fractal and fractional

Article

Hadamard-Type Fractional Integro-Differential Problem: A Note on Some Asymptotic Behavior of Solutions

Ahmad Mugbil [1,*] and Nasser-Eddine Tatar [2]

[1] Prep Year Math Program, College of General Studies, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
[2] Department of Mathematics, IRC for Intelligent Manufacturing and Robotics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia; tatarn@kfupm.edu.sa
* Correspondence: mugbil@kfupm.edu.sa

Abstract: As a follow-up to the inherent nature of Hadamard-Type Fractional Integro-differential problem, little is known about some asymptotic behaviors of solutions. In this paper, an integro-differential problem involving Hadamard fractional derivatives is investigated. The leading derivative is of an order between one and two whereas the nonlinearities may contain fractional derivatives of an order between zero and one as well as some non-local terms. Under some reasonable conditions, we prove that solutions are asymptotic to logarithmic functions. Our approach is based on a generalized version of Bihari–LaSalle inequality, which we prove. In addition, several manipulations and crucial estimates have been used. An example supporting our findings is provided.

Keywords: asymptotic behavior; fractional differential equation; Hadamard fractional derivative

check for updates

Citation: Mugbil, A.; Tatar, N.-E. Hadamard-Type Fractional Integro-Differential Problem: A Note on Some Asymptotic Behavior of Solutions. *Fractal Fract.* **2022**, *6*, 267. https://doi.org/10.3390/fractalfract6050267

Academic Editors: António M. Lopes, Alireza Alfi, Liping Chen and Sergio Adriani David

Received: 30 March 2022
Accepted: 13 May 2022
Published: 15 May 2022

1. Introduction

Of concern is the following general class of initial value problems modelled by:

$$\begin{cases} \left({}_H\mathcal{D}^{\alpha}_{t_0}u\right)'(t) = f\left(t, \left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(t), \int_{t_0}^{t} h\left(t,s,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(s)\right)ds\right), \; t > t_0 > 0, \\ \left({}_H\mathcal{I}^{1-\alpha}_{t_0}u\right)(t_0^+) = u_1, \;\; \left({}_H\mathcal{D}^{\alpha}_{t_0}u\right)(t_0^+) = u_2, \;\; u_1, u_2 \in \mathbb{R}, \end{cases} \tag{1}$$

where ${}_H\mathcal{D}^{\alpha}_{t_0}$, ${}_H\mathcal{D}^{\alpha_1}_{t_0}$ and ${}_H\mathcal{D}^{\alpha_2}_{t_0}$ are the Hadamard fractional derivatives of orders α, α_1 and α_2, respectively, $0 \le \alpha_1 < \alpha < 1$ and $0 \le \alpha_2 < \alpha < 1$. The operator ${}_H\mathcal{I}^{\rho}_{t_0}$ is the Hadamard fractional integral of order $\rho \ge 0$. The definitions of these operators are given in Section 2.

We shall investigate the asymptotic behavior of solutions for Problem (1). Sufficient conditions on the nonlinear source term guaranteeing the convergence of solutions to logarithmic functions, for large values of time, are established. The importance of using analytical techniques to study the asymptotic behavior of solutions for Problem (1) arises from the lack of explicit solutions.

It is known that solutions for many kinds of (integer-order) ordinary differential equations may approach a certain function as time goes to infinity; in particular they may decay to zero, oscillate, or blow up in finite time. Many results in this regard exist in the literature. For example, we refer the reader to the papers [1–6], in which various classes of linear and nonlinear ordinary differential equations have been studied. Generalizing the existing results from integer orders to non-integer fractional orders is of great importance due to their numerous applications; see for instance, [7–10]. Unfortunately, imitating the techniques verbatim is not straightforward. Many difficulties arise when trying to do so. Some of these difficulties are due to the nature of the fractional derivatives themselves as they involve by definition all the past memory of solutions as well as nonregular kernels. In addition, many fundamental properties of integer-order derivatives are not valid for fractional-order derivatives. The chain rule is an example of such invalid properties. We will go around these difficulties by utilizing some adequate estimations, like desingularization methods, to deal with singular terms and by modifying and/or generalizing some versions of Bihari–LaSalle inequality.

The study of the asymptotic behavior of solutions for fractional differential equations, with Riemann–Liouville or Caputo fractional derivatives, has been investigated by many researchers, see e.g., [11–20]. The authors of [15] considered the fractional differential equation:

$$\left(D_{0^+}^{\alpha}u'\right)(t)+f(t,u)=0,\ t>0,$$

under the condition

$$|f(t,u)|\leq\phi\left(t,\frac{|u|}{(1+t)^{\alpha}}\right),\ t\geq 0,\ u\in\mathbb{R},$$

where $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order α, $0<\alpha<1$. The function $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and the function $\phi:[0,\infty)\times[0,\infty)\to[0,\infty)$ is continuous in each argument and nondecreasing in the second one. They proved that the solutions can be represented asymptotically as $a_1+a_2t^{\alpha}+O(t^{\alpha-1})$, $a_1,a_2\in\mathbb{R}$.

In [20], the case when the source function f depends on the solution and its sub-first-order fractional derivative has been considered, namely,

$$\begin{cases}\left({}^{C}D_{t_0^+}^{\alpha}u\right)(t)=f\left(t,u(t),\left({}^{C}D_{t_0^+}^{\alpha_1}u\right)(t)\right),\ t>t_0>0,\\[2ex] \qquad u(t_0)=u_0,\end{cases}\tag{2}$$

where ${}^{C}D_{t_0^+}^{\alpha}$ is the Caputo fractional derivative of order α, $0<\alpha_1<\alpha<1$. The authors showed that any global solution of Problem (2) is asymptotic to ct^{α_1} for some real number c.

The present authors investigated the boundedness, power-type decay and asymptotic behavior of solutions for the initial value problems:

$$\begin{cases}\left({}^{C}D_{0^+}^{\alpha}u\right)(t)=f\left(t,u(t),\int_0^t h(t,s,u(s))ds\right),\ \ t>0,\ 0<\alpha\leq 1\\[2ex] \qquad u(0)=u_0,\ \ u_0\in\mathbb{R},\end{cases}$$

and

$$\begin{cases}\left(D_{0^+}^{\alpha+1}u\right)(t)=f\left(t,(D_{0^+}^{\alpha_1}u)(t),\int_0^t h\big(t,s,(D_{0^+}^{\alpha_2}u)(s)\big)ds\right),\ t>0,0\leq\alpha_1,\alpha_2\leq\alpha<1\\[2ex] \qquad \left(I_{0^+}^{1-\alpha}u\right)(0^+)=u_0,\ \left(D_{0^+}^{\alpha}u\right)(0^+)=u_1,\ \ u_0,u_1\in\mathbb{R}.\end{cases}$$

Several different classes of source functions f such as

$$\begin{aligned}|f(t,u,v)|&\leq k(t)P(|u|)+l(t)Q(|v|),\ \ |f(t,u,v)|\leq k(t)P(t^{\alpha_2}|u|)Q(|v|),\\ \text{or } |f(t,u,v)|&\leq k(t)P\left(t^{1-\alpha+\alpha_1}|u|\right)+l(t)Q(|v|),\end{aligned}$$

and on the kernel h such as:

$$\begin{aligned}|h(t,s,u)|&\leq w(s)K(|u|),\ |h(t,s,u)|\leq w(s)K(s^{\alpha_2}|u|),\ |h(t,s,u)|\leq w(t,s)K(|u|),\\ \text{or } |h(t,s,u)|&\leq w(s)K\left(t^{1-\alpha+\alpha_2}|u|\right),\end{aligned}$$

for some functions k,l,P,Q,w and K have been treated, see [11,12,21–23].

For fractional differential equations with Hadamard-type fractional derivatives, we found relatively few results in the literature tackling the long-time behavior of solutions of fractional initial value problems, see, [24–28]. The authors of [28] studied the stability and decay rate of the zero solution of the fractional differential problem:

$$\begin{cases} \left({}_H\mathcal{D}_{t_0}^{\alpha} u\right)(t) = f(u(t)),\ t > t_0 > 0, 0 < \alpha < 1 \\ \left({}_H\mathcal{I}_{t_0}^{1-\alpha} u\right)(t_0^+) = u_0,\ \ u_0 \in \mathbb{R}. \end{cases} \tag{3}$$

They considered first the linear case, $f(u) = cu$, $c \in \mathbb{R}$, and established a criteria for the decay rate of solutions and the Lyapunov stability of the zero equilibrium. For the nonlinear case, they obtained the stability and decay rate of the hyperbolic zero equilibrium. They used a modified Laplace transform to express solutions by Mittag–Leffler functions and then used the asymptotic expansions of these functions to discuss the stability and logarithmic decay of the solutions. In [25], the authors discussed the stability of logarithmic type for the initial value problem:

$$\begin{cases} \left({}_H\mathcal{D}_{t_0}^{\alpha} u\right)(t) = f\left(t, u(t), \left({}_H\mathcal{D}_{t_0}^{\alpha_1} u\right)(t)\right),\ t > t_0 > 0,\ 0 < \alpha_1 < \alpha < 1, \\ \left({}_H\mathcal{D}_{t_0}^{\alpha-1} u\right)(t_0^+) = u_0,\ u_0 \in \mathbb{R}. \end{cases} \tag{4}$$

Under some sufficient growth conditions of f, it has been shown that the solutions decay to zero as the logarithmic function $\left(\ln \frac{t}{t_0}\right)^{\alpha-1}$. Recently, the same authors considered Problem (4) in [26] with $\frac{d}{dt}\left({}_H\mathcal{D}_{t_0}^{\alpha} u\right)(t)$ on the left hand side and the additional initial condition $\left({}_H\mathcal{D}_{t_0}^{\alpha} u\right)(t_0^+) = u_1 \in \mathbb{R}$. They showed that solutions approach a logarithmic function as time goes to infinity.

To the best of our knowledge, the long-time behavior of solutions for the class of fractional integrodifferential equations with Hadamard fractional derivatives (1) has not been investigated so far. In this paper, we prove that solutions of (1) are asymptotic to the logarithmic function $\left(\ln \frac{t}{t_0}\right)^{\alpha}$ where α is the order of the involved fractional derivative. Under sufficient growth conditions, we show that there exist a real number r such that any solution for (1) in the space $u \in C_{1-\alpha,\ln}^{\alpha+1}[t_0,\infty)$, see (16), has the following property $\lim\limits_{t\to\infty} \frac{u(t)}{\left(\ln \frac{t}{t_0}\right)^{\alpha}} = r$.

The rest of this paper is organized as follows. In the next section, Section 2, we give some notations from fractional calculus and present some preliminary results. In Section 3, we introduce and prove our main results. Section 4 is devoted to an example that supports our results. A brief conclusion is presented at the end of the study in Section 5.

2. Preliminaries

This section is devoted to briefly introduce some basic definitions, notions, and properties from fractional calculus and fractional differential equations theory which will be used in further considerations.

Definition 1 ([7]). *We denote by $C_{\gamma,\ln}[t_0,T]$, $0 \leq \gamma < 1$, the following weighted space of continuous functions:*

$$C_{\gamma,\ln}[t_0,T] = \left\{\xi : (t_0,T] \to \mathbb{R} : \left(\ln \frac{t}{t_0}\right)^{\gamma} \xi(t) \in C[t_0,T]\right\}, \tag{5}$$

with the norm

$$\|\xi\|_{C_{\gamma,\ln}} = \left\|\left(\ln \frac{t}{t_0}\right)^{\gamma} \xi(t)\right\|_{C_{0,\ln}},$$

where $C[t_0, T] = C_{0,\ln}[t_0, T]$ *is the space of continuous functions on* $[t_0, T]$.

Definition 2 ([7])**.** *Let* $\delta = t\frac{d}{dt}$ *be the* δ*-derivative. For* $n \in \mathbb{N}$ *and* $0 \le \gamma < 1$*, the weighted space of continuously* δ*-differentiable functions up to order* $n-1$ *with* n*th* δ*-derivative in* $C_{\gamma,\ln}[t_0, T]$*, is denoted by* $C^n_{\delta,\gamma}[t_0, T]$ *and defined by:*

$$C^n_{\delta,\gamma}[t_0, T] = \Big\{\xi : (t_0, T] \to \mathbb{R} \mid \delta^i\xi \in C[t_0, T],\ i = 0, 1, 2, \ldots, n-1,\ \delta^n\xi \in C_{\gamma,\ln}[t_0, T]\Big\}, \tag{6}$$

with the norm

$$\|\xi\|_{C^n_{\delta,\gamma}} = \sum_{i=0}^{n-1}\left\|\delta^i\xi\right\|_C + \|\delta^n\xi\|_{C_{\gamma,\ln}}.$$

In particular, $C^0_{\delta,\gamma}[t_0, T] = C_{\gamma,\ln}[t_0, T]$.

A characterization of the space $C^n_{\delta,\gamma}[t_0, T]$ is given as follows [7]: The functions ξ in the space $C^n_{\delta,\gamma}[t_0, T]$, $n \in \mathbb{N}$ and $0 \le \gamma < 1$, can be represented as:

$$\xi(t) = \frac{1}{(n-1)!}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{n-1}\frac{h(s)}{s}ds + \sum_{i=0}^{n-1} b_i\left(\ln\frac{t}{t_0}\right)^i,$$

where $h \in C_{\gamma,\ln}[t_0, T]$ and b_i, $i = 0, 1, 2, \ldots, n-1$, are arbitrary constants. In fact, $h(t) = (\delta^n\xi)(t)$ and $b_i = \frac{(\delta^i\xi)(t_0)}{i!}$.

Definition 3 ([7])**.** *The Hadamard left-sided fractional integral of order* $\alpha > 0$ *is defined by:*

$$\left({}_H\mathcal{I}^{\alpha}_{t_0}w\right)(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{w(s)}{s}ds,\ t > t_0, \tag{7}$$

provided the right-hand side exists. We define ${}_H\mathcal{I}^0_{t_0}w = w$*. The function* Γ *is the Euler gamma function defined by* $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0$*, where* $t^{\alpha-1} = e^{(\alpha-1)\ln t}$.

Definition 4 ([7])**.** *The Hadamard left-sided fractional derivative of order* $\alpha > 0$*, is defined by:*

$$\left({}_H\mathcal{D}^{\alpha}_{t_0}w\right)(t) = \frac{1}{\Gamma(n-\alpha)}\delta^n\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{w(s)}{s}ds,\ t > t_0,$$

that is,

$$\left({}_H\mathcal{D}^{\alpha}_{t_0}w\right)(t) = \delta^n\left({}_H\mathcal{I}^{n-\alpha}_{t_0}w\right)(t),\ t > t_0, \tag{8}$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$, $n = -[-\alpha]$*. In particular, when* $\alpha = m \in \mathbb{N}_0$*, we have* ${}_H\mathcal{D}^m_{t_0}w = \delta^m w$.

The next lemma shows that the Hadamard fractional derivative (or integral) of a logarithmic function results in a multiple of the same logarithmic function with the order of the fractional derivative (or integral) subtracted from (or added to) its power.

Lemma 1 ([7])**.** *If* $\alpha > 0$, $\beta > 0$*, then:*

$$\left({}_H\mathcal{I}^{\alpha}_{t_0}\left(\ln\frac{s}{t_0}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\ln\frac{t}{t_0}\right)^{\beta+\alpha-1},\ t > t_0,$$

$$\left({}_H\mathcal{D}^{\alpha}_{t_0}\left(\ln\frac{s}{t_0}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\ln\frac{t}{t_0}\right)^{\beta-\alpha-1},\ t > t_0.$$

In particular, when $0 < \alpha < 1$, $\beta = 1$, then:

$$\left({}_H\mathcal{I}_{t_0}^{\alpha}1\right)(t) = \frac{1}{\Gamma(1+\alpha)}\left(\ln\frac{t}{t_0}\right)^{\alpha},\ t > t_0,$$

$$\left({}_H\mathcal{D}_{t_0}^{\alpha}1\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\ln\frac{t}{t_0}\right)^{-\alpha},\ t > t_0.$$

The last property shows that the Hadamard-type derivative of a constant is not zero.

The composite of the Hadamard operators of fractional differentiation and integration with different orders is given next.

Lemma 2 ([7])**.** *Let $0 < \beta < \alpha$ and $0 \le \gamma < 1$, then:*

$${}_H\mathcal{D}_{t_0}^{\beta}\,{}_H\mathcal{I}_{t_0}^{\alpha}w \ = \ {}_H\mathcal{I}_{t_0}^{\alpha-\beta}w$$

at every point in $(t_0, T]$ if $w \in C_{\gamma,\ln}[t_0, T]$ and at every point in $[t_0, T]$ if $w \in C[t_0, T]$. In particular, when $\alpha > \beta = m \in \mathbb{N}$, we obtain:

$${}_H\mathcal{D}_{t_0}^{m}\,{}_H\mathcal{I}_{t_0}^{\alpha}w \ = \ {}_H\mathcal{I}_{t_0}^{\alpha-m}w.$$

The Hadamard differentiation operator is the left inverse to the associated Hadamard integration operator [7]. That is, ${}_H\mathcal{D}_{t_0}^{\alpha}\,{}_H\mathcal{I}_{t_0}^{\alpha}w = w$ at every point in $(t_0, T]$ if $w \in C_{\gamma,\ln}[t_0, T]$. This property is not valid when the Hadamard fractional derivative and the Hadamard fractional integral are inverted as shown in the lemma below.

Lemma 3 ([7])**.** *Let $\alpha \ge 0$ and $n = -[-\alpha]$. If $w \in C_{\gamma,\ln}[t_0, T]$, $0 \le \gamma < 1$ and ${}_H\mathcal{I}_{t_0}^{n-\alpha}w \in C_{\delta,\gamma}^{n}[t_0, T]$, then:*

$$\left({}_H\mathcal{I}_{t_0}^{\alpha}\,{}_H\mathcal{D}_{t_0}^{\alpha}w\right)(t) = w(t) - \sum_{i=1}^{n}\frac{\left(\delta^{n-i}\left({}_H\mathcal{I}_{t_0}^{n-\alpha}w\right)\right)(t_0^+)}{\Gamma(\alpha-i+1)}\left(\ln\frac{t}{t_0}\right)^{\alpha-i}, \tag{9}$$

for all $t \in (t_0, T]$. In particular, for $0 \le \alpha < 1$, we have:

$$\left({}_H\mathcal{I}_{t_0}^{\alpha}\,{}_H\mathcal{D}_{t_0}^{\alpha}w\right)(t) = w(t) - \frac{\left({}_H\mathcal{I}_{t_0}^{1-\alpha}w\right)(t_0^+)}{\Gamma(\alpha)}\left(\ln\frac{t}{t_0}\right)^{\alpha-1} \tag{10}$$

at every point in $(t_0, T]$ if $w \in C_{\gamma,\ln}[t_0, T]$ and at every point in $[t_0, T]$ if $w \in C[t_0, T]$ and ${}_H\mathcal{I}_{t_0}^{1-\alpha}w \in C_{\delta,\gamma}^{1}[t_0, T]$.

For more about Hadamard fractional integral and derivative, we refer to the books [7,29,30].

The limit of the ratio of the Hadamard fractional integral $\left({}_H\mathcal{I}_{t_0}^{\alpha+1}sf(s,u(s),v(s))\right)(t)$ and the power function $\frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_0}\right)^{\alpha}$ as $t \to \infty$ is treated in the lemma below.

Lemma 4. *Let $f \in L^1(t_0, \infty)$, $t_0 > 0$. Suppose that u and v are real-valued functions defined on $[t_0, \infty)$, then:*

$$\begin{aligned}\lim_{t\to\infty}\frac{\left({}_H\mathcal{I}_{t_0}^{\alpha+1}sf(s,u(s),v(s))\right)(t)}{\frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_0}\right)^{\alpha}} &= \lim_{t\to\infty}\frac{1}{\left(\ln\frac{t}{t_0}\right)^{\alpha}}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha}f(s,u(s),v(s))ds \\ &= \int_{t_0}^{\infty}f(s,u(s),v(s))ds,\ \alpha > 0.\end{aligned}$$

Proof. It is easy to see that:

$$\begin{aligned}
&\left|\frac{\left({}_H\mathcal{I}_{t_0}^{\alpha+1} s f(s,u(s),v(s))\right)(t)}{\frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_0}\right)^{\alpha}}-\int_{t_0}^{\infty} f(s,u(s),v(s))ds\right| \\
&=\left|\frac{1}{\left(\ln\frac{t}{t_0}\right)^{\alpha}}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha} f(s,u(s),v(s))ds-\int_{t_0}^{\infty} f(s,u(s),v(s))ds\right| \\
&=\left|\int_{t_0}^{t}\left(\frac{\ln\frac{t}{t_0}-\ln\frac{s}{t_0}}{\ln\frac{t}{t_0}}\right)^{\alpha} f(s,u(s),v(s))ds-\int_{t_0}^{\infty} f(s,u(s),v(s))ds\right| \\
&=\left|\int_{t_0}^{\infty}\left[\left(1-\frac{\ln\frac{s}{t_0}}{\ln\frac{t}{t_0}}\right)^{\alpha}\chi_{[t_0,t]}(s)-1\right] f(s,u(s),v(s))ds\right| \\
&\leq \int_{t_0}^{\infty}\left|\left(1-\frac{\ln\frac{s}{t_0}}{\ln\frac{t}{t_0}}\right)^{\alpha}\chi_{[t_0,t]}(s)-1\right| |f(s,u(s),v(s))|ds,
\end{aligned}$$

where

$$\chi_{[t_0,t]}(s)=\begin{cases} 1 & \text{if } t_0\leq s\leq t, \\ 0, & \text{otherwise.}\end{cases}.$$

As $f \in L^1(t_0,\infty)$ and

$$\lim_{t\to\infty}\left(1-\frac{\ln\frac{s}{t_0}}{\ln\frac{t}{t_0}}\right)^{\alpha}\chi_{[t_0,t]}(s)=1, \text{ for } s<t,$$

we get from the Dominated Convergence Theorem [31],

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{\left({}_H\mathcal{I}_{t_0}^{\alpha+1} s f(s,u(s),v(s))\right)(t)}{\frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_0}\right)^{\alpha}}-\int_{t_0}^{\infty} f(s,u(s),v(s))ds\right| \\
&\leq \lim_{t\to\infty}\int_{t_0}^{\infty}\left|\left(1-\frac{\ln\frac{s}{t_0}}{\ln\frac{t}{t_0}}\right)^{\alpha}\chi_{[t_0,t]}(s)-1\right| |f(s,u(s),v(s))|ds \\
&=\int_{t_0}^{\infty}\lim_{t\to\infty}\left|\left(1-\frac{\ln\frac{s}{t_0}}{\ln\frac{t}{t_0}}\right)^{\alpha}\chi_{[t_0,t]}(s)-1\right| |f(s,u(s),v(s))|ds=0.
\end{aligned}$$

This completes the proof. □

The next lemma can be considered as a Hadamard fractional version of L'Hôpital's rule when applied to the solution of problem (1).

Lemma 5. *Let u be a solution of problem (1) with $f \in L^1(0,\infty)$. Then,*

$$\begin{aligned}
\lim_{t\to\infty}\frac{u(t)}{\left(\ln\frac{t}{t_0}\right)^{\alpha}} &= \lim_{t\to\infty}\frac{\left({}_H\mathcal{D}_{t_0}^{\alpha}u\right)(t)}{\Gamma(\alpha+1)} \\
&= \frac{1}{\Gamma(\alpha+1)}\left(u_2+\int_{t_0}^{\infty} f\left(s,\left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(s),\int_{t_0}^{s} h(s,\tau,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\tau))d\tau\right)ds\right).
\end{aligned}$$

Proof. Integrating both sides of the equation in (1) over the interval $[t_0,t]$, we obtain:

$$\left({}_H\mathcal{D}_{t_0}^{\alpha}u\right)(t)=u_2+\int_{t_0}^{t} f\left(s,\left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(s),\int_{t_0}^{s} h(s,\tau,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\tau))d\tau\right)ds. \tag{11}$$

Apply ${}_H\mathcal{I}^{\alpha}_{t_0}$ to (11) and use Lemmas 1 and 3, to find that:

$$u(t) - \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} = \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} \times \int_{t_0}^{s} f\left(\tau, \left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\sigma))d\sigma\right)d\tau\frac{ds}{s}, \tag{12}$$

for all $t > t_0 > 0$. Reorder the double integral on the right-hand side and integrate by substitution to have:

$$\begin{aligned} u(t) &= \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\int_{\tau}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} \\ &\quad \times f\left(\tau, \left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\sigma))d\sigma\right)\frac{ds}{s}d\tau \\ &= \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t}\left(\ln\frac{t}{\tau}\right)^{\alpha} \\ &\quad \times f\left(\tau, \left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\sigma))d\sigma\right)d\tau. \end{aligned} \tag{13}$$

Dividing both sides of (13) by $\left(\ln\frac{t}{t_0}\right)^{\alpha}$ gives:

$$\begin{aligned} \frac{u(t)}{\left(\ln\frac{t}{t_0}\right)^{\alpha}} &= \frac{u_1}{\Gamma(\alpha)\ln\frac{t}{t_0}} + \frac{u_2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\left(\ln\frac{t}{t_0}\right)^{\alpha}}\int_{t_0}^{t}\left(\ln\frac{t}{\tau}\right)^{\alpha} \\ &\quad \times f\left(\tau, \left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\sigma))d\sigma\right)d\tau, \end{aligned} \tag{14}$$

for all $t > t_0 > 0$. Taking the limit of the resulting ratio at infinity and applying Lemma 4 leads to the desired result. □

In the next lemma, we recall Bihari–LaSalle inequality which is a nonlinear generalization of the well-known Grönwall–Bellman inequality.

Lemma 6 ([32,33])**.** *Suppose that w and g are nonnegative continuous functions on $[t_0,\infty)$ and φ is a positive function on $(0,\infty)$, continuous and nondecreasing on $[0,\infty)$. If*

$$w(t) \le c + \int_{t_0}^{t} g(\tau)\varphi(w(\tau))d\tau,\ t \in [t_0,\infty),$$

where $c > 0$, then

$$w(t) \le \Phi^{-1}\left(\Phi(c) + \int_{t_0}^{t} g(\tau)d\tau\right),\ t \in [t_0,T_1],$$

where Φ^{-1} is the inverse function of Φ,

$$\Phi(x) = \int_{x_0}^{x}\frac{ds}{\varphi(s)},\ x > 0,\ x_0 > 0,$$

and T_1 is chosen so that $\Phi(c) + \int_{t_0}^{t} g(\tau)d\tau$ is in the domain of Φ^{-1} for all $t \in [t_0,T_1]$.

The next nonlinear integral inequality can be considered a generalization of Bihari–LaSalle inequality that has been recalled in Lemma 6.

Lemma 7 ([34]). *Assume that w and η_j, $j = 1, \dots, n$ are nonnegative continuous functions on $[t_0, T]$ and φ_j, $j = 1, \dots, n$ are nonnegative, continuous and nondecreasing on $[0, \infty)$ such that $\varphi_1 \propto \varphi_2 \propto \cdots \propto \varphi_n$ (that is, $\varphi_n/\varphi_{n-1}, \dots, \varphi_2/\varphi_1$ are nondecreasing functions). Assume further that c is a positive constant and*

$$w(t) \leq c + \sum_{j=1}^{n} \int_{t_0}^{t} \eta_j(s)\varphi_j(w(s))ds, \; t \in [t_0, T],$$

then, for all $t \in [t_0, T_1]$,

$$w(t) \leq \Phi_n^{-1}\left(\Phi_n(c_{n-1}) + \int_{t_0}^{t} \eta_n(s)ds\right),$$

where

1. *Φ_j^{-1} is the inverse function of Φ_j and $\Phi_j(x) = \int_{x_j}^{x} \frac{ds}{\varphi_j(s)}, x > 0, x_j > 0, , j = 1, \dots, n$.*
2. *The constants c_j are given by $c_0 = c$ and $c_j = \Phi_j^{-1}\left(\Phi_j(c_{j-1}) + \int_{t_0}^{T_1} \eta_j(\tau)d\tau\right), j = 1, \dots, n - 1$.*
3. *The number $T_1 \in [t_0, T]$ is the largest number such that:*

$$\int_{t_0}^{T_1} \eta_j(\tau)d\tau \leq \int_{c_{j-1}}^{\infty} \frac{d\tau}{\varphi_j(\tau)}, \; j = 1, \dots, n.$$

Lemma 8 ([34]). *Suppose that w and η_j, $j = 1, 2, 3$ are nonnegative continuous functions on $[t_0, T]$ and φ_j, $j = 1, 2, 3$ are nonnegative, continuous and nondecreasing on $[0, \infty)$ such that $\varphi_1 \propto \varphi_2 \propto \varphi_3$, (that is, φ_3/φ_2 and φ_2/φ_1 are nondecreasing functions). Assume further that c is a positive constant. If*

$$w(t) \leq c + \int_{t_0}^{t} \eta_1(\tau)\varphi_1(w(\tau))d\tau + \int_{t_0}^{t} \eta_2(\tau)\varphi_2\left(\int_{t_0}^{\tau} \eta_3(s)\varphi_3(w(s))ds\right)d\tau,$$

then, for all $t \in [t_0, T_1]$,

$$w(t) \leq \Phi_3^{-1}\left(\Phi_3(c_2) + \int_{t_0}^{t} \eta_3(\tau)d\tau\right),$$

where the functions Φ_j^{-1}, Φ_j, $j = 1, 2, 3$ and the constants c_0, c_1, c_2 are the same as those given in Lemma 7.

Remark 1. *By considering the following functions,*

$$\begin{aligned} \varpi_1(t) &:= \max_{\tau \in [0,t]} \{\varphi_1(\tau)\}, \\ \varpi_j(t) &:= \max_{\tau \in [0,t]} \left\{\frac{\varphi_j(\tau)}{\varpi_{j-1}(\tau)}\right\} \varpi_{j-1}(t), \; j = 2, 3, \end{aligned} \tag{15}$$

we can drop the ordering and monotonicity requirements in Lemma 8. It is clear that $\varphi_j(t) \leq \varpi_j(t)$, ϖ_j are nonnegative and nondecreasing functions on $[0, \infty)$ for all $t \in [0, \infty)$, $j = 1, 2, 3$ and $\varpi_1 \propto \varpi_2 \propto \varpi_3$.

3. Main Results

In this section, we study the asymptotic behavior of continuable solutions for the problem (1) in the space $C^{\alpha+1}_{1-\alpha,\ln}[t_0, T]$, $0 < t_0 < T \leq \infty$, defined by:

$$C^{\alpha+1}_{1-\alpha,\ln}[t_0, T] = \left\{ u : (0, T] \to \mathbb{R} \mid u \in C_{1-\alpha,\ln}[t_0, T], \; \frac{d\left({}_H\mathcal{D}^{\alpha}_{t_0} u(t)\right)}{dt} \in C_{1-\alpha,\ln}[t_0, T] \right\}, \tag{16}$$

where the space $C_{1-\alpha,\ln}[t_0, T]$ is defined in (5).

The following two types of functions are used repeatedly in the rest of this paper.

Definition 5. *A function g is said to be $\Omega_\varkappa$-type function if it is continuous, nonnegative on (t_0, ∞) and*

$$\int_{t_0}^{\infty} \left(\ln \frac{t}{t_0}\right)^{\varkappa} g(t)dt < \infty, \ \varkappa = 0 \text{ or } 1.$$

Definition 6. *A function ψ is said to be Ψ-type function if it is continuous, positive, nondecreasing on $(0, \infty)$,*

$$\psi(a) \leq b\psi\left(\frac{a}{b}\right) \text{ for all } \geq 1, b \geq 1,$$

and

$$\int_{\rho}^{t} \frac{ds}{\psi(s)} \to \infty \text{ as } t \to \infty \text{ for any } \rho > 0.$$

The two classes of functions introduced above are not empty, see the example provided in Section 4.

Consider the following nonlinear inequality:

$$w(t) \leq a_1 + \left(\ln \frac{t}{t_0}\right)\left(a_2 + a_3 \int_{t_0}^{t} \left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau\right)\right]ds\right), \tag{17}$$

for all $t > t_0$, where a_1, a_2 and a_3 are positive constants, g_1, g_3 are Ω_1-type functions, g_2 is Ω_0-type function and $\varphi_j, j = 1,2,3$ are Ψ-type functions with $\varphi_1 \propto \varphi_2 \propto \varphi_3$.

Let

$$K := \Phi_3(b_3) + \int_{t_0}^{t_0 e} g_3(s)ds, \tag{18}$$

and

$$K_1 := \Phi_3(c_2) + \int_{t_0 e}^{\infty} \left(\ln \frac{t}{t_0}\right) g_3(t)dt, \tag{19}$$

where

$$\begin{aligned}
b_3 &= \Phi_2^{-1}\left(\Phi_2(b_2) + a_3 \int_{t_0}^{t_0 e} g_2(s)ds\right), \ b_2 = \Phi_1^{-1}\left(\Phi_1(a_1 + a_2) + a_3 \int_{t_0}^{t_0 e} g_1(s)ds\right), \\
c_2 &= \Phi_2^{-1}\left(\Phi_2(c_1) + a_3 \int_{t_0 e}^{\infty} g_2(t)dt\right), \ c_1 = \Phi_1^{-1}\left(\Phi_1(K_2) + a_3 \int_{t_0 e}^{\infty} \left(\ln \frac{t}{t_0}\right) g_1(t)dt\right), \\
K_2 &= a_1 + a_2 + a_3\varphi_1\left(\Phi_3^{-1}(K)\right) \int_{t_0}^{t_0 e} g_1(s)ds + a_3\varphi_2\left(\varphi_3\left(\Phi_3^{-1}(K)\right) \int_{t_0}^{t_0 e} g_3(\tau)d\tau\right) \int_{t_0}^{t_0 e} g_2(s)ds,
\end{aligned} \tag{20}$$

and Φ_j^{-1} is the inverse functions of $\Phi_j(x) = \int_\rho^x \frac{ds}{\varphi_j(s)}, j = 1,2,3, \ x > \rho > 0$.

A generalized version of Lemma 8 is introduced below. We shall provide an estimate for an integral term that arises later in our present problem. Although this estimate is not the best possible, it ensures that such an integral is bounded, which is exactly what we will need to prove our results below.

Theorem 1. *Suppose that $w(t)$ is a continuous nonnegative function on (t_0, ∞) satisfying Inequality (17) for all $t > t_0$, a_1, a_2 and a_3 are positive constants, g_1, g_3 are Ω_1-type functions, g_2 is Ω_0-type function and $\varphi_j, j = 1,2,3$, are Ψ-type functions with $\varphi_1 \propto \varphi_2 \propto \varphi_3$. Then,*

$$w(t) \leq \Phi_3^{-1}(K) \text{ for all } t_0 < t < t_0 e, \tag{21}$$

and

$$w(t) \leq \Phi_3^{-1}(K_1)\left(\ln \frac{t}{t_0}\right) \text{ for all } t \geq t_0 e, \tag{22}$$

where K and K_1 are given in (18) and (19), respectively.

Proof. For $t_0 < t < t_0 e$, the inequality (17) becomes:

$$w(t) \leq a_1 + a_2 + a_3 \int_{t_0}^{t} \left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau \right) \right] ds, t_0 < t < t_0 e, \tag{23}$$

and Lemma 8 is applicable. The relation (21) follows immediately.

For the case $t \geq t_0 e$, we have $\ln \frac{t}{t_0} \geq 1$ and (17) may be rewritten as:

$$\begin{aligned} \frac{w(t)}{\ln \frac{t}{t_0}} &\leq a_1 + a_2 + a_3 \int_{t_0}^{t} \left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau \right) \right] ds \\ &\leq a_1 + a_2 + a_3 \int_{t_0}^{t_0 e} \left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau \right) \right] ds \\ &\quad + a_3 \int_{t_0 e}^{t} \left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau \right) \right] ds. \end{aligned} \tag{24}$$

In virtue of the estimate (21) together with the continuity and monotonicity of the functions $\varphi_i, i = 1,2,3$, we get:

$$\frac{w(t)}{\ln \frac{t}{t_0}} \leq K_2 + a_3 \int_{t_0 e}^{t} \left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau \right) \right] ds, \tag{25}$$

where K_2 is the constant given in (20).

Let

$$z := z_1 + z_2 + z_3, \tag{26}$$

where

$$\begin{aligned} z_1(t) \quad &: = K_2 + a_3 \int_{t_0 e}^{t} g_1(s)\varphi_1(w(s))ds, \ t \geq t_0 e, \\ z_2(t) \quad &: = a_3 \int_{t_0 e}^{t} g_2(s)\varphi_2(z_3(s))ds, \ t \geq t_0 e, \\ z_3(t) \quad &: = \int_{t_0}^{t} g_3(s)\varphi_3(w(s))ds, \ t > t_0. \end{aligned} \tag{27}$$

It is clear that

$$\frac{w(t)}{\ln \frac{t}{t_0}} < z(t) \text{ for all } t \geq t_0 e. \tag{28}$$

In light of the types of the functions g_i, and $\varphi_i, i = 1,2,3$, see Definitions 5 and 6, differentiating z yields:

$$\begin{aligned} z'(t) &= a_3 g_1(t)\varphi_1(w(t)) + a_3 g_2(t)\varphi_2(z_3(t)) + g_3(t)\varphi_3(w(t)) \\ &\leq a_3 g_1(t)\left(\ln \frac{t}{t_0}\right)\varphi_1\left(\frac{w(t)}{\ln \frac{t}{t_0}}\right) + a_3 g_2(t)\varphi_2(z(t)) + g_3(t)\left(\ln \frac{t}{t_0}\right)\varphi_3\left(\frac{w(t)}{\ln \frac{t}{t_0}}\right) \\ &\leq a_3\left(\ln \frac{t}{t_0}\right)g_1(t)\varphi_1(z(t)) + a_3 g_2(t)\varphi_2(z(t)) + \left(\ln \frac{t}{t_0}\right)g_3(t)\varphi_3(z(t)), t \geq t_0 e. \end{aligned} \tag{29}$$

Now, we integrate both sides of (29) over the interval $[t_0 e, t]$ to find:

$$\begin{aligned} z(t) \quad &\leq z(t_0 e) + a_3 \int_{t_0 e}^{t} \left(\ln \frac{s}{t_0}\right) g_1(s)\varphi_1(z(s))ds + a_3 \int_{t_0 e}^{t} g_2(s)\varphi_2(z(s))ds \\ &\quad + \int_{t_0 e}^{t} \left(\ln \frac{s}{t_0}\right) g_3(s)\varphi_3(z(s))ds. \end{aligned} \tag{30}$$

Applying Lemma 7 with $\eta_1(t) = a_3\left(\ln \frac{t}{t_0}\right)g_1(t)$, $\eta_2(t) = a_3 g_2(t)$, $\eta_3(t) = \left(\ln \frac{t}{t_0}\right)g_3(t)$ and $T_1 = \infty$ (by assumption $\int_\rho^\infty \frac{d\tau}{\varphi_i(\tau)} = \infty$, $i = 1,2,3$ for any $\rho > 0$), the inequality (30) leads to:

$$z(t) \le \Phi_3^{-1}(K_1), \text{ for all } t \ge t_0 e,$$

as desired. □

From now on, we assume that the functions f and h on the right hand-side of (1), satisfy the conditions below:

(A$_1$) $f(t,u,v) : (t_0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a $C_{1-\alpha,\ln}[t_0,\infty)$ function in $E_1 = \{(t,u,v) : t > t_0 > 0, u, v \in C_{1-\alpha,\ln}[t_0,\infty)\}$.

(A$_2$) $h(t,s,u)$ is a continuous function in $E_2 = \{(t,s,u) : t_0 \le s < t < \infty, u \in C_{1-\alpha,\ln}[t_0,\infty)\}$.

(A$_3$) There are Ω_1-type functions g_1, g_3, an Ω_0-type function g_2 and Ψ-type functions φ_i, $i = 1,2,3$ with $\varphi_1 \propto \varphi_2 \propto \varphi_3$ such that:

$$|f(t,u,v)| \le g_1(t)\varphi_1\left(\left(\ln \frac{t}{t_0}\right)^{1-\alpha+\alpha_1}|u|\right) + g_2(t)\varphi_2(|v|),\ (t,u,v) \in E_1,$$

$$|h(t,s,u)| \le g_3(s)\varphi_3\left(\left(\ln \frac{s}{t_0}\right)^{1-\alpha+\alpha_2}|u|\right),\ (t,s,u) \in E_2,$$

$$0 \le \alpha_1 < \alpha < 1 \text{ and } 0 \le \alpha_2 < \alpha < 1.$$

Functions satisfying the above hypotheses are given in Section 4.

In the next result, we shall show that the solution for Problem (1) satisfies the useful nonlinear inequality below.

Lemma 9. *Suppose that $u(t)$ is a $C^{\alpha+1}_{1-\alpha,\ln}[t_0,\infty)$-solution for Problem (1) and the functions f and h satisfy (**A$_1$**), (**A$_2$**) and (**A$_3$**). Then,*

$$\left(\ln \frac{t}{t_0}\right)^{1-\alpha}|u(t)| \le v(t),\ t > t_0, \tag{31}$$

and

$$\left(\ln \frac{t}{t_0}\right)^{1-\alpha+\alpha_j}\left|\left({}_H\mathcal{D}^{\alpha_j}_{t_0}u\right)(t)\right| \le v_j(t),\ t > t_0, \tag{32}$$

where

$$\begin{aligned} v(t) &= \frac{|u_1|}{\Gamma(\alpha)} + \frac{\ln \frac{t}{t_0}}{\Gamma(\alpha+1)}\left\{|u_2| + \int_{t_0}^t \left[g_1(s)\varphi_1\left(\left(\ln \frac{s}{t_0}\right)^{1-\alpha+\alpha_1}\left|\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(s)\right|\right)\right.\right. \\ &\quad \left.\left. + g_2(s)\varphi_2\left(\int_{t_0}^\tau g_3(\tau)\varphi_3\left(\left(\ln \frac{\tau}{t_0}\right)^{1-\alpha+\alpha_2}\left|\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\tau)\right|\right)d\tau\right)\right]ds\right\}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} v_j(t) &= \frac{|u_1|}{\Gamma(\alpha-\alpha_j)} + \frac{\ln \frac{t}{t_0}}{\Gamma(\alpha-\alpha_j+1)}\left\{|u_2| + \int_{t_0}^t \left[g_1(s)\varphi_1\left(\left(\ln \frac{s}{t_0}\right)^{1-\alpha+\alpha_1}\left|\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(s)\right|\right)\right.\right. \\ &\quad \left.\left. + g_2(s)\varphi_2\left(\int_{t_0}^\tau g_3(\tau)\varphi_3\left(\left(\ln \frac{\tau}{t_0}\right)^{1-\alpha+\alpha_2}\left|\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\tau)\right|\right)d\tau\right)\right]ds\right\}, \end{aligned} \tag{34}$$

for all $t > t_0$ and $j = 1,2$.

Proof. Firstly, we recall the relations (12) and (13) from the proof of Lemma 5,

$$\begin{aligned} u(t) &= \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} \\ &\times \int_{t_0}^{s} f\left(\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\sigma))d\sigma\right)d\tau\frac{ds}{s},\ t > t_0, \end{aligned} \tag{35}$$

$$\begin{aligned} u(t) &= \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t}\left(\ln\frac{t}{\tau}\right)^{\alpha} \\ &\times f\left(\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\sigma))d\sigma\right)d\tau,\ t > t_0. \end{aligned} \tag{36}$$

Equation (36) leads to the following bound on $|u(t)|$,

$$\begin{aligned} |u(t)| &\le \frac{|u_1|\left(\ln\frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|u_2|\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\left(\ln\frac{t}{t_0}\right)^{\alpha}}{\Gamma(\alpha+1)} \\ &\times \int_{t_0}^{t}\left|f\left(\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\sigma))d\sigma\right)\right|d\tau, t > t_0, \end{aligned} \tag{37}$$

or

$$\begin{aligned} \left(\ln\frac{t}{t_0}\right)^{1-\alpha}|u(t)| &\le \frac{|u_1|}{\Gamma(\alpha)} + \frac{|u_2|\ln\frac{t}{t_0}}{\Gamma(\alpha+1)} + \frac{\ln\frac{t}{t_0}}{\Gamma(\alpha+1)} \\ &\times \int_{t_0}^{t}\left|f\left(\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\sigma))d\sigma\right)\right|d\tau, \end{aligned} \tag{38}$$

for all $t > t_0$, which, in the light of the hypothesis ($\mathbf{A}_3$), leads to (31) as desired.

Now, apply the Hadamard differentiation operator ${}_H\mathcal{D}_{t_0}^{\alpha_j}$, $j = 1,2$ to (35), then employ Lemmas 1 and 2 (with $\beta = \alpha_j$ and $\gamma = 1-\alpha$, see (16)) to get:

$$\begin{aligned} \left({}_H\mathcal{D}_{t_0}^{\alpha_j}u\right)(t) &= \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j-1}}{\Gamma(\alpha-\alpha_j)} + \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)} + \frac{1}{\Gamma(\alpha-\alpha_j)}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-\alpha_j-1} \\ &\times \int_{t_0}^{s} f\left(\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\sigma))d\sigma\right)d\tau\frac{ds}{s},\ t > t_0. \end{aligned} \tag{39}$$

In a similar way to that used to obtain (36), the relation (39) reduces to:

$$\begin{aligned} \left({}_H\mathcal{D}_{t_0}^{\alpha_j}u\right)(t) &= \frac{u_1\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j-1}}{\Gamma(\alpha-\alpha_j)} + \frac{u_2\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)} + \frac{1}{\Gamma(\alpha-\alpha_j+1)}\int_{t_0}^{t}\left(\ln\frac{t}{\tau}\right)^{\alpha-\alpha_j} \\ &\times f\left(\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(\tau), \int_{t_0}^{\tau} h(\tau,\sigma,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\sigma))d\sigma\right)d\tau,\ t > t_0, \end{aligned} \tag{40}$$

and consequently,

$$\begin{aligned}\left|\left({}_H\mathcal{D}^{\alpha_j}_{t_0}u\right)(t)\right| &\leq \frac{|u_1|\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j-1}}{\Gamma(\alpha-\alpha_j)}+\frac{|u_2|\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}+\frac{1}{\Gamma(\alpha-\alpha_j+1)}\int_{t_0}^{t}\left(\ln\frac{t}{\tau}\right)^{\alpha-\alpha_j}\\ &\quad\times\left|f\left(\tau,\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(\tau),\int_{t_0}^{\tau}h(\tau,\sigma,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\sigma))d\sigma\right)\right|d\tau\\ &\leq \frac{|u_1|\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j-1}}{\Gamma(\alpha-\alpha_j)}+\frac{\left(\ln\frac{t}{t_0}\right)^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\Big[\,|u_2|+\\ &\quad\int_{t_0}^{t}\left|f\left(\tau,\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(\tau),\int_{t_0}^{\tau}h(\tau,\sigma,\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\sigma))d\sigma\right)\right|d\tau\Big],\ t>t_0,\end{aligned}\tag{41}$$

which yields, in light of the hypothesis ($\mathbf{A}_3$), the desired result in (32) and completes the proof. □

The main result of this section is given below.

Theorem 2. *Suppose that the hypotheses ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$) and ($\boldsymbol{A}_3$) are satisfied, then there exists a real number r such that any solution $u\in C^{\alpha+1}_{1-\alpha,\ln}[t_0,\infty)$ of Problem (1), has the following asymptotic property $\lim\limits_{t\to\infty}\frac{u(t)}{\left(\ln\frac{t}{t_0}\right)^{\alpha}}=r$.*

Proof. Let us start by recalling the relations (31) and (32) in Lemma 9,

$$\left(\ln\frac{t}{t_0}\right)^{1-\alpha}|u(t)|\leq v(t),\ t>t_0,\tag{42}$$

and

$$\left(\ln\frac{t}{t_0}\right)^{1-\alpha+\alpha_j}\left|\left({}_H\mathcal{D}^{\alpha_j}_{t_0}u\right)(t)\right|\leq v_j(t),\ t>t_0,\ j=1,2,\tag{43}$$

where u is a $C^{\alpha+1}_{1-\alpha,\ln}[t_0,\infty)$-solution for Problem (1), $v(t)$ and $v_j(t)$, $j=1,2$ are given in (33) and (34), respectively.

Let

$$\begin{aligned}a_1 &= |u_1|\max_{j=1,2}\left\{\frac{1}{\Gamma(\alpha)},\frac{1}{\Gamma(\alpha-\alpha_j)}\right\},\ a_2=|u_2|\max_{j=1,2}\left\{\frac{1}{\Gamma(\alpha+1)},\frac{1}{\Gamma(\alpha-\alpha_j+1)}\right\},\\ a_3 &= \max_{j=1,2}\left\{\frac{1}{\Gamma(\alpha+1)},\frac{1}{\Gamma(\alpha-\alpha_j+1)}\right\},\end{aligned}\tag{44}$$

and

$$\begin{aligned}w(t) &= a_1+\left(\ln\frac{t}{t_0}\right)\left\{a_2+a_3\int_{t_0}^{t}\left[g_1(s)\varphi_1\left(\left(\ln\frac{s}{t_0}\right)^{1-\alpha+\alpha_1}\left|\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u\right)(s)\right|\right)\right.\right.\\ &\quad\left.\left.+g_2(s)\varphi_2\left(\int_{t_0}^{\tau}g_3(\tau)\varphi_3\left(\left(\ln\frac{\tau}{t_0}\right)^{1-\alpha+\alpha_2}\left|\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u\right)(\tau)\right|\right)d\tau\right)\right]ds\right\},\ t>t_0,\end{aligned}\tag{45}$$

then,

$$v(t)\leq w(t)\text{ and }v_j(t)\leq w(t)\text{ for all }t>t_0,\ j=1,2.\tag{46}$$

From the nondecreasingness property of the functions φ_i, $i=1,2,3$, we have:

$$
\begin{aligned}
w(t) &= a_1 + \left(\ln \frac{t}{t_0}\right)\left\{a_2 + a_3 \int_{t_0}^{t}\left[g_1(s)\varphi_1(v_1(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{\tau} g_3(\tau)\varphi_3(v_2(\tau))d\tau\right)\right]ds\right\} \\
&\leq a_1 + \left(\ln \frac{t}{t_0}\right)\left\{a_2 + a_3 \int_{t_0}^{t}\left[g_1(s)\varphi_1(w(s)) + g_2(s)\varphi_2\left(\int_{t_0}^{\tau} g_3(\tau)\varphi_3(w(\tau))d\tau\right)\right]ds\right\},
\end{aligned}
$$

for all $t > t_0$, which is Inequality (17). Therefore, we get from Theorem 1 that:

$$
w(t) \leq \left(\ln \frac{t}{t_0}\right)\Phi_3^{-1}(K_1) \text{ for all } t \geq t_0 e, \tag{47}
$$

and as a consequence of the estimates (42) and (46), we find:

$$
\left(\ln \frac{t}{t_0}\right)^{1-\alpha}|u(t)| \leq \left(\ln \frac{t}{t_0}\right)\Phi_3^{-1}(K_1) \text{ for all } t > t_0 e, \tag{48}
$$

or

$$
\frac{|u(t)|}{\left(\ln \frac{t}{t_0}\right)^{\alpha}} \leq K_3 := \Phi_3^{-1}(K_1) \text{ for all } t > t_0 e, \tag{49}
$$

where K_1 is as in (19).

Now, in the light of Hypothesis (**A**$_3$), and the inequalities (43) and (46), we deduce that:

$$
\begin{aligned}
\mathcal{J} \ :&= \left|\int_{t_0}^{t} f\left(s, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(s), \int_{t_0}^{s} h\left(s,\tau,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\tau)\right)d\tau\right)ds\right| \\
&\leq \int_{t_0}^{t}\left|f\left(s, \left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(s), \int_{t_0}^{s} h\left(s,\tau,\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\tau)\right)d\tau\right)\right|ds \\
&\leq \int_{t_0}^{t} g_1(s)\varphi_1\left(\left(\ln \frac{s}{t_0}\right)^{1-\alpha+\alpha_1}\left|\left({}_H\mathcal{D}_{t_0}^{\alpha_1}u\right)(s)\right|\right)ds + \int_{t_0}^{t} g_2(s) \\
&\quad \times \varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3\left(\left(\ln \frac{\tau}{t_0}\right)^{1-\alpha+\alpha_2}\left|\left({}_H\mathcal{D}_{t_0}^{\alpha_2}u\right)(\tau)\right|\right)d\tau\right)ds \\
&\leq \int_{t_0}^{t_0 e} g_1(s)\varphi_1(w(s))ds + \int_{t_0}^{t_0 e} g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau\right)ds \\
&\quad + \int_{t_0 e}^{t} g_1(s)\varphi_1(w(s))ds + \int_{t_0 e}^{t} g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau\right)ds \\
&= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned} \tag{50}
$$

The first integral,

$$
\begin{aligned}
\mathcal{J}_1 \ :&= \int_{t_0}^{t_0 e} g_1(s)\varphi_1(w(s))ds + \int_{t_0}^{t_0 e} g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau\right)ds \\
&\leq \int_{t_0}^{t_0 e} g_1(s)\varphi_1\left(\Phi_3^{-1}(K)\right)ds + \int_{t_0}^{t_0 e} g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3\left(\Phi_3^{-1}(K)\right)d\tau\right)ds \\
&\leq \varphi_1(K_4)\int_{t_0}^{t_0 e} g_1(s)ds + \int_{t_0}^{t_0 e} g_2(s)\varphi_2\left(\varphi_3(K_4)\int_{t_0}^{s} g_3(\tau)d\tau\right)ds
\end{aligned} \tag{51}
$$

is finite by (21) from Theorem 1, $K_4 := \Phi_3^{-1}(K)$ and K is given in (18).

The second integral,

$$
\mathcal{J}_2 := \int_{t_0 e}^{t} g_1(s)\varphi_1(w(s))ds + \int_{t_0 e}^{t} g_2(s)\varphi_2\left(\int_{t_0}^{s} g_3(\tau)\varphi_3(w(\tau))d\tau\right)ds,\ t \geq t_0 e, \tag{52}
$$

can be estimated in view of (47), Theorem 1 and the type of the functions $\varphi_i, i = 1,2,3$ as follows:

$$
\begin{aligned}
\mathcal{J}_2 \quad &\leq \quad \int_{t_0 e}^{t} \left(\ln \frac{s}{t_0}\right) g_1(s) \varphi_1(K_3) ds + \int_{t_0 e}^{t} g_2(s) \varphi_2 \left(\int_{t_0}^{t_0 e} g_3(\tau) \varphi_3(K_4) d\tau \right. \\
&\quad \left. + \int_{t_0 e}^{s} \left(\ln \frac{\tau}{t_0}\right) g_3(\tau) \varphi_3(K_3) d\tau \right) ds \\
&\leq \quad \varphi_1(K_3) \int_{t_0 e}^{t} \left(\ln \frac{s}{t_0}\right) g_1(s) ds + \int_{t_0 e}^{t} g_2(s) \varphi_2 \left(\varphi_3(K_4) \int_{t_0}^{t_0 e} g_3(\tau) d\tau \right. \\
&\quad \left. + \varphi_3(K_3) \int_{t_0 e}^{s} \left(\ln \frac{\tau}{t_0}\right) g_3(\tau) d\tau \right) ds, \text{ for all } t \geq t_0 e. \qquad (53)
\end{aligned}
$$

Since g_1 and g_3 are Ω_1-type functions and g_2 is an Ω_0-type function, we see that the integral $\mathcal{J}_2$ is uniformly bounded and so is the integral $\mathcal{J}$.

Therefore, the integral $\int_{t_0}^{t} f\left(s, \left({}_H\mathcal{D}_{t_0}^{\alpha_1} u\right)(s), \int_{t_0}^{s} h(s,\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_2} u\right)(\tau)) d\tau\right) ds$ absolutely convergent and, consequently,

$$
\lim_{t \to \infty} \int_{t_0}^{t} f\left(s, \left({}_H\mathcal{D}_{t_0}^{\alpha_1} u\right)(s), \int_{t_0}^{s} h(s,\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_2} u\right)(\tau)) d\tau\right) ds < \infty, \qquad (54)
$$

Using Lemma 5, there exits a finite real number r such that:

$$
\lim_{t \to \infty} \frac{u(t)}{\left(\ln \frac{t}{t_0}\right)^{\alpha}} = \frac{1}{\Gamma(\alpha+1)} \left(u_2 + \lim_{t \to \infty} \int_{t_0}^{t} f\left(s, \left({}_H\mathcal{D}_{t_0}^{\alpha_1} u\right)(s), \int_{t_0}^{s} h(s,\tau, \left({}_H\mathcal{D}_{t_0}^{\alpha_2} u\right)(\tau)) d\tau\right) ds \right) = r, \qquad (55)
$$

as wanted. □

Remark 2. *If the condition (**A**$_3$) is replaced by the condition:*

(A$_4$) *There are Ω_1-type functions ξ_1, ξ_2 and Ψ-type functions ϕ_i, $i = 1,2,3$ with $\phi_1 \phi_2 \propto \phi_3$ such that*

$$
|f(t,u,v)| \leq \xi_1(t) \phi_1 \left(\left(\ln \frac{t}{t_0}\right)^{1-\alpha+\alpha_1} |u| \right) \phi_2(|v|), \ (t,u,v) \in E_1, \ 0 \leq \alpha_1 < \alpha < 1,
$$

and

$$
|h(t,s,u)| \leq \xi_2(s) \phi_3 \left(\left(\ln \frac{s}{t_0}\right)^{1-\alpha+\alpha_2} |u| \right), \ (t,s,u) \in E_2, \ 0 \leq \alpha_2 < \alpha < 1,
$$

then the conclusion of Theorem 2 is still valid. We state this fact below.

Theorem 3. *Suppose that the functions f and h satisfy the conditions (**A**$_1$), (**A**$_2$) and (**A**$_4$). Then, there exists a real number r such that any solution $u \in C_{1-\alpha,\ln}^{\alpha+1}[t_0,\infty)$ for Problem (1) satisfies* $\lim_{t \to \infty} \frac{u(t)}{\left(\ln \frac{t}{t_0}\right)^{\alpha}} = r$.

The proof is skipped as it can be shown in a similar manner to that used in the proof of Theorem 2.

Remark 3. *The problem,*

$$\begin{cases} \left({}_H\mathcal{D}^{\alpha}_{t_0}u\right)'(t) = f\left(t, u(t), \int_{t_0}^{t} h(t,s,u(s))ds\right),\ t > t_0 > 0,\ 0 < \alpha < 1, \\ \left({}_H\mathcal{I}^{1-\alpha}_{t_0}u\right)(t_0^+) = u_1,\ \left({}_H\mathcal{D}^{\alpha}_{t_0}u\right)(t_0^+) = u_2,\ u_1, u_2 \in \mathbb{R}. \end{cases} \tag{56}$$

is a special case of Problem (1) when $\alpha_1 = \alpha_2 = 0$. Therefore, we can conclude that there exists a constant $r \in \mathbb{R}$ such that any solution $u \in C^{\alpha+1}_{1-\alpha,\ln}[t_0,\infty)$ for Problem (56) satisfies $\lim_{t\to\infty} \frac{u(t)}{\left(\ln\frac{t}{t_0}\right)^{\alpha}} = r$.

4. Example

Consider the initial value problem:

$$\begin{cases} \left({}_H\mathcal{D}^{\alpha}_{t_0}u\right)'(t) = t^{-\sigma_1}\left(\ln\frac{t}{t_0}\right)^{\beta_1}\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u(t)\right)^{\gamma_1} + t^{-\sigma_2}\left(\ln\frac{t}{t_0}\right)^{\beta_2}\left[\int_{t_0}^{t}(t+s)^{-\sigma_3}\right. \\ \qquad\qquad \left. \times\left(\ln\frac{s}{t_0}\right)^{\beta_3}\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u(s)\right)^{\gamma_3}ds\right]^{\gamma_2}, t > t_0 > 0, \\ \left({}_H\mathcal{I}^{1-\alpha}_{t_0}u\right)(t_0^+) = u_1,\ \left({}_H\mathcal{D}^{\alpha}_{t_0}u\right)(t_0^+) = u_2,\ u_1, u_2 \in \mathbb{R}, \end{cases} \tag{57}$$

where $0 \le \alpha_1 < \alpha < 1$, $0 \le \alpha_2 < \alpha < 1$, $\sigma_j > 1$, $j = 1,2,3$, $\beta_1 > (1-\alpha+\alpha_1)\gamma_1 - 1$, $\beta_2 > -1$, $\beta_3 > (1-\alpha+\alpha_2)\gamma_3 - 1$ and $0 < \gamma_1 \le \gamma_2 \le \gamma_3 < 1$.

The source function,

$$f(t,u,v) = t^{-\sigma_1}\left(\ln\frac{t}{t_0}\right)^{\beta_1}u^{\gamma_1} + t^{-\sigma_2}\left(\ln\frac{t}{t_0}\right)^{\beta_2}v^{\gamma_2},\ t > t_0 > 0, \tag{58}$$

satisfies

$$\begin{aligned} & \left| f\left(t, {}_H\mathcal{D}^{\alpha_1}_{t_0}u(t), \int_{t_0}^{t} h\left(t,s,{}_H\mathcal{D}^{\alpha_2}_{t_0}u(s)\right)ds\right)\right| \\ = & \left| t^{-\sigma_1}\left(\ln\frac{t}{t_0}\right)^{\beta_1}\left({}_H\mathcal{D}^{\alpha_1}_{t_0}u(t)\right)^{\gamma_1} + t^{-\sigma_2}\left(\ln\frac{t}{t_0}\right)^{\beta_2}\left(\int_{t_0}^{t}(t+s)^{-\sigma_3}\right.\right. \\ & \left.\left. \times\left(\ln\frac{s}{t_0}\right)^{\beta_3}\left({}_H\mathcal{D}^{\alpha_2}_{t_0}u(s)\right)^{\gamma_3}ds\right)^{\gamma_2}\right| \\ \le & \ g_1(t)\varphi_1\left(\left(\ln\frac{t}{t_0}\right)^{1-\alpha+\alpha_1}\left|{}_H\mathcal{D}^{\alpha_1}_{t_0}u(t)\right|\right) + g_2(t)\varphi_2\left(\int_{t_0}^{t} g_3(s)\right. \\ & \left. \times\varphi_3\left(\left(\ln\frac{s}{t_0}\right)^{1-\alpha+\alpha_2}\left|{}_H\mathcal{D}^{\alpha_2}_{t_0}u(s)\right|\right)ds\right), \end{aligned}$$

with

$$\begin{aligned} h(t,s,u) &= (t+s)^{-\sigma_3}\left(\ln\frac{s}{t_0}\right)^{\beta_3}u^{\gamma_3},\ g_1(t) = t^{-\sigma_1}\left(\ln\frac{t}{t_0}\right)^{\beta_1+(\alpha-\alpha_1-1)\gamma_1}, \\ g_2(t) &= t^{-\sigma_2}\left(\ln\frac{t}{t_0}\right)^{\beta_2},\ g_3(t) = t^{-\sigma_3}\left(\ln\frac{t}{t_0}\right)^{\beta_3+(\alpha-\alpha_2-1)\gamma_3},\ \varphi_j(t) = t^{\gamma_j},\ t > t_0 > 0, \end{aligned}$$

where $\beta_1 > (1-\alpha+\alpha_1)\gamma_1 - 1$, $\beta_2 > -1$, $\beta_3 > (1-\alpha+\alpha_2)\gamma_3 - 1$, $\sigma_j > 1$, $j = 1,2,3$ and $0 < \gamma_1 \le \gamma_2 \le \gamma_3 \le 1$.

The functions g_j, $j = 1, 2, 3$ are continuous, nonnegative on (t_0, ∞),

$$\begin{aligned}\int_{t_0}^{t_0 e} g_1(t)dt &= \int_{t_0}^{t_0 e}\left(\ln\frac{t}{t_0}\right)^{\beta_1+(\alpha-\alpha_1-1)\gamma_1}\frac{dt}{t^{\sigma_1}} \le \frac{1}{t_0^{\sigma_1-1}}\int_{t_0}^{t_0 e}\left(\ln\frac{t}{t_0}\right)^{\beta_1+(\alpha-\alpha_1-1)\gamma_1}\frac{dt}{t}\\ &= \frac{1}{t_0^{\sigma_1-1}}\int_0^1 s^{\beta_1+(\alpha-\alpha_1-1)\gamma_1}ds = \frac{1}{(\beta_1+(\alpha-\alpha_1-1)\gamma_1+1)t_0^{\sigma_1-1}} < \infty,\end{aligned}$$

$$\int_{t_0}^{t_0 e} g_3(t)dt = \int_{t_0}^{t_0 e}\left(\ln\frac{t}{t_0}\right)^{\beta_3+(\alpha-\alpha_2-1)\gamma_1}\frac{dt}{t^{\sigma 3}} \le \frac{1}{(\beta_3+(\alpha-\alpha_3-1)\gamma_3+1)t_0^{\sigma_3-1}} < \infty,$$

and

$$\int_{t_0}^{t_0 e} g_2(t)dt = \int_{t_0}^{t_0 e}\left(\ln\frac{t}{t_0}\right)^{\beta_2}\frac{dt}{t^{\sigma_2}} \le \frac{1}{t_0^{\sigma_2-1}}\int_{t_0}^{t_0 e}\left(\ln\frac{t}{t_0}\right)^{\beta_2}\frac{dt}{t} = \frac{1}{(\beta_2+1)t_0^{\sigma_2-1}} < \infty.$$

The function g_2 is an Ω_0-type because:

$$\begin{aligned}\int_{t_0 e}^{\infty} g_2(t)dt &= \int_{t_0 e}^{\infty}\left(\ln\frac{t}{t_0}\right)^{\beta_2}\frac{dt}{t^{\sigma_2}} = \frac{1}{t_0^{\sigma_2-1}}\int_{t_0 e}^{\infty}\left(\ln\frac{t}{t_0}\right)^{\beta_2}\left(\frac{t_0}{t}\right)^{\sigma_2-1}\frac{dt}{t}\\ &\le \frac{1}{t_0^{\sigma_2-1}}\int_{t_0}^{\infty}\left(\ln\frac{t}{t_0}\right)^{\beta_2}e^{-(\sigma_2-1)\ln\frac{t}{t_0}}\frac{dt}{t} = \frac{1}{t_0^{\sigma_2-1}}\int_0^{\infty}s^{\beta_2}e^{-(\sigma_2-1)s}ds\\ &= \frac{1}{(\sigma_2-1)^{\beta_2+1}t_0^{\sigma_2-1}}\int_0^{\infty}\tau^{\beta_2}e^{-\tau}d\tau = \frac{\Gamma(\beta_2+1)}{(\sigma_2-1)^{\beta_2+1}t_0^{\sigma_2-1}} < \infty,\end{aligned}$$

whereas the functions g_1 and g_3 are Ω_1-type functions since

$$\begin{aligned}\int_{t_0 e}^{\infty}\left(\ln\frac{t}{t_0}\right)g_1(t)dt &= \int_{t_0 e}^{\infty}\left(\ln\frac{t}{t_0}\right)^{\beta_1+(\alpha-\alpha_1-1)\gamma_1+1}\frac{dt}{t^{\sigma_1}}\\ &\le \frac{1}{t_0^{\sigma_1-1}}\int_{t_0}^{\infty}\left(\ln\frac{t}{t_0}\right)^{\beta_1+(\alpha-\alpha_1-1)\gamma_1+1}e^{-(\sigma_1-1)\ln\frac{t}{t_0}}\frac{dt}{t}\\ &= \frac{1}{t_0^{\sigma_1-1}}\int_0^{\infty}s^{\beta_1+(\alpha-\alpha_1-1)\gamma_1+1}e^{-(\sigma_1-1)s}ds\\ &= \frac{\Gamma(\beta_1+(\alpha-\alpha_1-1)\gamma_1+2)}{(\sigma_1-1)^{\beta_1+(\alpha-\alpha_1-1)\gamma_1+2}t_0^{\sigma_1-1}} < \infty,\end{aligned}$$

and

$$\int_{t_0 e}^{\infty}\left(\ln\frac{t}{t_0}\right)g_3(t)dt = \int_{t_0 e}^{\infty}\left(\ln\frac{t}{t_0}\right)^{\beta_3+(\alpha-\alpha_2-1)\gamma_3+1}\frac{dt}{t^{\sigma_3}} \le \frac{\Gamma(\beta_3+(\alpha-\alpha_2-1)\gamma_3+2)}{(\sigma_3-1)^{\beta_3+(\alpha-\alpha_2-1)\gamma_3+2}t_0^{\sigma_3-1}} < \infty.$$

The functions $\varphi_j(t) = t^{\gamma_j}$, $j = 1, 2, 3$, $0 < \gamma_1 \le \gamma_2 \le \gamma_3 \le 1$, are Ψ-type functions with $\varphi_1 \propto \varphi_2 \propto \varphi_3$. These three functions are continuous, positive, nondecreasing on $(0, \infty)$,

$$\varphi_j(a) = a^{\gamma_j} \le b\left(\frac{a}{b}\right)^{\gamma_j} = b\varphi_j\left(\frac{a}{b}\right) \text{ for all } a \ge 1,\ b \ge 1,\ 0 < \gamma_j \le 1,$$

and

$$\int_{\rho}^{x}\frac{dt}{\varphi_j(t)} = \int_{\rho}^{x}\frac{dt}{t^{\gamma_j}} \to \infty \text{ as } x \to \infty \text{ for any } \rho > 0.$$

As $0 < \gamma_1 \le \gamma_2 \le \gamma_3 \le 1$, then φ_3/φ_2 and φ_2/φ_1 are nondecreasing functions and so $\varphi_1 \propto \varphi_2 \propto \varphi_3$.

Obviously, Conditions ($\mathbf{A}_1$), ($\mathbf{A}_2$) and ($\mathbf{A}_3$) are fulfilled with these functions and the hypotheses of Theorem 2 as well. Therefore, there exists a real constant $r \in \mathbb{R}$ such that solutions of Problem (57) satisfy $\lim_{t\to\infty} \frac{u(t)}{\left(\ln \frac{t}{t_0}\right)^{\alpha}} = r$.

5. Conclusions

In this article, we considered a fractional integro-differential problem with Hadamard-Type fractional derivatives. The nonlinear source function depends on a lower-order Hadamard fractional derivative of the state and an integral involving another lower-order Hadamard fractional derivative of the state. We assumed the boundedness of the nonlinearities in question by some special kinds of functions in some appropriate spaces. Under these nonlinear growth conditions, we demonstrated that solutions of the initial value fractional problem under consideration are not only bounded by logarithmic functions but actually they converge asymptotically to logarithmic functions.

Author Contributions: Conceptualization, A.M. and N.-E.T.; methodology, A.M. and N.-E.T.; validation, A.M. and N.-E.T.; formal analysis, A.M. and N.-E.T.; investigation, A.M.; writing original draft preparation, A.M.; writing, review and editing, N.-E.T.; Funding acquisition, A.M.; project administration, A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research and the APC were funded by King Fahd University of Petroleum and Minerals through project number SB191023.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to acknowledge the support provided by King Fahd University of Petroleum and Minerals (KFUPM) through project number SB191023.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Halanay, A. On the asymptotic behavior of the solutions of an integro-differential equation. *J. Math. Anal. Appl.* **1965**, *10*, 319–324. [CrossRef]
2. Cohen, D.S. The asymptotic behavior of a class of nonlinear differential equations. *Proc. Am. Math. Soc.* **1967**, *18*, 607–609. [CrossRef]
3. Dannan, F.M. Integral inequalities of Gronwall-Bellman-Bihari type and asymptotic behavior of certain second order nonlinear differential equations. *J. Math. Anal. Appl.* **1985**, *108*, 151–164. [CrossRef]
4. Constantin, A. On the asymptotic behavior of second order nonlinear differential equations. *Rendiconti di Matematica e delle sue Applicazioni* **1993**, *13*, 627–634.
5. Philos, C.G.; Tsamatos, P.C. Solutions approaching polynomials at infinity to nonlinear ordinary differential equations. *Electron. J. Differ. Equ.* **2005**, *2005*, 1–25.
6. Mustafa, O.G.; Rogovchenko, Y.V. Asymptotic integration of a class of nonlinear differential equations. *Appl. Math. Lett.* **2006**, *19*, 849–853. [CrossRef]
7. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Limited: Amsterdam, The Netherlands, 2006; Volume 204.
8. Magin, R.L.; Abdullah, O.; Baleanu, D.; Zhou, X.J. Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation. *J. Magn. Reson.* **2008**, *190*, 255–270. [CrossRef]
9. Sabatier, J.; Agrawal, O.P.; Machado, J.T. *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*; Springer: Berlin/Heidelberg, Germany, 2007. [CrossRef]
10. Shah, N.A.; Elnaqeeb, T.; Animasaun, I.; Mahsud, Y. Insight into the natural convection flow through a vertical cylinder using caputo time-fractional derivatives. *Int. J. Appl. Comput. Math.* **2018**, *4*, 1–18. [CrossRef]
11. Ahmad, A.M.; Furati, K.M.; Tatar, N.E. Asymptotic power type behavior of solutions to a nonlinear fractional integro-differential equation. *Electron. J. Differ. Equ.* **2017**, *2017*, 1–16.
12. Ahmad, A.M.; Furati, K.M.; Tatar, N.E. Asymptotic Behavior of Solutions for a Class of Fractional Integro-differential Equations. *Mediterr. J. Math.* **2018**, *15*, 188. [CrossRef]

13. Băleanu, D.; Mustafa, O.G. On the asymptotic integration of a class of sublinear fractional differential equations. *J. Math. Phys.* **2009**, *50*, 123520. [CrossRef]
14. Băleanu, D.; Mustafa, O.G.; Agarwal, R.P. Asymptotic integration of (1+ α)-order fractional differential equations. *Comput. Math. Appl.* **2011**, *62*, 1492–1500. [CrossRef]
15. Băleanu, D.; Agarwal, R.P.; Mustafa, O.G.; Coşulschi, M. Asymptotic integration of some nonlinear differential equations with fractional time derivative. *J. Phys. A Math. Theor.* **2011**, *44*, 055203. [CrossRef]
16. Brestovanska, E.; Medved', M. Asymptotic behavior of solutions to second-order differential equations with fractional derivative perturbations. *Electron. J. Differ. Equ.* **2014**, *2014*, 1–10.
17. Grace, S.R. On the oscillatory behavior of solutions of nonlinear fractional differential equations. *Appl. Math. Comput.* **2015**, *266*, 259–266. [CrossRef]
18. Medved', M. On the asymptotic behavior of solutions of nonlinear differential equations of integer and also of non-integer order. *Electron. J. Qual. Theory Differ. Equ.* **2012**, *10*, 1–9.
19. Medved', M. Asymptotic integration of some classes of fractional differential equations. *Tatra Mt. Math. Publ.* **2013**, *54*, 119–132. [CrossRef]
20. M. Medved' and M. Pospíšil. Asymptotic Integration of Fractional Differential Equations with Integrodifferential Right-Hand Side. *Math. Model. Anal.* **2015**, *20*, 471–489. [CrossRef]
21. Ahmad, A.M.; Furati, K.M.; Tatar, N.E. Boundedness and power-type decay of solutions for a class of generalized fractional Langevin equations. *Arab. J. Math.* **2019**, *8*, 79–94. [CrossRef]
22. Ahmad, A.M. On the Asymptotic Behavior of Solutions for a Fractional Differential Equation with a Singular Kernel. In Proceedings of the 24th International Conference on Circuits, Systems, Communications and Computers (CSCC), Chania, Greece, 19–22 July 2020; pp. 130–134. [CrossRef]
23. Ahmad, A.M. The Asymptotic Behavior of Solutions of a Fractional Integro-differential Equation. *WSEAS Trans. Syst. Control* **2020**, *15*, 341–348. [CrossRef]
24. Graef, J.R.; Grace, S.R.; Tunç, E. Asymptotic behavior of solutions of nonlinear fractional differential equations with Caputo-Type Hadamard derivatives. *Fract. Calc. Appl. Anal.* **2017**, *20*, 71–87. [CrossRef]
25. Kassim, M.D.; Tatar, N.E. Stability of logarithmic type for a Hadamard fractional differential problem. *J. Pseudo-Differ. Oper. Appl.* **2020**, *11*, 447–466. [CrossRef]
26. Kassim, M.D.; eddine Tatar, N. Asymptotic behavior of solutions of fractional differential equations with Hadamard fractional derivatives. *Fract. Calc. Appl. Anal.* **2021**, *24*, 483–508. [CrossRef]
27. Li, M.; Wang, J. Analysis of nonlinear Hadamard fractional differential equations via properties of Mittag-Leffler functions. *J. Appl. Math. Comput.* **2016**, *51*, 487–508. doi: [CrossRef]
28. Li, C.; Li, Z. Stability and Logarithmic Decay of the Solution to Hadamard-Type Fractional Differential Equation. *J. Nonlinear Sci.* **2021**, *31*, 1–60. [CrossRef]
29. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017. [CrossRef]
30. Băleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus: Models and Numerical Methods*; World Scientific: Singapore, 2012; Volume 3.
31. De Barra, G. *Measure Theory and Integration*; Elsevier: Amsterdam, The Netherlands, 2003.
32. Bihari, I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Hung.* **1956**, *7*, 81–94. [CrossRef]
33. LaSalle, J. Uniqueness theorems and successive approximations. *Ann. Math.* **1949**, *50*, 722–730. [CrossRef]
34. Pinto, M. Integral inequalities of Bihari-type and applications. *Funkc. Ekvacioj* **1990**, *33*, 387–403.