



Article

Investigating a Generalized Fractional Quadratic Integral Equation

Basim N. Abood ¹, Saleh S. Redhwan ^{2,*} , Omar Bazighifan ^{3,4} and Kamsing Nonlaopon ⁵

¹ Department of Mathematics, College of Education of Pure Science, University of Wasit, Kut 52001, Iraq; basim.nasih@yahoo.com

² Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431001, India

³ Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com

⁴ Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen

⁵ Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand; nkamsi@kku.ac.th

* Correspondence: saleh.redhwan909@gmail.com

Abstract: In this article, we investigate the analytical and approximate solutions for a fractional quadratic integral equation in the frame of the generalized Riemann–Liouville fractional integral operator with respect to another function. The existence and uniqueness results obtained. Moreover, some new special results corresponding to suitable values of the parameters ζ and q are given. The main results are proved by applying Banach’s fixed point theorem, the Adomian decomposition method, and Picard’s method. In the end, we present a numerical example to justify our results.

Keywords: fractional differential equations; fixed point theorems; ζ -fractional derivative; monotone operator



Citation: Abood, B.N.; Redhwan, S.S.; Bazighifan, O.; Nonlaopon, K.

Investigating a Generalized Fractional Quadratic Integral Equation. *Fractal Fract.* **2022**, *6*, 251. <https://doi.org/10.3390/fractalfract6050251>

Academic Editors: António M. Lopes, Alireza Alfi, Liping Chen and Sergio Adriani David

Received: 17 March 2022

Accepted: 27 April 2022

Published: 1 May 2022

Publisher’s Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Fractional differential equations (FDEs) with initial/boundary conditions arise from a set of applications included in different fields of science and engineering, e.g., practical problems, conservative systems, concerning mechanics, physics, harmonic oscillator, biology, economy, control systems, chemistry, atomic energy, medicine, information theory, nonlinear oscillations, the engineering technique fields; this is because FDEs characterize many real-world processes linked to memory and hereditary properties of different materials more carefully as compared to classical order differential equations. For further details [1–5].

In [6], Hilfer was given a generalization of fractional derivatives (FDs) of Riemann–Liouville (RL) and Caputo, with the so-called Hilfer FD of order q and a type p , $0 < p < 1$. More specifics on this FD mentioned above can be found in [7,8]. In Ref. [9], the researchers introduced the FD with another function in the frame of Hilfer FD, with the so-called ζ –Hilfer FD. For some new results of ζ –Hilfer type initial value problems (IVPs), see [10–13] and, for boundary value problems (BVPs), see [14–16].

In recent decades, there has been a lot of enthusiasm for the Adomian decomposition method (ADM), which is an analytical technique for solving broad types of functional equations. The method was successfully applied to a lot of employments in applied sciences. Here, we also refer to some recent works [17–20] dealing with the technique and its application.

In [21], Picard’s Method (PM) creates a sequence of increasingly specific algebraic approximations of the curtailed precise solution of the first-order differential equation with an initial value.

First, they compare the PM method with the ADM by [20,22] on a group of examples. In [23], the researchers contrasted the two techniques for a quadratic integral equation (QIE).

The QIEs can be widely applicable in more applications like the dynamic theory of gases, the theory of radiative exchange, the traffic theory, etc. The QIEs have been the focus of several papers and monographs, see [23–28].

For example, the researchers in [23] proved the existence and uniqueness of the solution for

$$\varkappa(\vartheta) = h(\vartheta) + \mathfrak{g}(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \mathcal{F}(v, \varkappa(v)) dv,$$

by using the Adomian method and Picard method. In [29], the investigators discussed the analytical and approximate solutions for the fractional quadratic integral equation (FQIE)

$$x(\vartheta) = h(\vartheta) + \mathfrak{g}(\vartheta, x(\vartheta)) \mathbb{I}_{0+}^{q,\rho} \mathcal{F}(\vartheta, x(\vartheta)), \quad \vartheta \in \mathcal{J} = [0, 1], \quad q > 0,$$

where $\mathbb{I}_{0+}^{q,\rho}$ is the Katugampola fractional integral.

In this article, we give the analytical and approximate solutions for the following fractional quadratic integral equation (FQIE)

$$\varkappa(\vartheta) = h(\vartheta) + g(\vartheta, \varkappa(\vartheta)) \mathbb{I}_{0+}^{q,\zeta} \mathcal{F}(\vartheta, \varkappa(\vartheta)), \quad \vartheta \in \mathcal{J} = [0, 1], \quad q > 0, \quad (1)$$

where $\mathbb{I}_{0+}^{q,\zeta}$ is the left sided ζ -RL fractional integral of order q defined by

$$\mathbb{I}_{0+}^{q,\zeta} \mathcal{F}(\vartheta, \varkappa(\vartheta)) = \frac{1}{\Gamma(q)} \int_0^\vartheta \zeta'(v) (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa(v)) dv.$$

Observe that the considered equation is investigated under the Riemann–Liouville integral of fractional order and with respect to another function. In fact, for problem (1), the existence and uniqueness of solutions can be proved readily by using fixed point theorems. However, in general, it is difficult to obtain the exact solutions of (1) directly, due to the Riemann–Liouville operator not having good regularities. Relying on this motivation, recently Kilbas et al. [1] and Almeida [30] provided generalized definitions of fractional calculus involving another function. In this regard, we first give recent results on existence and uniqueness of (1) based on Banach’s fixed point theorem and then apply the Adomian decomposition method and Picard method to obtain an approximate solution for (1). Particularly, if $\zeta(\vartheta) = \vartheta$, $\zeta(\vartheta) = \log(\vartheta)$, and $\zeta(\vartheta) = \vartheta^\rho$, then our results will reduce to the classical Riemann–Liouville, Hadamard, and Katugampola fractional quadratic integral equation, respectively.

The article is formed as follows. In Section 2, we present some notations and definitions used all through the article. Our main results for the generalized FQIE (1) are addressed in Section 3. An example to explain the acquired results is constructed in Section 4.

2. Preliminaries

In this section, we set some notations and introductory facts that will be applied in the proofs of the subsequent results.

Let $C(\mathcal{J}, \mathbb{R})$ be the Banach space of continuous functions and $L(\mathcal{J}, \mathbb{R})$ are Lebesgue integrable functions from $\mathcal{J}$ into $\mathbb{R}$ with the norms

$$\|z\|_\infty = \sup\{|z(\vartheta)| : \vartheta \in \mathcal{J}\},$$

and

$$\|z\|_L = \int_a^b |z(\vartheta)| d\vartheta,$$

respectively.

For $\varsigma = q + 2p - qp$, where $1 < q < 2$, and $0 \leq p \leq 1$, then $1 < \varsigma \leq 2$. Let $\zeta \in C^1(\mathcal{J}, \mathbb{R})$ be an increasing function with $\zeta'(\vartheta) \neq 0$, for all $\vartheta \in \mathcal{J}$.

Definition 1 ([1]). Let $q > 0$ and $\mathcal{F} \in L^1(\mathcal{J}, \mathbb{R})$. The ζ -RL fractional integral of order q of a function $\mathcal{F}$ is given by

$$\mathbb{I}^{q;\zeta} \mathcal{F}(\vartheta, \varkappa(\vartheta)) = \frac{1}{\Gamma(q)} \int_a^\vartheta \zeta'(v) (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(\vartheta, \varkappa(v)) dv,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Lemma 1 ([1,9]). Let $q, \eta, \delta > 0$. Then,

1. $\mathbb{I}^{q;\zeta} \mathbb{I}^{\eta;\zeta} \mathcal{F}(\vartheta, \varkappa(\vartheta)) = \mathbb{I}^{q+\eta;\zeta} \mathcal{F}(\vartheta, \varkappa(\vartheta))$.
2. $\mathbb{I}^{q;\zeta} (\zeta(\vartheta) - \zeta(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(q+\delta)} (\zeta(\vartheta) - \zeta(a))^{q+\delta-1}$.

Here, we can suffice to refer to Banach's fixed point theorem [31] and Krasnoselskii's fixed point theorem [31].

3. Main Results

Let us introduce the following hypotheses which are used to investigate the FQDE (1).

1. $h : \mathcal{J} \rightarrow \mathbb{R}$ is a continuous function on $\mathcal{J}$.
2. $\mathcal{F}, g : \mathbb{J} \times \mathbb{R} \rightarrow \mathbb{R}$ are a bounded and continuous function with $\mu_1 = \sup_{(\vartheta, \varkappa) \in \mathcal{J} \times \mathbb{R}} |g(\vartheta, \varkappa)|$, and $\mu_2 = \sup_{(\vartheta, \varkappa) \in \mathcal{J} \times \mathbb{R}} |\mathcal{F}(\vartheta, \varkappa)|$.
3. There exist two constants $\hbar_1, \hbar_2 > 0$ such that

$$\begin{aligned} |g(\vartheta, \varkappa) - g(\vartheta, y)| &\leq \hbar_1 |\varkappa - y|, \\ |\mathcal{F}(\vartheta, \varkappa) - \mathcal{F}(\vartheta, y)| &\leq \hbar_2 |\varkappa - y|, \end{aligned}$$

for all $\vartheta \in \mathcal{J}$ and $\varkappa, y \in \mathbb{R}$.

Our first result is based on Banach's fixed point theorem to obtain the uniqueness solution of the nonlinear FQIE (1).

3.1. Existence and Uniqueness of Solutions

Theorem 1. Suppose (1), (2) and (3) hold. If

$$\Upsilon := \left(\frac{\hbar_1 \mu_2 + \hbar_2 \mu_1}{\Gamma(q+1)} \right) < 1,$$

then the nonlinear FQIE (1) has a unique solution $\varkappa \in C(\mathcal{J})$.

Proof. It is easy to see that $\Pi : C(\mathcal{J}) \rightarrow C(\mathcal{J})$, where

$$(\Pi \varkappa)(\vartheta) = h(\vartheta) + g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa(v)) dv, \quad \vartheta \in \mathcal{J}, \quad q > 0.$$

Now, let $\mathcal{B}_r \subset C(\mathcal{J})$ where $\mathcal{B}_r$ is defined as

$$\mathcal{B}_r = \{\varkappa(\vartheta) \in C(\mathcal{J}) : |\varkappa(\vartheta) - h(\vartheta)| \leq r, \text{ for } \vartheta \in \mathcal{J}\}.$$

If we choose $r = \frac{\mu_1\mu_2}{\Gamma(q+1)}$, then the operator $\Pi : \mathcal{B}_r \rightarrow \mathcal{B}_r$. Indeed, for $\varkappa \in \mathcal{B}_r$, we have

$$\begin{aligned} |\varkappa(\vartheta) - h(\vartheta)| &\leq |g(\vartheta, \varkappa(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \varkappa(v))| dv \\ &\leq \mu_1\mu_2 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \\ &\leq \frac{\mu_1\mu_2}{\Gamma(q+1)} (\zeta(\vartheta))^q \\ &\leq \frac{\mu_1\mu_2}{\Gamma(q+1)} = r. \end{aligned}$$

In addition, $\mathcal{B}_r$ is a closed subset of $C(\mathcal{J})$. In order to prove that Π is a contraction, we have

$$\begin{aligned} (\Pi\varkappa)(\vartheta) - (\Pi y)(\vartheta) &= g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa(v)) dv \\ &\quad - g(\vartheta, y(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, y(v)) dv \\ &\quad + g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, y(v)) dv \\ &\quad - g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, y(v)) dv \\ &= [g(\vartheta, \varkappa(\vartheta)) - g(\vartheta, y(\vartheta))] \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, y(v)) dv \\ &\quad + g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} [\mathcal{F}(v, \varkappa(v)) - \mathcal{F}(v, y(v))] dv. \end{aligned}$$

Then,

$$\begin{aligned} |(\Pi\varkappa)(\vartheta) - (\Pi y)(\vartheta)| &\leq |g(\vartheta, \varkappa(\vartheta)) - g(\vartheta, y(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, y(v))| dv \\ &\quad + |g(\vartheta, \varkappa(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \varkappa(v)) - \mathcal{F}(v, y(v))| dv \\ &\leq \frac{\hbar_1\mu_2}{\Gamma(q+1)} (\zeta(\vartheta))^q |\varkappa(\vartheta) - y(\vartheta)| + \hbar_2\mu_1 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\varkappa(v) - y(v)| dv \\ &\leq \frac{\hbar_1\mu_2}{\Gamma(q+1)} \|\varkappa - y\| + \hbar_2\mu_1 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \|\varkappa(v) - y(v)\| dv, \end{aligned}$$

which implies $\square$

$$\begin{aligned} \|(\Pi\varkappa)(\vartheta) - (\Pi y)(\vartheta)\| &= \sup_{\vartheta \in \mathcal{J}} |(\Pi\varkappa)(\vartheta) - (\Pi y)(\vartheta)| \\ &\leq \frac{\hbar_1\mu_2}{\Gamma(q+1)} \|\varkappa - y\| + \hbar_2\mu_1 \|\varkappa - y\| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \\ &\leq \frac{\hbar_1\mu_2}{\Gamma(q+1)} \|\varkappa - y\| + \frac{\hbar_2\mu_1}{\Gamma(q+1)} \|\varkappa - y\| \\ &= Y \|\varkappa - y\|. \end{aligned}$$

Since $Y < 1$, the operator Π is a contraction mapping. Hence, as a consequence of Banach's fixed point theorem, the FQIE (1) has a unique solution $\varkappa \in C(\mathcal{J})$. This complete the proof.

3.2. Picard Method (PM)

By applying the PM to the FQEI (1), the solution is framed by the sequence

$$\begin{cases} \varkappa_n(\vartheta) = h(\vartheta) + g(\vartheta, \varkappa_{n-1}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-1}(v)) dv, & n = 1, 2, \dots \\ \varkappa_0(\vartheta) = h(\vartheta) \end{cases} \quad (2)$$

The functions $\varkappa_n$ can be written as

$$\varkappa_n = \varkappa_0 + \sum_{j=1}^n [\varkappa_j - \varkappa_{j-1}],$$

where the functions $\{\varkappa_n(\vartheta)\}_{n \geq 1}$ are continuous.

If the infinite series $\sum [\varkappa_j - \varkappa_{j-1}]$ converges, then the sequence functions $\varkappa_n(\vartheta)$ will converge to $\varkappa(\vartheta)$. Consequently, the solution will be

$$\varkappa(\vartheta) = \lim_{n \rightarrow \infty} \varkappa_n(\vartheta).$$

Now, we show that $\{\varkappa_n(\vartheta)\}_{n \geq 1}$ has uniform convergence. Consider the infinite series

$$\sum_{n=1}^{\infty} [\varkappa_n(\vartheta) - \varkappa_{n-1}(\vartheta)].$$

From (2) for $n = 1$, we achieve

$$\varkappa_1(\vartheta) - \varkappa_0(\vartheta) = g(\vartheta, \varkappa_0(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_0(v)) dv.$$

Consequently,

$$|\varkappa_1(\vartheta) - \varkappa_0(\vartheta)| \leq \mu_1 \mu_2 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \leq \frac{\mu_1 \mu_2}{\Gamma(q+1)} (\zeta(\vartheta))^q. \quad (3)$$

Here, we find the expression $\varkappa_n(\vartheta) - \varkappa_{n-1}(\vartheta)$, for $n \geq 2$ as

$$\begin{aligned} & \varkappa_n(\vartheta) - \varkappa_{n-1}(\vartheta) \\ = & g(\vartheta, \varkappa_{n-1}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-1}(v)) dv \\ & - g(\vartheta, \varkappa_{n-2}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-2}(v)) dv \\ & + g(\vartheta, \varkappa_{n-1}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-2}(v)) dv \\ & - g(\vartheta, \varkappa_{n-1}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-2}(v)) dv \\ = & g(\vartheta, \varkappa_{n-1}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} [\mathcal{F}(v, \varkappa_{n-1}(v)) - \mathcal{F}(v, \varkappa_{n-2}(v))] dv \\ & + [g(\vartheta, \varkappa_{n-1}(\vartheta)) - g(\vartheta, \varkappa_{n-2}(\vartheta))] \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-2}(v)) dv. \end{aligned}$$

Using Hypotheses (2) and (3), we attain

$$\begin{aligned}
 & |\varkappa_n(\vartheta) - \varkappa_{n-1}(\vartheta)| \\
 \leq & |g(\vartheta, \varkappa_{n-1}(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \varkappa_{n-1}(v)) - \mathcal{F}(v, \varkappa_{n-2}(v))| dv \\
 & + |g(\vartheta, \varkappa_{n-1}(\vartheta)) - g(\vartheta, \varkappa_{n-2}(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \varkappa_{n-2}(v))| dv \\
 \leq & \hbar_2 \mu_1 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\varkappa_{n-1}(v) - \varkappa_{n-2}(v)| dv \\
 & + \hbar_1 \mu_2 |\varkappa_{n-1}(\vartheta) - \varkappa_{n-2}(\vartheta)| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv.
 \end{aligned}$$

By taking $n = 2$, and using (3), we obtain

$$\begin{aligned}
 |\varkappa_2(\vartheta) - \varkappa_1(\vartheta)| & \leq \hbar_2 \mu_1 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\varkappa_1(v) - \varkappa_0(v)| dv \\
 & + \hbar_1 \mu_2 |\varkappa_1(\vartheta) - \varkappa_0(\vartheta)| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \\
 & \leq \frac{\hbar_2 \mu_1^2 \mu_2}{\Gamma(q+1)} \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} (\zeta(v))^q dv \\
 & + \frac{\hbar_1 \mu_2^2 \mu_1}{\Gamma(q+1)} (\zeta(\vartheta))^q \frac{(\zeta(\vartheta))^q}{\Gamma(q+1)} \\
 & \leq \frac{\hbar_2 \mu_1^2 \mu_2}{\Gamma(q+1)} \frac{\Gamma(q+1)}{\Gamma(2q+1)} (\zeta(\vartheta))^{2q} \\
 & + \frac{\hbar_1 \mu_2^2 \mu_1}{\Gamma(q+1)\Gamma(q+1)} (\zeta(\vartheta))^{2q} \\
 & \leq \frac{\mu_1 \mu_2}{\Gamma(q+1)} \left[\hbar_2 \mu_1 \frac{\Gamma(q+1)}{\Gamma(2q+1)} + \frac{\hbar_1 \mu_2}{\Gamma(q+1)} \right] (\zeta(\vartheta))^{2q}.
 \end{aligned}$$

Similarly, for $n = 3$,

$$\begin{aligned}
 |\varkappa_3(\vartheta) - \varkappa_2(\vartheta)| & \leq \hbar_2 \mu_1 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\varkappa_2(v) - \varkappa_1(v)| dv \\
 & + \hbar_1 \mu_2 |\varkappa_2(\vartheta) - \varkappa_1(\vartheta)| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \\
 & \leq \frac{\mu_1 \mu_2}{\Gamma(q+1)} \left(\hbar_2 \mu_1 \frac{\Gamma(q+1)}{\Gamma(2q+1)} + \frac{\hbar_1 \mu_2}{\Gamma(q+1)} \right) \\
 & \times \left(\hbar_2 \mu_1 \frac{\Gamma(2q+1)}{\Gamma(3q+1)} + \frac{\hbar_1 \mu_2}{\Gamma(q+1)} \right) (\zeta(\vartheta))^{3q}.
 \end{aligned}$$

Repeating this process, we obtain

$$\begin{aligned}
 |\varkappa_n(\vartheta) - \varkappa_{n-1}(\vartheta)| &\leq \frac{\mu_1\mu_2}{\Gamma(q+1)} \left(\hbar_2\mu_1 \frac{\Gamma(q+1)}{\Gamma(2q+1)} + \frac{\hbar_1\mu_2}{\Gamma(q+1)} \right) \\
 &\quad \times \left(\hbar_2\mu_1 \frac{\Gamma(2q+1)}{\Gamma(3q+1)} + \frac{\hbar_1\mu_2}{\Gamma(q+1)} \right) \times \dots \\
 &\quad \times \left(\hbar_2\mu_1 \frac{\Gamma((n-1)q+1)}{\Gamma(nq+1)} + \frac{\hbar_1\mu_2}{\Gamma(q+1)} \right) (\zeta(\vartheta))^{nq} \\
 &\leq \frac{\mu_1\mu_2}{\Gamma(q+1)} \left(\hbar_2\mu_1 \frac{\Gamma(q+1)}{\Gamma(q+1)} + \frac{\hbar_1\mu_2}{\Gamma(q+1)} \right) \\
 &\quad \times \left(\hbar_2\mu_1 \frac{\Gamma(2q+1)}{\Gamma(q+1)} + \frac{\hbar_1\mu_2}{\Gamma(q+1)} \right) \times \dots \\
 &\quad \times \left(\hbar_2\mu_1 \frac{\Gamma((n-1)q+1)}{\Gamma((n-1)q+1)} + \frac{\hbar_1\mu_2}{\Gamma(q+1)} \right) \\
 &\leq \frac{\mu_1\mu_2}{\Gamma(q+1)} ((\hbar_2\mu_1 + \hbar_1\mu_2)) \times ((\hbar_2\mu_1 + \hbar_1\mu_2)) \times \dots \\
 &\quad \times ((\hbar_2\mu_1 + \hbar_1\mu_2)) \\
 &\leq \frac{\mu_1\mu_2}{\Gamma(q+1)} ((\hbar_2\mu_1 + \hbar_1\mu_2))^n.
 \end{aligned}$$

Since $\left(\frac{\hbar_1\mu_2 + \hbar_2\mu_1}{\Gamma(q+1)}\right) < 1$, then the series $\sum_{n=1}^{\infty} [\varkappa_n(\vartheta) - \varkappa_{n-1}(\vartheta)]$ and the sequence $\{\varkappa_n(\vartheta)\}$ are uniformly convergent.

Because $\mathcal{F}(\vartheta, \varkappa)$ and $g(\vartheta, \varkappa)$ are continuous in $\varkappa$, it follows that

$$\begin{aligned}
 \varkappa(\vartheta) &= \lim_{n \rightarrow \infty} g(\vartheta, \varkappa_n(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_n(v)) dv \\
 &= g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa(v)) dv.
 \end{aligned}$$

This shows the existence of a solution. Here, we need to show that this solution is unique; let $y(\vartheta)$ be a continuous solution of the FQEI (1) that is

$$y(\vartheta) = h(\vartheta) + g(\vartheta, y(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, y(v)) dv, \quad \vartheta \in [0, 1], \quad q > 0.$$

Hence,

$$\begin{aligned}
 y(\vartheta) - \varkappa_n(\vartheta) &= g(\vartheta, y(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, y(v)) dv \\
 &\quad - g(\vartheta, \varkappa_{n-1}(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-1}(v)) dv \\
 &\quad + g(\vartheta, y(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-1}(v)) dv \\
 &\quad - g(\vartheta, y(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-1}(v)) dv \\
 &= g(\vartheta, y(\vartheta)) \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} [\mathcal{F}(v, y(v)) - \mathcal{F}(v, \varkappa_{n-1}(v))] dv \\
 &\quad + [g(\vartheta, y(\vartheta)) - g(\vartheta, \varkappa_{n-1}(\vartheta))] \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} \mathcal{F}(v, \varkappa_{n-1}(v)) dv.
 \end{aligned}$$

By using assumptions (2) and (3), we obtain

$$\begin{aligned}
|y(\vartheta) - \varkappa_n(\vartheta)| &\leq |g(\vartheta, y(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, y(v)) - \mathcal{F}(v, \varkappa_{n-1}(v))| dv \\
&\quad + |g(\vartheta, y(\vartheta)) - g(\vartheta, \varkappa_{n-1}(\vartheta))| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \varkappa_{n-1}(v))| dv \\
&\leq \hbar_2 \mu_1 \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |y(v) - \varkappa_{n-1}(v)| dv \\
&\quad + \hbar_1 \mu_2 |y(\vartheta) - \varkappa_{n-1}(\vartheta)| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv.
\end{aligned} \tag{4}$$

However, we have

$$|y(\vartheta) - h(\vartheta)| \leq \frac{\mu_1 \mu_2}{\Gamma(q+1)} \zeta(\vartheta)^q.$$

Hence, using (4), we obtain

$$|y(\vartheta) - \varkappa_n(\vartheta)| \leq \frac{\mu_1 \mu_2}{\Gamma(q+1)} [(\hbar_2 \mu_1 + \hbar_1 \mu_2)]^n.$$

Consequently,

$$\lim_{n \rightarrow \infty} \varkappa_n(\vartheta) = y(\vartheta) = \varkappa(\vartheta).$$

This ends the proof.

Corollary 1. Under the assumptions of Theorem 1, if $\zeta(\vartheta) = \vartheta^\rho$, then the FQEI (1) reduces to

$$\varkappa(\vartheta) = h(\vartheta) + g(\vartheta, \varkappa(\vartheta)) \int_0^\vartheta \frac{v^{\rho-1}}{\Gamma(q)} \left(\frac{\vartheta^\rho - v^\rho}{\rho} \right)^{q-1} \mathcal{F}(v, \varkappa(v)) dv,$$

which has a unique solution; see [29].

3.3. AD Method (ADM)

In this section, we will analyze ADM for the FQEI (1). The solution algorithm of the FQEI (1) by applying ADM is

$$\varkappa_0(\vartheta) = h(\vartheta), \tag{5}$$

$$\varkappa_\ell(\vartheta) = \omega_{(\ell-1)}(\vartheta) \mathbb{I}_{0+}^{q, \zeta} \omega_{(\ell-1)}(\vartheta), \tag{6}$$

where ω_ℓ and ω_ℓ are Adomian polynomials of the nonlinear terms $g(\vartheta, \varkappa)$ and $\mathcal{F}(v, \varkappa)$, respectively, which forms as follows:

$$\omega_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(g \left(\vartheta, \sum_{\ell=0}^{\infty} \lambda^\ell \varkappa_\ell \right) \right) \right]_{\lambda=0}, \tag{7}$$

$$\omega_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\mathcal{F} \left(\vartheta, \sum_{\ell=0}^{\infty} \lambda^\ell \varkappa_\ell \right) \right) \right]_{\lambda=0}. \tag{8}$$

Now, we will show the solution as

$$\varkappa(\vartheta) = \sum_{\ell=0}^{\infty} \varkappa_\ell. \tag{9}$$

3.4. Convergence Analysis

Theorem 2. Let $\varkappa(\vartheta)$ be a solution of the FQIE (1) and there exists a positive constant $\mathbb{M}$ satisfying $|\varkappa_1(\vartheta)| < \mathbb{M}$. Then, solution (9) of the FQIE (1) applying ADM converges.

Proof. Let $\{\mathbb{S}_{\varrho_1}\}$ be a sequence such that $\{\mathbb{S}_{\varrho_1}\} = \sum_{\ell=0}^{\varrho_1} \varkappa_\ell$ is a sequence of partial sums from the series (9) and we have

$$\begin{aligned} g(\vartheta, \varkappa) &= \sum_{\ell=0}^{\infty} \omega_\ell, \\ \mathcal{F}(\vartheta, \varkappa) &= \sum_{\ell=0}^{\infty} \omega_\ell. \end{aligned}$$

Set $\mathbb{S}_{\varrho_1}$ and let $\mathbb{S}_{\varrho_2}$ be two partial sums with $\varrho_1 > \varrho_2$. Now, we show that $\mathbb{S}_{\varrho_1}$ is a Cauchy sequence in $C(\mathcal{J})$.

$$\begin{aligned} \mathbb{S}_{\varrho_1} - \mathbb{S}_{\varrho_2} &= \sum_{\ell=0}^{\varrho_1} \varkappa_\ell - \sum_{\ell=0}^{\varrho_2} \varkappa_\ell \\ &= \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) - \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \right) \\ &= \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) - \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) \\ &\quad + \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) - \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \right) \\ &= \left[\sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) - \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \right] \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) \\ &\quad + \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \left[\sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) - \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \right] \right) \end{aligned}$$

However,

$$\begin{aligned} \|\mathbb{S}_{\varrho_1} - \mathbb{S}_{\varrho_2}\| &\leq \max_{\vartheta \in \mathcal{J}} \left| \sum_{\ell=\varrho_2+1}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) \right| \\ &\quad + \max_{\vartheta \in \mathcal{J}} \left| \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \left(\mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=\varrho_2+1}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right) \right| \\ &\leq \max_{\vartheta \in \mathcal{J}} \left| \sum_{\ell=\varrho_2}^{\varrho_1-1} \omega_\ell(\vartheta) \right| \left| \mathbb{I}_{0+}^{q;\zeta} \sum_{\ell=0}^{\varrho_1} \omega_{(\ell-1)}(\vartheta) \right| + \max_{\vartheta \in \mathcal{J}} \left| \sum_{\ell=0}^{\varrho_2} \omega_{(\ell-1)}(\vartheta) \right| \left| \sum_{\ell=\varrho_2}^{\varrho_1-1} \omega_\ell(\vartheta) \right| \\ &\leq \max_{\vartheta \in \mathcal{J}} |g(\vartheta, \mathbb{S}_{\varrho_1-1}) - g(\vartheta, \mathbb{S}_{\varrho_2-1})| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \mathbb{S}_{\varrho_1})| dv \\ &\quad + \max_{\vartheta \in \mathcal{J}} |g(\vartheta, \mathbb{S}_{\varrho_2})| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} |\mathcal{F}(v, \mathbb{S}_{\varrho_1-1}) - \mathcal{F}(v, \mathbb{S}_{\varrho_2-1})| dv \\ &\leq \hbar_1 \mu_2 \max_{\vartheta \in \mathcal{J}} |\mathbb{S}_{\varrho_1-1} - \mathbb{S}_{\varrho_2-1}| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \\ &\quad + \hbar_2 \mu_1 \max_{\vartheta \in \mathcal{J}} |\mathbb{S}_{\varrho_1-1} - \mathbb{S}_{\varrho_2-1}| \int_0^\vartheta \frac{\zeta'(v)}{\Gamma(q)} (\zeta(\vartheta) - \zeta(v))^{q-1} dv \\ &\leq \frac{1}{\Gamma(q+1)} [(\hbar_2 \mu_1 + \hbar_1 \mu_2)] \max_{\vartheta \in \mathcal{J}} |\mathbb{S}_{\varrho_1-1} - \mathbb{S}_{\varrho_2-1}| \\ &\leq Y \|\mathbb{S}_{\varrho_1-1} - \mathbb{S}_{\varrho_2-1}\|. \end{aligned}$$

Let $\varrho_1 = \varrho_2 + 1$; then,

$$\|\mathbb{S}_{\varrho_2+1} - \mathbb{S}_{\varrho_2}\| \leq Y \|\mathbb{S}_{\varrho_2} - \mathbb{S}_{\varrho_2-1}\| \leq Y^2 \|\mathbb{S}_{\varrho_2-1} - \mathbb{S}_{\varrho_2-2}\| \leq \dots \leq Y^{\varrho_2} \|\mathbb{S}_1 - \mathbb{S}_0\|.$$

In addition, we have

$$\begin{aligned}
 \|\mathbb{S}_{\varrho_1} - \mathbb{S}_{\varrho_2}\| &\leq \|\mathbb{S}_{\varrho_2+1} - \mathbb{S}_{\varrho_2}\| + \|\mathbb{S}_{\varrho_2+2} - \mathbb{S}_{\varrho_2+1}\| + \cdots + \|\mathbb{S}_{\varrho_1} - \mathbb{S}_{\varrho_1-1}\| \\
 &\leq \left[Y^{\varrho_2} + Y^{\varrho_2+1} + \cdots + Y^{\varrho_1-1} \right] \|\mathbb{S}_1 - \mathbb{S}_0\| \\
 &\leq Y^{\varrho_2} \left[1 + Y + \cdots + Y^{\varrho_1-\varrho_2-1} \right] \|\mathbb{S}_1 - \mathbb{S}_0\| \\
 &\leq Y^{\varrho_2} \left[\frac{1 - Y^{\varrho_1-\varrho_2}}{1 - Y} \right] \|\varkappa_1\|.
 \end{aligned}$$

The assumptions $0 < Y < 1$, and $\varrho_1 > \varrho_2$ lead to $(1 - Y^{\varrho_1-\varrho_2}) \leq 1$. Hence,

$$\begin{aligned}
 \|\mathbb{S}_{\varrho_1} - \mathbb{S}_{\varrho_2}\| &\leq \frac{Y^{\varrho_2}}{1 - Y} \|\varkappa_1\| \\
 &\leq \frac{Y^{\varrho_2}}{1 - Y} \max_{\vartheta \in \mathcal{J}} |\varkappa_1(\vartheta)|.
 \end{aligned}$$

However, $|\varkappa_1(\vartheta)| < \mathbb{M}$ and as $\varrho_2 \rightarrow \infty$, then $\|\mathbb{S}_{\varrho_1} - \mathbb{S}_{\varrho_2}\| \rightarrow 0$ and hence $\{\mathbb{S}_{\varrho_1}\}$ is a Cauchy sequence in $C(\mathcal{J})$, and the series $\sum_{\ell=0}^{\infty} \varkappa_{\ell}(\vartheta)$ converges. $\square$

4. Numerical Example

In this part, we will study numerical example via Picard and ADM methods.

Example 1. Consider the following nonlinear FQIE:

$$\varkappa(\vartheta) = \left(\vartheta^3 - \frac{104\vartheta^{\frac{17}{2}}}{750} \right) + \frac{1}{4} \varkappa(\vartheta) \mathbb{I}_{0+}^{\frac{1}{2}, \zeta} \varkappa^4(\vartheta). \quad (10)$$

Here, the $\varkappa(\vartheta) = \vartheta^3$ is the exact solution for (10).

Taking $\zeta = \frac{1}{2}$, and applying PM to (10), we obtain

$$\begin{aligned}
 \varkappa_n(\vartheta) &= \left(\vartheta^3 - \frac{104\vartheta^{\frac{17}{2}}}{750} \right) + \frac{1}{4} \varkappa_{n-1}(\vartheta) \mathbb{I}_{0+}^{\frac{1}{2}, \frac{1}{2}} \varkappa_{n-1}^4(\vartheta), \quad n = 1, 2, \dots, \\
 \varkappa_0(\vartheta) &= \left(\vartheta^3 - \frac{104\vartheta^{\frac{17}{2}}}{750} \right).
 \end{aligned}$$

and the solution will be in the form

$$\varkappa(\vartheta) = \varkappa_n(\vartheta).$$

Again, applying ADM to (10), we obtain

$$\varkappa_0(\vartheta) = \left(\vartheta^3 - \frac{104\vartheta^{\frac{17}{2}}}{750} \right),$$

$$\varkappa_i(\vartheta) = \frac{1}{4} \varkappa_{i-1}(\vartheta) \mathbb{I}_{0+}^{\frac{1}{2}, \frac{1}{2}} \omega_{i-1}(\vartheta), \quad i = 1, 2, \dots$$

where ω_i are Adomian polynomials of the nonlinear term $\varkappa^4$, and the solution will be

$$\varkappa(\vartheta) = \sum_{i=0}^p \varkappa_i(\vartheta).$$

5. Conclusions

In this article, we have considered the FQIE (1) in the frame of the generalized Riemann–Liouville fractional integral operator. First, the existence and uniqueness of solutions

for the proposed equations were obtained. Next, we have given some special results corresponding to suitable values of the parameters ζ and q . Moreover, the main results have been proven based on Banach's fixed point theorem, the Adomian decomposition method, and Picard's method. Finally, we have presented an example. The present results are new for some special cases. The proposed techniques can be extended to other ζ -Hilfer fractional quadratic integral equations [32].

Author Contributions: B.N.A.: Formal Analysis, Methodology, Visualization, Writing—original draft; S.S.R.: Supervision, Investigation, Review & Editing; O.B.: Formal Analysis of revised manuscript, Improve English writing for the revised manuscript, Supervision for revised manuscript; K.N.: Supervision for the revised manuscript, Investigation of the revised manuscript, Visualization, Editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
2. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Elsevier: Amsterdam, The Netherlands, 1998.
3. Redhwan, S.S.; Shaikh, S.L.; Abdo, M.S. Implicit fractional differential equation with anti-periodic boundary condition involving Caputo-Katugampola type. *Aims Math.* **2020**, *5*, 3714–3730. [\[CrossRef\]](#)
4. Redhwan, S.S.; Shaikh, S.L.; Abdo, M.S. Some properties of Sadik transform and its applications of fractional-order dynamical systems in control theory. *Adv. Theory Nonlinear Anal. Appl.* **2019**, *4*, 51–66. [\[CrossRef\]](#)
5. Redhwan, S.S.; Shaikh, S.L.; Abdo, M.S. coupled non-separated system of Hadamard-type fractional differential equations. *Adv. Theory Nonlinear Anal. Its Appl.* **2021**, *6*, 33–44.
6. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
7. Hilfer, R. Experimental evidence for fractional time evolution in glass forming materials. *J. Chem. Phys.* **2002**, *284*, 399–408. [\[CrossRef\]](#)
8. Sousa, J.V.D.C.; de Oliveira, E.C. A Gronwall inequality and the Cauchy-type problem by means of ζ -Hilfer operator. *arXiv* **2017**, arXiv:1709.03634.
9. Sousa, J.V.D.C.; De Oliveira, E.C. On the ζ -Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [\[CrossRef\]](#)
10. da Vanterler, C.; Sousa, J.; Kucche, K.D.; Capelas de Oliveira, E. On the Ulam-Hyers stabilities of the solutions of ζ -Hilfer fractional differential equation with abstract Volterra operator. *Math. Methods Appl. Sci.* **2019**, *42*, 3021–3032.
11. Luo, D.; Shah, K.; Luo, Z. On the Novel Ulam–Hyers Stability for a Class of Nonlinear ζ -Hilfer Fractional Differential Equation with Time-Varying Delays. *Mediterr. J. Math.* **2019**, *16*, 1–15. [\[CrossRef\]](#)
12. Almalahi, M.A.; Abdo, M.S.; Panchal, S.K. ζ -Hilfer fractional functional differential equation by Picard operator method. *J. Appl. Nonlinear Dyn* **2020**, *9*, 685–702. [\[CrossRef\]](#)
13. Redhwan, S.S.; Shaikh, S.L.; Abdo, M.S.; Shatanawi, W.; Abodayeh, K.; Almalahi, M.A.; Aljaaidi, T. Investigating a generalized Hilfer-type fractional differential equation with two-point and integral boundary conditions. *AIMS Math.* **2022**, *7*, 1856–1872. [\[CrossRef\]](#)
14. Asawasamrit, S.; Kijjathanakorn, A.; Ntouyas, S.K.; Tariboon, J. Nonlocal boundary value problems for Hilfer fractional differential equations. *Bull. Korean Math. Soc.* **2018**, *55*, 1639–1657.
15. Mali, A.D.; Kucche, K.D. Nonlocal boundary value problem for generalized Hilfer implicit fractional differential equations. *Math. Methods Appl. Sci.* **2020**, *43*, 8608–8631. [\[CrossRef\]](#)
16. Ntouyas, S.K.; Vivek, D. Existence and uniqueness results for sequential ζ -Hilfer fractional differential equations with multi-point boundary conditions. *Acta Math. Univ.* **2021**, *90*, 171–185.
17. Adomian, G. *Stochastic System*; Academic Press: New York, NY, USA, 1983.
18. Adomian, G. *Nonlinear Stochastic System Theory and Applications to Physics*; Springer Dordrecht: New York, NY, USA, 1989.
19. Cherruault, Y. Convergence of Adomian's Method. *Kybernetes* **1989**, *18*, 31–38. [\[CrossRef\]](#)
20. Rach, R. On the Adomian (decomposition) method and comparisons with Picard's method. *J. Math. Anal. Appl.* **1987**, *128*, 480–483. [\[CrossRef\]](#)

21. Curtain, R.F.; Pritchard, A.J. *Functional Analysis in Modern Applied Mathematics*; Academic Press: Cambridge, MA, USA, 1977.
22. Bellomo, N.; Sarafyan, D. On Adomian's decomposition method and some comparisons with Picard's iterative scheme. *J. Math. Anal. Appl.* **1987**, *123*, 389–400. [[CrossRef](#)]
23. El-Sayed, A.M.A.; Hashem, H.H.G.; Ziada, E.A.A. Picard and Adomian methods for quadratic integral equation. *Comput. Appl. Math.* **2010**, *29*, 447–463. [[CrossRef](#)]
24. Banaś, J.; Lecko, M.; El-Sayed, W. G. Existence theorems for some quadratic integral equations. *J. Math. Anal. Appl.* **1998**, *222*, 276–285. [[CrossRef](#)]
25. El-Sayed, A.M.A.; Saleh, M.M.; Ziada, E.A.A. Numerical and analytic solution for nonlinear quadratic integral equations. *Math. Sci. Res. J.* **2008**, *12*, 183–191.
26. El-Sayed, A.; Hashem, H.H.G. Integrable and continuous solutions of a nonlinear quadratic integral equation. *Electron. Qual. Theory Differ. Equ.* **2008**, *25*, 1–10. [[CrossRef](#)]
27. Mohamed Abdalla, D. On quadratic integral equation of fractional orders. *J. Math. Anal. Appl.* **2005**, *311*, 112–119.
28. Mohamed, D.; Henderson, J. Existence and asymptotic stability of solutions of a perturbed quadratic fractional integral equation. *Fract. Calc. Appl. Anal.* **2009**, *12*, 71–86.
29. Abood, B.N.; Redhwan, S.S.; Abdo, M.S. Analytical and Approximate solutions for a generalized fractional quadratic integral equation. *Nonlinear Funct. Anal. Appl.* **2021**, *26*, 497–512.
30. Almeida R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [[CrossRef](#)]
31. Burton, T. A.; Kirk, C. A fixed point theorem of Krasnoselskii Schaefer type. *Math. Nachrichten* **1998**, *189*, 23–31. [[CrossRef](#)]
32. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: Hoboken, NJ, USA, 1993.