$ such that there is a unique solution $(x,y,z) \in \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e)$ for the problems in Equation (3) with* [45]

$$
\left\|(x,y,z) - (\hat{x},\hat{y},\hat{z})\right\| \leq \mathcal{K}_1\pi_1 + \mathcal{K}_2\pi_2 + \mathcal{K}_3\pi_3,
$$

For every solution, $(\hat{x},\hat{y},\hat{z})$ belongs to $\mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e)$ of the inequality.

Theorem 3. *Suppose that $(\mathcal{H}_2)$ holds. Then, the BVPs in Equaiton (3) are Hyers–Ulam (H-U) stable.*

Proof. Let $(x,y,z) \in \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e) \times \mathcal{C}([0,1],\mathcal{R}_e)$ be for Equation (3) the solutions of the problems that satisfy the main results. Let $(\hat{x},\hat{y},\hat{z})$ be any satisfying solution for Equation (19):

$$
\begin{cases}
(\mathcal{D}^{\alpha} + \varphi\mathcal{D}^{\alpha-1})x(\tau) = f(\tau, x(\tau), y(\tau), z(\tau)) + \mathcal{Z}_1(x,y,z)(\tau), \\
(\mathcal{D}^{\beta} + \varphi\mathcal{D}^{\beta-1})y(\tau) = g(\tau, x(\tau), y(\tau), z(\tau)) + \mathcal{Z}_2(x,y,z)(\tau), \\
(\mathcal{D}^{\gamma} + \varphi\mathcal{D}^{\gamma-1})z(\tau) = h(\tau, x(\tau), y(\tau), z(\tau)) + \mathcal{Z}_3(x,y,z)(\tau),
\end{cases} \tag{20}
$$

for $\tau \in [0,1]$. Therefore,

$$
\begin{aligned}
\hat{x}(\tau) =& \mathcal{E}_1(\hat{x},\hat{y},\hat{z})(\tau) \\
&+ \left(\frac{1 - e^{-\varphi\tau}}{\varphi \Upsilon}\right)\Bigg\{\mathcal{A}_4\mathcal{A}_5\Bigg(\mathcal{B}_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j - s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}\mathcal{Z}_2(x,y,z)du\right)ds \\
&+ \int_0^1 e^{-\varphi(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}\mathcal{Z}_1(x,y,z)du\right)ds\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{A}_5\mathcal{A}_2\Bigg(\mathcal{B}_2\sum_{j=1}^{k-2}v_j\int_0^{\varpi_j}e^{-\varphi(\varpi_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}\mathcal{Z}_3(x,y,z)du\bigg)ds\\
&+\int_0^1 e^{-\varphi(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}\mathcal{Z}_2(x,y,z)du\bigg)ds\Bigg)\\
&+\mathcal{A}_2\mathcal{A}_3\Bigg(\mathcal{B}_3\sum_{j=1}^{k-2}\vartheta_j\int_0^{\rho_j}e^{-\varphi(\rho_j-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}\mathcal{Z}_1(x,y,z)du\bigg)ds\\
&+\int_0^1 e^{-\varphi(1-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}\mathcal{Z}_3(x,y,z)du\bigg)ds\Bigg)\Bigg\}\\
&+\int_0^\tau e^{-\varphi(\tau-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}\mathcal{Z}_1(x,y,z)du\bigg)ds,
\end{aligned}
$$

$$
\begin{aligned}
&|\widehat{x}(\tau)-\mathcal{E}_1(\widehat{x},\widehat{y},\widehat{z})(\tau)|\\
&\le\bigg(\frac{1-e^{-\varphi\tau}}{\varphi\Upsilon}\bigg)\Bigg\{\mathcal{A}_4\mathcal{A}_5\Bigg(\mathcal{B}_1\sum_{j=1}^{k-2}w_j\int_0^{\varrho_j}e^{-\varphi(\varrho_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}\pi_2du\bigg)ds\\
&+\int_0^1 e^{-\varphi(1-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}\pi_1du\bigg)ds\Bigg)\\
&+\mathcal{A}_5\mathcal{A}_2\Bigg(\mathcal{B}_2\sum_{j=1}^{k-2}v_j\int_0^{\varpi_j}e^{-\varphi(\varpi_j-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}\pi_3du\bigg)ds\\
&+\int_0^1 e^{-\varphi(1-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}\pi_2du\bigg)ds\Bigg)\\
&+\mathcal{A}_2\mathcal{A}_3\Bigg(\mathcal{B}_3\sum_{j=1}^{k-2}\vartheta_j\int_0^{\rho_j}e^{-\varphi(\rho_j-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}\pi_1du\bigg)ds\\
&+\int_0^1 e^{-\varphi(1-s)}\bigg(\int_0^s\frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)}\pi_3du\bigg)ds\Bigg)\Bigg\}\\
&+\int_0^\tau e^{-\varphi(\tau-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}\pi_1du\bigg)ds,\\
&\le\bigg(\frac{1-e^{-\varphi\tau}}{\Upsilon\varphi}\bigg)\Bigg[\Bigg\{\mathcal{A}_4\mathcal{A}_5\bigg[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\alpha)}\bigg]+\mathcal{A}_2\mathcal{A}_3\bigg[\mathcal{B}_3\sum_{j=1}^{k-2}|\vartheta_j|\rho_j^{\alpha-1}\frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\alpha)}\bigg]\pi_1\Bigg\}\\
&+\Bigg\{\mathcal{A}_4\mathcal{A}_5\bigg(\mathcal{B}_1\sum_{j=1}^{k-2}|w_j|\varrho_j^{\beta-1}\frac{(1-e^{-\varphi\varrho_j})}{\varphi\Gamma(\beta)}\bigg)+\bigg(\mathcal{A}_2\mathcal{A}_5\bigg[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\beta)}\bigg]\bigg)\Bigg\}\pi_2\\
&+\Bigg\{\mathcal{A}_3\mathcal{A}_2\bigg[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\gamma)}\bigg]+\mathcal{A}_2\mathcal{A}_5\bigg(\mathcal{B}_2\sum_{j=1}^{k-2}|v_j|\varpi_j^{\gamma-1}\frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\gamma)}\bigg)\Bigg\}\pi_3\Bigg]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\alpha)},\\
&\le(\mathcal{U}_1\pi_1+\mathcal{V}_1\pi_2+\mathcal{W}_1\pi_3).
\end{aligned}
$$

Likewise, we get

$$
\begin{aligned}
\widehat{y}(\tau)=&\mathcal{E}_2(\widehat{x},\widehat{y},\widehat{z})(\tau)\\
&+\bigg(\frac{1-e^{-\varphi\tau}}{\varphi\mathcal{A}_2}\bigg)\Bigg\{\mathcal{B}_1\sum_{j=1}^{k-2}w_j\int_0^{\varrho_j}e^{-\varphi(\varrho_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}G_2(x,y,z)(u)du\bigg)ds\\
&+\int_0^1 e^{-\varphi(1-s)}\bigg(\int_0^s\frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}G_1(x,y,z)(u)du\bigg)ds\\
&+\frac{1}{\Upsilon}\Bigg[\mathcal{A}_1\mathcal{A}_4\mathcal{A}_5\Bigg(\mathcal{B}_1\sum_{j=1}^{k-2}w_j\int_0^{\varrho_j}e^{-\varphi(\varrho_j-s)}\bigg(\int_0^s\frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)}G_2(x,y,z)(u)du\bigg)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 e^{-\varphi(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} G_1(x,y,z)(u)du\right)ds\Bigg) \\
&+\mathcal{A}_1\mathcal{A}_5\mathcal{A}_2\Bigg(\mathcal{B}_2\sum_{j=1}^{k-2} v_j\int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} G_3(x,y,z)(u)du\right)ds \\
&+\int_0^1 e^{-\varphi(1-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} G_2(x,y,z)(u)du\right)ds\Bigg) \\
&+\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\Bigg(\mathcal{B}_3\sum_{j=1}^{k-2} \vartheta_j\int_0^{\rho_j} e^{-\varphi(\rho_j-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} G_1(x,y,z)(u)du\right)ds \\
&+\int_0^1 e^{-\varphi(1-s)}\left(\int_0^s \frac{(s-u)^{\gamma-2}}{\Gamma(\gamma-1)} G_3(x,y,z)(u)du\right)ds\Bigg)\Bigg]\Bigg\} \\
&+\int_0^\tau e^{-\varphi(\tau-s)}\left(\int_0^s \frac{(s-u)^{\beta-2}}{\Gamma(\beta-1)} G_2(x,y,z)(u)du\right)ds,
\end{aligned}
$$

$$
\begin{aligned}
&|\widehat{y}(\tau)-\mathcal{E}_2(\widehat{x},\widehat{y},\widehat{z})(\tau)| \le \left(\frac{1-e^{-\varphi\tau}}{\varphi\mathcal{A}_2}\right) \\
&\times\Bigg[\Bigg\{\frac{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}{\Upsilon}\left(\mathcal{B}_3\sum_{j=1}^{k-2}|\vartheta_j|\rho_j^{\alpha-1}\frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\alpha)}\right)+\frac{\mathcal{A}_1\mathcal{A}_4\mathcal{A}_5}{\Upsilon}\left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\alpha)}\right)+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\alpha)}\Bigg\}\pi_1 \\
&+\Bigg\{\left(\mathcal{B}_1\sum_{j=1}^{k-2}|w_j|\varrho_j^{\beta-1}\frac{(1-e^{-\varphi\varrho_j})}{\varphi\Gamma(\beta)}\right)+\frac{\mathcal{A}_1\mathcal{A}_4\mathcal{A}_5}{\Upsilon}\left(\mathcal{B}_1\sum_{j=1}^{k-2}|w_j|\varrho_j^{\beta-1}\frac{(1-e^{-\varphi\varrho_j})}{\varphi\Gamma(\beta)}\right) \\
&+\frac{\mathcal{A}_1\mathcal{A}_4\mathcal{A}_5}{\Upsilon}\left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\beta)}\right]\Bigg\}\pi_2+\Bigg\{\frac{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}{\Upsilon}\left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\gamma)}\right) \\
&+\frac{\mathcal{A}_1\mathcal{A}_4\mathcal{A}_5}{\Upsilon}\left(\mathcal{B}_2\sum_{j=1}^{k-2}|v_j|\varpi_j^{\gamma-1}\frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\gamma)}\right)\Bigg\}\pi_3\Bigg]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\beta)}, \\
&\le(\mathcal{U}_2\pi_1+\mathcal{V}_2\pi_2+\mathcal{W}_2\pi_3).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&|\widehat{z}(\tau)-\mathcal{E}_3(\widehat{x},\widehat{y},\widehat{z})(\tau)| \\
&\le\left(\frac{(\varphi\tau-1+e^{-\varphi\tau})}{\varphi^2\Upsilon}\right)\Bigg[\Bigg\{\mathcal{A}_4\mathcal{A}_1\left(\mathcal{B}_3\sum_{j=1}^{k-2}|\vartheta_j|\rho_j^{\alpha-1}\frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\alpha)}\right)+\mathcal{A}_4\mathcal{A}_6\left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\alpha)}\right)\Bigg\}\pi_1 \\
&+\Bigg\{\left(\mathcal{B}_1\sum_{j=1}^{k-2}|w_j|\varrho_j^{\beta-1}\frac{(1-e^{-\varphi\varrho_j})}{\varphi\Gamma(\beta)}\right)+\mathcal{A}_2\mathcal{A}_6\left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\beta)}\right)\Bigg\}\pi_2 \\
&+\Bigg\{\mathcal{A}_2\mathcal{A}_6\left(\mathcal{B}_2\sum_{j=1}^{k-2}|v_j|\varpi_j^{\gamma-1}\frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\gamma)}\right)+\mathcal{A}_1\mathcal{A}_4\left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\gamma)}\right)\Bigg\}\pi_3\Bigg]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\gamma)}, \\
&\le(\mathcal{U}_3\pi_1+\mathcal{V}_3\pi_2+\mathcal{W}_3\pi_3),
\end{aligned}
$$

where we describe $\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3,\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{W}_1,\mathcal{W}_2$, and $\mathcal{W}_3$. As a result, the $\mathcal{E}$ operator, as specified in the main results, can be excluded from the fixed point property in the following manner. We now have

$$
\begin{aligned}
|x(\tau)-\widehat{x}(\tau)| &= |x(\tau)-\mathcal{E}_1(\widehat{x},\widehat{y},\widehat{z})(\tau)+\mathcal{E}_1(\widehat{x},\widehat{y},\widehat{z})(\tau)-\widehat{x}(\tau)|, \\
&\le |\mathcal{E}_1(x,y,z)(\tau)-\mathcal{E}_1(\widehat{x},\widehat{y},\widehat{z})(\tau)|+|\mathcal{E}_1(\widehat{x},\widehat{y},\widehat{z})(\tau)-\widehat{x}(\tau)|, \\
&\le (\mathcal{U}_1\sigma_1+\mathcal{V}_1\varepsilon_1+\mathcal{W}_1\kappa_1)+(\mathcal{U}_1\sigma_2+\mathcal{V}_1\varepsilon_2+\mathcal{W}_1\kappa_2)+(\mathcal{U}_1\sigma_3+\mathcal{V}_1\varepsilon_3+\mathcal{W}_1\kappa_3) \\
&\quad \|(x,y,z)-(\widehat{x},\widehat{y},\widehat{z})\|+(\mathcal{U}_1\pi_1+\mathcal{V}_1\pi_2+\mathcal{W}_1\pi_3),
\end{aligned} \tag{21}
$$

which implies that

$$\begin{aligned}
|y(\tau)-\widehat{y}(\tau)| &= |y(\tau)-\mathcal{E}_2(\widehat{x},\widehat{y},\widehat{z})(\tau)+\mathcal{E}_2(\widehat{x},\widehat{y},\widehat{z})(\tau)-\widehat{y}(\tau)|,\\
&\leq |\mathcal{E}_2(x,y,z)(\tau)-\mathcal{E}_2(\widehat{x},\widehat{y},\widehat{z})(\tau)|+|\mathcal{E}_2(\widehat{x},\widehat{y},\widehat{z})(\tau)-\widehat{y}(\tau)|,\\
&\leq (\mathcal{U}_2\sigma_1+\mathcal{V}_2\varepsilon_1+N_2\kappa_1)+(\mathcal{U}_2\sigma_2+\mathcal{V}_2\varepsilon_2+\mathcal{W}_2\kappa_2)+(\mathcal{U}_2\sigma_3+\mathcal{V}_2\varepsilon_3+\mathcal{W}_2\kappa_3)\\
&\quad \|(x,y,z)-(\widehat{x},\widehat{y},\widehat{z})\|+(\mathcal{U}_2\pi_1+\mathcal{V}_2\pi_2+\mathcal{W}_2\pi_3).
\end{aligned} \tag{22}$$

Likewise, we can obtain

$$\begin{aligned}
|z(\tau)-\widehat{z}(\tau)| &= |z(\tau)-\mathcal{E}_3(\widehat{x},\widehat{y},\widehat{z})(\tau)+\mathcal{E}_3(\widehat{x},\widehat{y},\widehat{z})(\tau)-\widehat{z}(\tau)|,\\
&\leq |\mathcal{E}_3(x,y,z)(\tau)-\mathcal{E}_3(\widehat{x},\widehat{y},\widehat{z})(\tau)|+|\mathcal{E}_3(\widehat{x},\widehat{y},\widehat{z})(\tau)-\widehat{z}(\tau)|,\\
&\leq (\mathcal{U}_3\sigma_1+\mathcal{V}_3\varepsilon_1+\mathcal{W}_3\kappa_1)+(\mathcal{U}_3\sigma_2+\mathcal{V}_3\varepsilon_2+\mathcal{W}_3\kappa_2)+(\mathcal{U}_3\sigma_3+\mathcal{V}_3\varepsilon_3+\mathcal{W}_3\kappa_3)\\
&\quad \|(x,y,z)-(\widehat{x},\widehat{y},\widehat{z})\|+(\mathcal{U}_3\pi_1+\mathcal{V}_3\pi_2+\mathcal{W}_3\pi_3),
\end{aligned}. \tag{23}$$

It follows from Equations (21)–(23) that

$$\begin{aligned}
\|(x,y,z)-(\widehat{x},\widehat{y},\widehat{z})\| \leq &(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\pi_1+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)\pi_2+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)\pi_3\\
&+(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)(\sigma_1+\varepsilon_1+\kappa_1)\\
&+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)(\sigma_2+\varepsilon_2+\kappa_2)\\
&+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)(\sigma_3+\varepsilon_3+\kappa_3)\|(x,y,z)-(\widehat{x},\widehat{y},\widehat{z})\|,
\end{aligned}$$

$$\begin{aligned}
&\|(x,y,z)-(\widehat{x},\widehat{y},\widehat{z})\|\\
&\leq \frac{(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\pi_1+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)\pi_2+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)\pi_3}{1-((\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)(\sigma_1+\varepsilon_1+\kappa_1)+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)(\sigma_2+\varepsilon_2+\kappa_2)+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)(\sigma_3+\varepsilon_3+\kappa_3))}\\
&\leq \mathcal{K}_1\pi_1+\mathcal{K}_2\pi_2+\mathcal{K}_3\pi_3,
\end{aligned}$$

with

$$\begin{aligned}
&\mathcal{K}_1\\
&=\frac{(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)}{1-((\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)(\sigma_1+\varepsilon_1+\kappa_1)+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)(\sigma_2+\varepsilon_2+\kappa_2)+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)(\sigma_3+\varepsilon_3+\kappa_3))},\\
&\mathcal{K}_2\\
&=\frac{(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)}{1-((\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)(\sigma_1+\varepsilon_1+\kappa_1)+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)(\sigma_2+\varepsilon_2+\kappa_2)+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)(\sigma_3+\varepsilon_3+\kappa_3))},\\
&\mathcal{K}_3\\
&=\frac{(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)}{1-((\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)(\sigma_1+\varepsilon_1+\kappa_1)+(\mathcal{V}_1+\mathcal{V}_2+\mathcal{V}_3)(\sigma_2+\varepsilon_2+\kappa_2)+(\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3)(\sigma_3+\varepsilon_3+\kappa_3))}.
\end{aligned}$$

Therefore, the BVP in Equation (3) is H-U stable. □

5. Example

Let us consider the following mixed-type coupled fractional differential system:

$$\begin{cases} ({}^cD^{\alpha} + \lambda {}^cD^{\alpha-1})x(\tau) = f(\tau, x(\tau), y(\tau), z(\tau)), \quad 2 < \alpha \leq 3, \\ ({}^cD^{\beta} + \lambda {}^cD^{\beta-1})y(\tau) = g(\tau, x(\tau), y(\tau), z(\tau)), \quad 2 < \beta \leq 3, \\ ({}^cD^{\gamma} + \lambda {}^cD^{\gamma-1})z(\tau) = h(\tau, x(\tau), y(\tau), z(\tau)), \quad 3 < \gamma \leq 4, \\ x(0) = 0, \quad x'(0) = 0, \quad x(1) = \mathcal{A}_1 \sum_{j=1}^{k-2} p_j y(\theta_j) + \mathcal{B}_1 \mathcal{I}^{\omega} y(\varpi), \\ y(0) = 0, \quad y'(0) = 0, \quad y(1) = \mathcal{A}_2 \sum_{j=1}^{k-2} q_j z(\delta_j) + \mathcal{B}_2 \mathcal{I}^{\psi} z(\varrho), \\ z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 0, \quad z(1) = \mathcal{A}_3 \sum_{j=1}^{k-2} r_j x(\eta_j) + \mathcal{B}_3 \mathcal{I}^{\varsigma} x(\vartheta). \end{cases} \tag{24}$$

Here, $\alpha = \frac{9}{4}, \beta = \frac{5}{2}, \gamma = \frac{7}{2}$; $\varsigma = 13/20; \psi = 11/20; \vartheta = 53/50; \omega = 9/20$; $\varpi = 93/50$; $\varrho = 73/50$; $\mathcal{A}_1 = 17/400$; $\mathcal{A}_2 = 15/300$; $\mathcal{A}_3 = 13/200$; $\mathcal{B}_1 = 17/200$; $\mathcal{B}_2 = 8/125$; $\mathcal{B}_3 = 6/68; p_1 = 1/20$; $p_2 = 2/20; q_1 = 1/100$; $q_2 = 1/50; r_1 = 1/1000$; $r_2 = 1/500$; $\delta_1 = 7/5$; $\delta_2 = 3/2$; $\theta_1 = 13/10$; $\theta_2 = 14/10$; $k = 4$; and $\Delta = 0.07296323051370401$ with the given data. Therefore, it is found that

$$\begin{aligned} \mathcal{U}_1 &= 1.68831521003326, \quad \mathcal{U}_2 = 0.114571581986479, \quad \mathcal{U}_3 = 0.0798362345742745, \\ \mathcal{V}_1 &= 0.368806892892658, \quad \mathcal{V}_2 = 1.44239273375484, \quad \mathcal{V}_3 = 0.0222719550302514, \\ \mathcal{W}_1 &= 0.114644044759300, \quad \mathcal{W}_2 = 0.453485339282843, \quad \mathcal{W}_3 = 0.573371662060637. \end{aligned}$$

(I) In order to illustrate Theorem 1, we take

$$\begin{aligned} H_1(\tau, x, y, z) &= e^{-2\tau} + \frac{1}{16} x \cos y + \frac{e^{-\tau}}{9} y \sin z + \frac{e^{-\tau}}{8} z \cos x \\ H_2(\tau, x, y, z) &= \tau\sqrt{\tau^3 + 9} + \frac{1}{9\Pi} x \tan^{-1} y + \frac{1}{8e} y + \frac{1}{81e^{\tau}} z \sin x \\ H_3(\tau, x, y, z) &= \frac{e^{-\tau}}{10} + \frac{e^{-\tau}}{8} x + \frac{e^{-\tau}}{4+\tau} y + \frac{e^{-2\tau}}{16} z \cos y. \end{aligned} \tag{25}$$

It is easy to check that condition $(\mathcal{H}_2)$ is satisfied with $\sigma_0 = \frac{1}{e^2}$, $\sigma_1 = \frac{1}{16}$, $\sigma_2 = \frac{1}{9e}$, $\sigma_3 = \frac{1}{8e}$, $\varepsilon_0 = 3$, $\varepsilon_1 = \frac{1}{9\Pi}$, $\varepsilon_2 = \frac{1}{8e}$, $\varepsilon_3 = \frac{1}{81e^{\tau}}$, $\kappa_0 = \frac{e^{-\varrho}}{10}$, $\kappa_1 = \frac{1}{8e}$, $\kappa_2 = \frac{1}{5e}$, $and \kappa_3 = \frac{e^{-2\tau}}{16}$.
Furthermore, we have

$$\begin{aligned} (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_1 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_1 &\simeq 0.235040718032698 < 1, \\ (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_2 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_2 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_2 &\simeq 0.245256248763410 < 1, \\ (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\sigma_3 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\varepsilon_3 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_3 &\simeq 0.104559339619535 < 1, \end{aligned}$$

Clearly, the hypotheses of Theorem 1 are satisfied, and hence the conclusion of Theorem 1 applies to Equation (24), with x, y, and z given by Equation (25).

(II) To demonstrate Theorem 2, we will use the following:

$$\begin{aligned} H_1(\tau, x, y, z) &= \frac{e^{-2}}{\sqrt{8 + \tau^2}} \cos x + \cos \tau \\ H_1(\tau, x, y, z) &= \frac{1}{25 + \tau^2} (\sin x + |y|) + e^{-\tau} \\ H_1(\tau, x, y, z) &= \frac{e^{-\tau}}{9} \sin z + \tan^{-1} \tau, \end{aligned} \tag{26}$$

which clearly satisfies condition $(\hat{\mathcal{H}}_2)$ with $\Theta_1 = \frac{1}{9e^2}, \Theta_2 = \frac{1}{26}$ and $\Theta_3 = \frac{1}{9e}$. Moreover,

$$(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\Theta_2 + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\Theta_3 \simeq 0.145488540089847 < 1.$$

There is a unique solution to Equation (24) on $[0,1]$ with Equation (26) and the x, y, and z given by Theorem 2.

6. Conclusions

In this article, we studied a tripled system of sequential fractional differential equations of the order χ. Within the proposed system, the existence and uniqueness of the solutions are determined by tripled integral boundary conditions. Our major findings were demonstrated using the Banach and Krasnoselskii fixed point theorems. The stability of the solutions involved in the Hyers–Ulam-type was investigated. We provided some examples to demonstrate the generalization of the study results. The work described in this article is novel and considerably adds to the established literature of knowledge on the subject. When the parameters in problems $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ were specified, our results conformed to a few special cases. Assume that we formulated the problem in Equation (3) by taking $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ in the presented findings:

$$\begin{cases} x(0) = 0, \quad x'(0) = 0, \quad x(\mathcal{T}) = \mathcal{B}_1 \mathcal{I}^{\omega} y(\varpi), \\ y(0) = 0, \quad y'(0) = 0, \quad y(\mathcal{T}) = \mathcal{B}_2 \mathcal{I}^{\psi} z(\varrho), \\ z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 0, \quad z(\mathcal{T}) = \mathcal{B}_3 \mathcal{I}^{\varsigma} x(\vartheta). \end{cases}$$

We can then solve the above problem (Equation (3)) by using the methodology employed in the previous section. Future research may concentrate on various concepts of stability and existence as they relate to a neutral time delay system or inclusion and a time delay system or inclusion with a finite delay.

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