fractal and fractional

Article

New Explicit Solutions of the Extended Double (2+1)-Dimensional Sine-Gorden Equation and Its Time Fractional Form

Gangwei Wang [1,*,†], Li Li [2], Qi Wang [1,†] and Juan Geng [1]

[1] School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China; wangqi80617@163.com (Q.W.); stgengjuan@heuet.edu.cn (J.G.)
[2] Library of Hebei University of Economics and Business, Shijiazhuang 050061, China; lili@heuet.edu.cn
* Correspondence: wanggangwei@heuet.edu.cn
† These authors contributed equally to this work.

Abstract: In this paper, the extended double (2+1)-dimensional sine-Gorden equation is studied. First of all, using the symmetry method, the corresponding vector fields, Lie algebra and infinitesimal generators are derived. Then, from infinitesimal generators, the symmetry reductions are presented. In addition, these reduced equations are converted into the corresponding partial differential equations, which including classical double (1+1)-dimensional sine-Gorden equation. Moreover, based on the Lie symmetry method again, these reduced equations are investigated. Meanwhile, based on traveling wave transformation, some explicit solutions of the extended double (2+1)-dimensional sine-Gorden equation are obtained. Consequently, a conservation law is derived via conservation law multiplier method. Finally, especially with the help of the fractional complex transform, some solutions of double time fractional (2+1)-dimensional sine-Gorden equation are also derived. These results might explain complex nonlinear phenomenon.

Keywords: extended double (2+1)-dimensional sine-Gorden equation; time fractional form; Lie symmetry; symmetry reductions; explicit solutions; conservation laws

Citation: Wang, G.; Li, L.; Wang, Q.; Geng, J. New Explicit Solutions of the Extended Double (2+1)-Dimensional Sine-GORDEN Equation and Its Time Fractional Form. *Fractal Fract.* **2022**, *6*, 166. https://doi.org/10.3390/fractalfract6030166

Academic Editors: Adem Kilicman and Haci Mehmet Baskonus

Received: 5 February 2022
Accepted: 15 March 2022
Published: 17 March 2022

1. Introduction

Recently, Wang, from extended Lax pairs, derived a (2+1)-dimensional sine-Gorden Equation [1],

$$u_{xx} - u_{xy} - u_{xt} + u_{yt} = \sin u, \tag{1}$$

they studied the kink wave and anti-kink wave solutions, also derived conservation laws. Based on the results of ref. [1], this paper focuses on the following extended double (2+1)-dimensional sine-Gorden equation:

$$u_{xx} - u_{xy} - u_{xt} + u_{yt} = \sin u + a \sin 2u, \tag{2}$$

and time fractional form as follows:

$$u_{xx} - u_{xy} - (u_t^{\alpha})_x + (u_t^{\alpha})_y = \sin u + a \sin 2u, \tag{3}$$

where $D_t^{\alpha}(\cdot)$ is the modified Riemann–Liouville derivative [2–4], $0 < \alpha \leq 1$, a is a constant. Some preliminaries of the modified Riemann–Liouville derivative refer to [2–4]. If $a = 0$, Equation (2) reduced to Equation (1). Furthermore, it can be seen that if $\alpha = 1$, Equation (3) becomes Equation (2). This paper extends our previous work in [1] to study the extended double (2+1)-dimensional sine-Gorden equation. The sine-Gorden-type equation appears in many science fields [5,6], such as nonlinear optics, quantum field theory, differential geometry, solid state physics and so on. The author [7] derived multiple optical kink wave solutions of sine-Gordon-type equations. In paper [8], they got some exact solitary wave solutions for sine-Gordon-type equations using an auxiliary ordinary differential equation method. In [9], they considered initial value problem for the sine-Gordon equation, this equation is solved by the inverse-scattering method. The author, in paper [10], using tanh method and a variable separated ODE method, solved double sine-Gordon equation, and some exact solutions are derived.

There are many methods to study the nonlinear evolution equations, such as the Lie symmetry method [11–22], Hamiltonian system [23–26], Hirota's bilinear direct method [27], B*ä*cklund transformation [28,29], inverse scattering transformation [6], Darboux transformations [30], Lax pairs [31,32] and so on. On the other hand, it is well known that fractional order differential equations are well suited to characterize materials and processes with memory and genetic properties, and their description of complex systems has the advantages of simple modeling, clear physical meaning of parameters, and accurate description, thus becoming one of the important tools for mathematical modeling of complex mechanics and physical processes [33–36]. With the development of science and technology, most of the problems in real-life natural phenomena such as optical and thermal systems, rheological and material and mechanical systems, signal processing and system identification, control and robotics, and other applications can be described by fractional order differential equations [37–42]. Therefore, the research on fractional order differential equations has also received extensive attention from more and more authors, especially the fractional order differential equations abstracted from practical problems have become a hot research topic for many mathematicians. As fractional order differential equations appear in more and more scientific fields, the study of both theoretical analysis and numerical computation of fractional order differential equations is particularly urgent [37–42]. This is because it can better explain the complex natural phenomena.

Lie symmetry method provides a powerful and fundamental framework to the investigation of differential equations, it can link between different differential equations, it also can construct conservation laws for differential equations. As this equation is obtained from the extended Lax pair and this equation can be reduced to the classical sine-Gorden equation, it is necessary to study this equation to provide stronger theoretical support for solving practical problems. The current paper is divided into the following main sections, in Section 2, the extended double (2+1)-dimensional sine-Gorden equation is studied using symmetry method. In Section 3, symmetry reductions and analytical solutions of the extended double (2+1)-dimensional sine-Gorden equation are presented. A conservation law is given by Section 4. Explicit solutions of the extended double time fractional (2+1)-dimensional sine-Gorden equation are obtained in Section 5. The last Section is the conclusion of this paper.

2. Symmetry Analysis of the Extended Double (2+1)-Dimensional Sine-Gorden Equation (2)

Given the following vector fields [11–20]

$$V = \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \eta(x,y,t,u)\frac{\partial}{\partial u}, \tag{4}$$

where

$$\begin{cases} t^* = t + \epsilon\xi^3(x,y,t,u) + O(\epsilon^2),\ x^* = x + \epsilon\xi^1(x,y,t,u) + O(\epsilon^2), \\ y^* = y + \epsilon\xi^2(x,y,t,u) + O(\epsilon^2),\ u^* = u + \epsilon\eta(x,y,t,u) + O(\epsilon^2). \end{cases} \tag{5}$$

Consider the connection of vector field (4) and symmetry of the extended double (2+1)-dimensional sine-Gorden equation, V needs to satisfy Lie's symmetry condition

$$pr^{(2)}V(\Delta_1)|_{\Delta_1=0} = 0, \tag{6}$$

where $\Delta_1 = u_{xx} - u_{xy} - u_{xt} + u_{yt} - \sin u - a\sin 2u$.

Given the second prolongation

$$pr^{(2)}V = \begin{cases} V + \eta^x(x,y,t,u)\frac{\partial}{\partial u_x} + \eta^y(x,y,t,u)\frac{\partial}{\partial u_y} + \eta^t(x,y,t,u)\frac{\partial}{\partial u_t} + \eta^{xt}(x,y,t,u)\frac{\partial}{\partial u_{xt}} \\ +\eta^{xy}(x,y,t,u)\frac{\partial}{\partial u_{xy}} + \eta^{xt}(x,y,t,u)\frac{\partial}{\partial u_{xt}} + \eta^{yt}(x,y,t,u)\frac{\partial}{\partial u_{yt}}, \end{cases} \tag{7}$$

$pr^{(2)}V$ to the (2), the following infinitesimal criterion reads as

$$\eta^{xx} - \eta^{xy} - \eta^{xt} + \eta^{yt} - \eta\cos u - 2a\cos 2u\eta = 0, \tag{8}$$

where

$$\begin{aligned} \eta^t &= D_t(\eta) - u_x D_t(\xi^1) - u_y D_t(\xi^2) - u_t D_t(\xi^3), \\ \eta^x &= D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2) - u_t D_x(\xi^3), \\ \eta^y &= D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2) - u_t D_y(\xi^3), \\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\xi^3) - u_{xx} D_x(\xi^1) - u_{xy} D_x(\xi^2), \\ \eta^{yy} &= D_y(\eta^y) - u_{yt} D_y(\xi^3) - u_{xy} D_y(\xi^1) - u_{yy} D_y(\xi^2), \end{aligned} \tag{9}$$

and so on and here D_i is the following total derivative operator

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots \qquad i = 1,2,3, \tag{10}$$

and $(x^1, x^2, x^3) = (t, x, y)$.

After calculation, the following results are derived:

$$\begin{aligned} &\eta = 0, \xi^3 = (x+y)F_2(x+t+y) + F_3(x+t+y), \\ &\xi^1 = (-x-2y)F_2(x+t+y) + F_4(x+t+y), \\ &\xi^2 = F_1(x+t+y) + yF_2(x+t+y), \end{aligned} \tag{11}$$

where $F_i (i = 1,2,3,4)$ are arbitrary functions of $x + t + y$. Therefore, from Equation (4), the following infinitesimal generators are obtained:

$$V = V_1 + V_2 + V_3 + V_4, \tag{12}$$

where

$$\begin{aligned} V_1 &= F_4(x+t+y)\frac{\partial}{\partial x}, \\ V_2 &= F_1(x+t+y)\frac{\partial}{\partial y}, \\ V_3 &= F_3(x+t+y)\frac{\partial}{\partial t}, \\ V_4 &= (x+y)F_2(x+t+y)\frac{\partial}{\partial t} + (-x-2y)F_2(x+t+y)\frac{\partial}{\partial x} + yF_2(x+t+y)\frac{\partial}{\partial y}. \end{aligned} \tag{13}$$

It is clear that if the appropriate functions for F_1, F_2, F_3, and F_4 are selected, an infinite number of symmetries are presented. Obviously, there are basic infinitesimal generators

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial x}, \\
V_2 &= \frac{\partial}{\partial y}, \\
V_3 &= \frac{\partial}{\partial t}, \\
V_4 &= (x+y)\frac{\partial}{\partial t} + (-x-2y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.
\end{aligned} \tag{14}$$

Their Lie algebra and commutative relations are given by

$$\begin{aligned}
&[V_1, V_1] = 0, [V_2, V_2] = 0, [V_3, V_3] = 0, [V_4, V_4] = 0, \\
&[V_1, V_2] = -[V_2, V_1] = 0, [V_1, V_3] = -[V_3, V_1] = 0, \\
&[V_2, V_3] = -[V_3, V_2] = 0, [V_1, V_4] = -[V_4, V_1] = V_3 - V_1, \\
&[V_2, V_4] = -[V_4, V_2] = V_3 - 2V_1 + V_2, \\
&[V_3, V_4] = -[V_4, V_3] = 0.
\end{aligned} \tag{15}$$

Based on the Lie group method, if $u = f(x, y, t)$ is a solution of the extended double (2+1)-dimensional sine-Gorden equation, so are

$$\begin{aligned}
u^1 &= f(x-\varepsilon, y, t), \\
u^2 &= f(x, y-\varepsilon, y), \\
u^3 &= f(x, y, t-\varepsilon),
\end{aligned} \tag{16}$$

and so on. In next section, some analytical solutions will be investigated.

3. Symmetry Reductions and Analytical Solutions of the Extended Double (2+1)-Dimensional Sine-Gorden Equation (2)

3.1. Symmetry Reductions

3.1.1. V_1

For generator V_1, the invariant function is $u = f(\tau, \eta)$, where $\tau = t, \eta = y$ are invariants; therefore, the following partial differential equation (PDEs) are derived:

$$u_{\tau\eta} = \sin u + a \sin 2u. \tag{17}$$

It needs to be emphasized that this equation is just famous double (1+1)-dimensional sine-Gorden equation.

3.1.2. V_2

For vector V_2, the invariant function is $u = f(\xi, \tau)$ and invariants are $\tau = t, \xi = x$. Thus, they generate following PDE

$$-u_{\tau\xi} = \sin u + a \sin 2u. \tag{18}$$

3.1.3. V_3

Considering V_3, the invariant function and invariants are $u = f(\xi, \eta)$ and $\xi = x, \eta = y$, respectively. For this case, following PDE is presented

$$u_{\xi\xi} - u_{\xi\eta} = \sin u + a \sin 2u. \tag{19}$$

3.1.4. Traveling Wave Transformation

Given $u = f(\xi)$ and $\xi = x + ky - ct$, one can get the following PDE

$$(1 - kc - k + c)u_{\xi\xi} = \sin u + a \sin 2u. \tag{20}$$

From the above analysis, it is found that (17)–(19) are PDEs and (20) is ordinary differential equation(ODE). Of course, (17)–(19) also can reduced to ODE via traveling wave transformation. Now, as (17)–(19) still are PDEs, these equations are studied again using again Lie symmetry method.

3.2. Symmetry Analysis of Reduced Equations

Base on the previous steps, we consider (17)–(19) using the Lie symmetry method again.

3.2.1. Lie Symmetry of Equations (17) and (18)

For these PDEs, one can obtain the infinitesimal generators as follows:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}. \end{aligned} \tag{21}$$

3.2.2. Lie Symmetry of Equation (19)

For this PDE, the following infinitesimal generators are derived

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= (-2t - x)\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}. \end{aligned} \tag{22}$$

Of course, these equations can continue to be handled in the same way and with the same steps as before. For the sake of simplicity, they are not listed in detail.

3.3. Analytical Solutions of the Extended Double (2+1)-Dimensional Sine-Gorden Equation (2)

In previous section, a ODE is derived via traveling wave transformation. Now, one can study this ODE (20).

For the extended double sine-Gordon equation, using the traveling wave transformation $\xi = x + ky - ct$

$$u_{xx} - u_{xy} - u_{xt} + u_{yt} = \sin u + a \sin 2u, \tag{23}$$

that can be converted to the following ODE

$$(1 - kc - k + c)u_{\xi\xi} = \sin u + a \sin 2u, \tag{24}$$

or equivalently form

$$u'' = \frac{1}{1 - kc - k + c}\sin(u) + \frac{a}{1 - kc - k + c}\sin(2u). \tag{25}$$

From the above analysis, it can be seen that in order to guarantee the existence of solutions, it is required that k is not equal to 1. Assuming that $u(\xi)$ satisfies the following ODE:

$$u' = a_0 + \sum_{j=1}^{n} (a_j \cos u + b_j \sin u), \tag{26}$$

where $a_j (j = 0, 1, 2, \cdots)$ and $b_j (j = 1, 2, \cdots)$ are constants need to be fixed, also the positive integer n can be fixed via the leading-order analysis method. In this way, one can get $n = 1$, thus we have

$$u' = a_0 + a_1 \cos u + b_1 \sin u, \tag{27}$$

where a_1 and b_1 are constants that will be determined. Differentiating (27) with respect to ξ gives

$$\begin{aligned} u''(\xi) &= -a_1 \sin u u' + b_1 \cos u u' \\ &= -a_1 \sin u(a_0 + a_1 \cos u + b_1 \sin u) + b_1 \cos u(a_0 + a_1 \cos u + b_1 \sin u). \end{aligned} \tag{28}$$

Comparing (28) with (25) it can be found that b_1 has to equal to zero. Therefore, one can obtain

$$-a_0 a_1 = \frac{1}{1 - kc - k + c}, -\frac{a_1^2}{2} = \frac{a}{1 - kc - k + c}, \tag{29}$$

that is to say,

$$a_0 = \sqrt{\frac{1}{2a(kc + k - c - 1)}}, a_1 = \sqrt{\frac{2a}{kc + k - c - 1}}. \tag{30}$$

It can be seen that Equation (27) is separable, hence one can have

$$\frac{1}{a_0 + a_1 \cos(u)} du = d\xi, \tag{31}$$

where by integrating both sides, the following solutions [8] are derived

$$u_{1,2} = \begin{cases} 2 \arctan\left[\pm\sqrt{\frac{a_1 + a_0}{a_1 - a_0}} \tan \frac{\sqrt{a_1^2 - a_0^2}}{2}(\xi + \xi_0)\right] \\ 2 \arctan\left[\pm\sqrt{\frac{a_1 + a_0}{a_1 - a_0}} \cot \frac{\sqrt{a_1^2 - a_0^2}}{2}(\xi + \xi_0)\right] \end{cases} \text{for } a_0^2 > a_1^2, \tag{32}$$

$$u_{3,4} = \begin{cases} 2 \arctan\left[\pm\sqrt{\frac{a_1 + a_0}{a_1 - a_0}} \tanh \frac{\sqrt{a_1^2 - a_0^2}}{2}(\xi + \xi_0)\right] \\ 2 \arctan\left[\pm\sqrt{\frac{a_1 + a_0}{a_1 - a_0}} \coth \frac{\sqrt{a_1^2 - a_0^2}}{2}(\xi + \xi_0)\right] \end{cases} \text{for } a_0^2 < a_1^2, \tag{33}$$

$$\begin{aligned} u_5 &= 2 \arctan a_0(\xi + \xi_0) \text{ for } a_1 = a_0, \\ u_6 &= -2 \operatorname{arccot} a_0(\xi + \xi_0) \text{ for } a_1 = -a_0, \end{aligned} \tag{34}$$

where ξ_0 is the constant of integration. These solutions are solitary wave solutions, they have obvious physical implications, that is to say, the wave form does not change with time in the propagation process, and outside the zone of interaction (the collision zone), the solitary wave retains its shape and velocity. Therefore, in turn, it can be derived that the following analytical solutions

$$u_{1,2} = \begin{cases} 2\arctan\left[\pm\sqrt{\frac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\tan\right. \\ \left.\sqrt{\frac{4a^2-1}{8a(kc+k-c-1)}}(x+ky-ct+\xi_0)\right] \\ 2\arctan\left[\pm\sqrt{\frac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\cot\right. \\ \left.\sqrt{\frac{4a^2-1}{8a(kc+k-c-1)}}(x+ky-ct+\xi_0)\right] \end{cases} \tag{35}$$

for $a_0^2 > a_1^2$,

$$u_{3,4} = \begin{cases} 2\arctan\left[\pm\sqrt{\frac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\tanh\right. \\ \left.\sqrt{\frac{4a^2-1}{8a(kc+k-c-1)}}(x+ky-ct+\xi_0)\right] \\ 2\arctan\left[\pm\sqrt{\frac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\coth\right. \\ \left.\sqrt{\frac{4a^2-1}{8a(kc+k-c-1)}}(x+ky-ct+\xi_0)\right] \end{cases} \tag{36}$$

for $a_0^2 < a_1^2$,

$$\begin{aligned} u_5 &= 2\arctan\sqrt{\frac{1}{2a(kc+k-c-1)}}(x+ky-ct+\xi_0) \text{ for } a_1 = a_0, \\ u_6 &= -2\,\mathrm{arccot}\sqrt{\frac{1}{2a(kc+k-c-1)}}(x+ky-ct+\xi_0) \text{ for } a_1 = -a_0. \end{aligned} \tag{37}$$

4. Conservation Laws of Equation (2)

A conservation law express the following form:

$$D_tT + D_xX + D_yY = 0. \tag{38}$$

By using conservation law multiplier method [13], let

$$R[u] \triangleq u_{xx} - u_{xy} - u_{xt} + u_{yt} - \sin u - a\sin 2u = 0. \tag{39}$$

Therefore, one can get

$$D_i\Phi^i[u] \triangleq D_1\Phi^1[u] + D_2\Phi^2[u] + D_3\Phi^3[u] = 0, \tag{40}$$

the total derivative operators satisfy solutions of Equation (39). Solving $\Lambda[u]R[u] \equiv D_i\Phi^i[u]$, for every u in Equation (39), one should get a multiplier $\Lambda[u] = \Lambda(x, y, t, u)$.

Theorem 1 ([13])**.** *Consider the divergence expression $D_i\Phi^i[u]$, one can have*

$$E_u\left(D_i\Phi^i[u]\right) \equiv 0, \tag{41}$$

where E_u is the Euler operator given by

$$E_u = \frac{\partial}{\partial u} - D_i\frac{\partial}{\partial u} + \cdots + (-1)^s D_{i_1}\cdots D_{i_s}\frac{\theta}{\partial u_{i1}\cdots i_s} + \cdots \tag{42}$$

Theorem 2 ([13])**.** *A divergence expression for Equation (39) can be derived by a conservation law multiplier $\Lambda(x, y, t, u)$ if and only if*

$$E_u(\Lambda(x, y, t, u)R[u]) \equiv 0, \tag{43}$$

holds for every u in Equation (39).

Theorem 3. *Consider second order multiplier $\{\Lambda\} = \{x, t, y, u_x, u_{xx}\}$, and arbitrary constant a, Equation (39) has a local conservation law*

$$\begin{aligned}
T &= \left(-\frac{1}{2}u_x^2 + \frac{1}{2}u_x u_y\right), \\
X &= \left(\frac{1}{2}uu_{ty} - \frac{1}{2}uu_{xy} + \frac{1}{2}u_x^2 - \frac{1}{2}u_x u_y + a(\cos u)^2 + \cos u - a - 1\right), \\
Y &= \left(-\frac{1}{2}uu_{tx} + \frac{1}{2}uu_{xx}\right),
\end{aligned} \tag{44}$$

in other words,

$$\begin{aligned}
&D_t\left(-\frac{1}{2}u_x^2 + \frac{1}{2}u_x u_y\right) + D_y\left(-\frac{1}{2}uu_{tx} + \frac{1}{2}uu_{xx}\right) \\
&+ D_x\left(\frac{1}{2}uu_{ty} - \frac{1}{2}uu_{xy} + \frac{1}{2}u_x^2 - \frac{1}{2}u_x u_y + a(\cos u)^2 + \cos u - a - 1\right) = 0.
\end{aligned} \tag{45}$$

Proof. Consider Equation (43), one gets

$$\Lambda_t = \Lambda_x = \Lambda_y = \Lambda_u = \Lambda_{u_{xx}} = 0, u_x\Lambda_{u_x} = \Lambda.$$

Solving these equations, one can obtain

$$\Lambda = c_1 u_x,$$

where c_1 is arbitrary constant. Thus, it shows that Theorem 3 holds. □

5. Analytical Solutions of the Extended Double Time Fractional (2+1)-Dimensional Sine-Gorden Equation (3)

For the extended double time fractional (2+1)-dimensional sine-Gorden Equation (3), using the fractional complex transform [3,4] $\xi = x + ky - c\frac{t^\alpha}{\Gamma(1+\alpha)}$, and $u(x, y, t) = u(\xi)$, substitute them into Equation (3),

$$u_{xx} - u_{xy} - (u_t^\alpha)_x + (u_t^\alpha)_y = \sin u + a\sin 2u, \tag{46}$$

should be converted to the following ODE as follows

$$(1-kc-k+c)u_{\xi\xi}=\sin u+a\sin 2u. \tag{47}$$

If we assume $\xi=k_1x+k_2y-c\frac{t^\alpha}{\Gamma(1+\alpha)}$, this requirement k_1 is not equal to k_2. Without loss in generality, we assume $k_1=1$. It can be found that k also cannot be equal to 1 in order to guarantee the existence of the solution. This means that after fractional complex transform, it also becomes the same ordinary differential equation as Equation (24). If α is equal to 1, it is the classical traveling wave transform. In this way, using the results obtained above, one can directly obtain solutions of the fractional order differential equation as described below

$$u_{1,2}=\begin{cases}2\arctan\left[\pm\sqrt{\dfrac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\tan\sqrt{\dfrac{4a^2-1}{8a(kc+k-c-1)}}\right.\\ \left.\left(x+ky-c\dfrac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)\right]\\ 2\arctan\left[\pm\sqrt{\dfrac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\cot\sqrt{\dfrac{4a^2-1}{8a(kc+k-c-1)}}\right.\\ \left.\left(x+ky-c\dfrac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)\right]\end{cases} \tag{48}$$

for $a_0^2>a_1^2$,

$$u_{3,4}=\begin{cases}2\arctan\left[\pm\sqrt{\dfrac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\tanh\sqrt{\dfrac{4a^2-1}{8a(kc+k-c-1)}}\right.\\ \left.\left(x+ky-c\dfrac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)\right]\\ 2\arctan\left[\pm\sqrt{\dfrac{\sqrt{\frac{2a}{kc+k-c-1}}+\sqrt{\frac{1}{2a(kc+k-c-1)}}}{\sqrt{\frac{2a}{kc+k-c-1}}-\sqrt{\frac{1}{2a(kc+k-c-1)}}}}\coth\sqrt{\dfrac{4a^2-1}{8a(kc+k-c-1)}}\right.\\ \left.\left(x+ky-c\dfrac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)\right]\end{cases} \tag{49}$$

for $a_0^2<a_1^2$,

$$\begin{aligned}u_5&=2\arctan\sqrt{\frac{1}{2a(kc+k-c-1)}}\left(x+ky-c\frac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)\ \text{for}\ a_1=a_0,\\ u_6&=-2\,\mathrm{arccot}\sqrt{\frac{1}{2a(kc+k-c-1)}}\left(x+ky-c\frac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)\ \text{for}\ a_1=-a_0.\end{aligned} \tag{50}$$

The above analysis shows that if $\alpha=1$, these solutions are transformed into Equations (35)–(37).

Take the following solution $u_5=2\arctan\sqrt{\frac{1}{2a(kc+k-c-1)}}\left(x+ky-c\frac{t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right)$ as an example to describe how the solution changes with different α. Let $k=c=2, a=\frac{1}{6}, \xi_0=0$, one can get $u_5 = 2\arctan\left(x + 2y - 2\frac{t^\alpha}{\Gamma(1+\alpha)}\right)$. Figures 1–4 depict the solution u_5 for different α at $y = 0$.

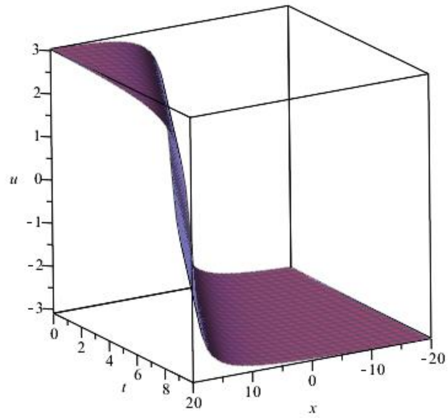

Figure 1. $\alpha = 1$.

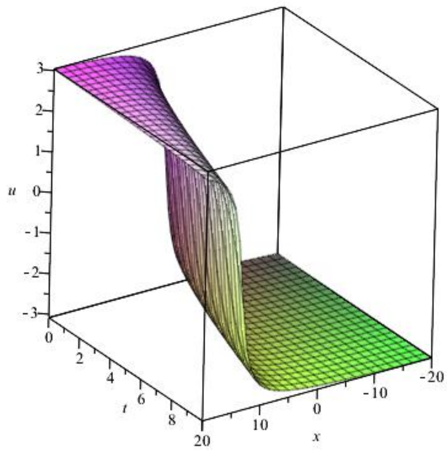

Figure 2. $\alpha = 0.8$.

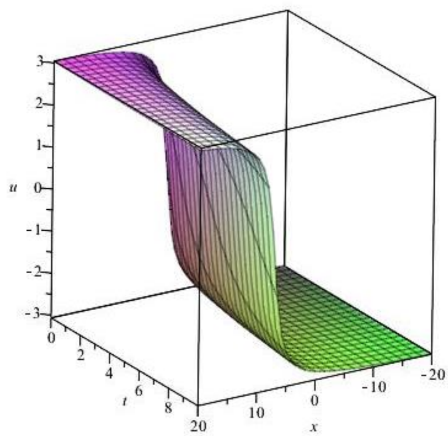

Figure 3. $\alpha = 0.5$.

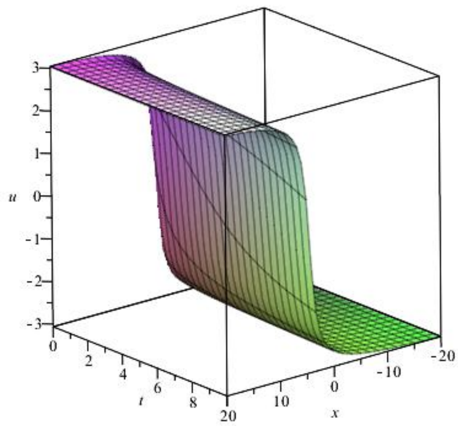

Figure 4. $\alpha = 0.3$.

The following Figure 5 shown the solution with different α at $y = 0, t = 2$. As can be seen in Figure 5, the location of the solution changes as α changes. The larger the value of α, the more forward the solution is located, and conversely, the smaller the value of α, the more backward the solution is located.

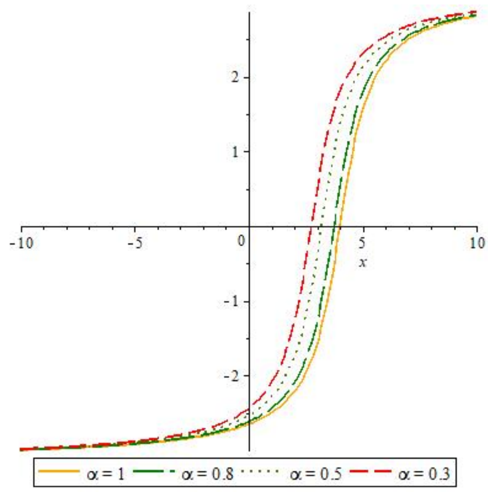

Figure 5. Plots of solutions with different α at $y = 0, t = 2$.

6. Conclusions

In this paper, the extended double (2+1)-dimensional sine-Gorden equation and its time fractional form are studied. It is clear that this paper generalizes the results in the literature [1]. This equation not only has extra terms $a \sin 2u$, but also has the time fractional order form of this equation. More importantly, some explicit solutions are obtained.

The extended double (2+1)-dimensional sine-Gorden equation is reduced to a double (1+1)-dimensional sine-Gorden equation by the group method. In addition, it can be found that this equation has basic symmetries. By these basic symmetries, symmetry reductions are obtained. Furthermore, some exact solutions of the extended double (2+1)-dimensional sine-Gorden equation are obtained using traveling wave transform. After that, a conservation law of this equation is obtained based on the conservation law multiplier method. Meanwhile, using the fractional complex transform, some new explicit solutions of the extended double time fractional (2+1)-dimensional sine-Gorden equation also presented.

Consequently, the extended double (2+1)-dimensional sine-Gorden equation is an interesting new equation in nonlinear mathematical physics fields, it should be noted that there are still many issues worth investigating, such as non-local symmetry, non-local conservation laws and many more explicit solutions.

Author Contributions: Writing-original draft preparation: G.W.; Formal analysis, L.L.; Investigation, Q.W.; Writing—review and editing, J.G. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by Natural Science Foundation of Hebei Province of China (No. A2018207030), Youth Key Program of Hebei University of Economics and Business (2018QZ07), Key Program of Hebei University of Economics and Business (2020ZD11), Youth Team Support Program of Hebei University of Economics and Business.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to thank the anonymous reviewers and editors for their valuable comments and suggestions, which led to the improvement of the quality of this paper.

Conflicts of Interest: The authors declares no conflict of interest.

References

1. Wang, G.; Yang, K.; Gu, H.; Guan, F.; Kara, A.H. A (2+1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions. *Nucl. Phys. B* **2020**, *953*, 114956. [CrossRef]
2. Jumarie, G. Modified Riemann–Liouville derivative and fractional taylor series of nondifferentiable functions further results. *Comput. Math. Appl.* **2006**, *51*, 1367–1376. [CrossRef]
3. Li, C.; Guo, Q.; Zhao, M. On the solutions of (2+1)-dimensional time-fractional Schrödinger Equation. *Appl. Math. Lett.* **2020**, *94*, 238–243. [CrossRef]
4. Li, Z.B.; He, J.H. Fractional complex transform for fractional differential equations. *Math. Comput. Appl.* **2010**, *15*, 970–973. [CrossRef]
5. Perring, J.K.; Skyrme, T.H. A model unified field equation. *Nucl. Phys.* **1962**, *31*, 550. [CrossRef]
6. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981.
7. Wazwaz, A.M. Multiple optical kink solutions for new Painlevé integrable (3+1)-dimensional sine-Gordon equations with constant and time-dependent coefficients. *Optik* **2020**, *219*, 165003. [CrossRef]
8. Xie, Y.X.; Tang, J.S. A unified approach in seeking the solitary wave solutions to sine-Gordon type equations. *Chin. Phys. B.* **2005**, *14*, 1303–1306.
9. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. Method for solving the sine-Gordon equation. *Phys. Rev. Lett.* **1973**, *30*, 1262. [CrossRef]
10. Wazwaz, A.M. The tanh method and a variable separated ODE method for solving double sine-Gordon equation. *Phys. Lett. A* **2006**, *350*, 367–370. [CrossRef]
11. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
12. Olver, P.J. *Application of Lie Group to Differential Equation*; Springer: New York, NY, USA, 1986.
13. Bluman, G.W.; Cheviakov, A.; Anco, S. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: New York, NY, USA, 2010.
14. Wang, G.W.; Kara, A.H. A (2+1)-dimensional KdV equation and mKdV equation: Symmetries, group invariant solutions and conservation laws. *Phys. Lett. A* **2019**, *383*, 728–731. [CrossRef]
15. Wang, G.; Vega-Guzman, J.; Biswas, A.; Alzahrani, A.K.; Kara, A.H. (2+1)-dimensional Boiti-Leon-Pempinelli equation-Domain walls, invariance properties and conservation laws. *Phys. Lett. A* **2020**, *384*, 126255. [CrossRef]
16. Wang, G.; Liu, Y.; Wu, Y.; Su, X. Symmetry analysis for a seventh-order generalized KdV equation and its fractional version in fluid mechanics. *Fractals* **2020**, *28*, 2050044. [CrossRef]
17. Wang, G.W.; Liu, X.Q.; Ying, Y.Y. Lie symmetry analysis to the time fractional generalized fifth-order KdV equation. *Commun. Nonlinear Sci. Numer. Simulat.* **2013**, *18*, 2321–2326. [CrossRef]
18. Wang, G.W.; Wazwaz, A.M. On the modified Gardner type equation and its time fractional form, Chaos. *Solitons Fractals* **2022**, *155*, 111694. [CrossRef]
19. Wang, G.W. A novel (3+1)-dimensional sine-Gorden and sinh-Gorden equation: Derivation, symmetries and conservation laws. *Appl. Math. Lett.* **2021**, *113*, 106768. [CrossRef]
20. Wang, G.W. A new (3+1)-dimensional Schrödinger equation: Derivation, soliton solutions and conservation laws. *Nonlinear Dyn.* **2021**, *104*, 1595–1602. [CrossRef]
21. Lou, S.Y.; Ma, H.C. Non-Lie symmetry groups of (2+1)-dimensional nonlinear systems obtained from a simple direct method. *J. Phys. A Math. Gen.* **2005**, *38*, L129–L137. [CrossRef]

22. Zhao, Z.L.; He, L. Lie symmetry, nonlocal symmetry analysis, and interaction of solutions of a (2+1)-dimensional KdV-mKdV equation. *Theor. Math. Phys.* **2021**, *206*, 142–162. [CrossRef]
23. Hu, W.; Wang, Z.; Zhao, Y.; Deng, Z. Symmetry breaking of infinite-dimensional dynamic system. *Appl. Math. Lett.* **2020**, *103*, 106207. [CrossRef]
24. Hu, W.P.; Zhang, C.Z.; Deng, Z.C. Vibration and elastic wave propagation in spatial flexible damping panel attached to four special springs. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *84*, 105199. [CrossRef]
25. Hu, W.P.; Ye, J.; Deng, Z.C. Internal resonance of a flexible beam in a spatial tethered system. *J. Sound Vib.* **2020**, *475*, 115286 [CrossRef]
26. Hu, W.P.; Yu, L.J.; Deng, Z.C. Minimum Control Energy of Spatial Beam with Assumed Attitude Adjustment Target. *Acta Mech. Solida Sin.* **2020**, *33*, 51–60. [CrossRef]
27. Hirota, R. *The Direct Method in Soliton Theory*; Cambridge University Press: Cambridge, UK, 2004.
28. Rogers, C.; Schief, W.K. *Bäcklund and Darboux Transformations, Geometry and Morden Applications in Soliton Theory*; Cambridge University Press: Cambridge, UK, 2002.
29. Hu, X.B.; Zhu, Z.N. A Bäcklund transformation and nonlinear superposition formula for the Belov-Chaltikian lattice. *J. Phys. A* **1998**, *31*, 4755. [CrossRef]
30. Gu, C.; Hu, H.; Hu, A.; Zhou, Z. *Darboux Transformations in Integrable Systems Theory and Their Applications to Geometry*; Springer: Berlin/Heidelberg, Germany, 2005.
31. Lax, P.D. Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.* **1968**, *21*, 467–490. [CrossRef]
32. Kudryashov, N.A. Lax Pairs and special polynomials associated with self-similar reductions of Sawada-Kotera and Kupershmidt equations. *Regul. Chaotic Dyn.* **2020**, *25*, 59–77. [CrossRef]
33. Moaddy, K.; Momani, S.; Hashim, I. The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics. *Comput. Math. Appl.* **2011**, *61*, 1209–1216. [CrossRef]
34. He, J. Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. *Methods. Appl. Mech. Eng.* **1998**, *167*, 57–68. [CrossRef]
35. Grigorenko, L.; Grigorenko, E. Chaotic dynamics of the fractional lorenz system. *Phys. Rev. Lett.* **2003**, *91*, 034101. [CrossRef] [PubMed]
36. Mainardi, F. *Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics, Fractals and Fractional Calculus in Continuum Mechanics*; Springer: New York, NY, USA, 1997.
37. Abbasb, Y.S.; Kazem, S.; Alhuthali, M.S.; Alsulami, H.H. Application of the operational matrix of fractional-order Legendre functions for solving the time-fractional convection-diffusion equation. *Appl. Math. Comput.* **2015**, *266*, 31–40.
38. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: San Diego, CA, USA, 2006.
39. Huang, F.H.; Guo, B.L. General solutions to a class of time fractional partial differential equations. *Appl. Math. Mech.* **2010**, *31*, 815–826. [CrossRef]
40. Momani, S.; Noor, M.A. Numerical methods for fourth-order fractional integro-differential equations. *Appl. Math. Comput.* **2006**, *182*, 754–760. [CrossRef]
41. Jiang, Z.; Zhang, Z.G.; Li, J.J.; Yang, H.W. Analysis of Lie symmetries with conservation laws and solutions of generalized (4+1)-dimensional time-fractional Fokas equation. *Fractal Fract.* **2022**, *6*, 108. [CrossRef]
42. Sene, N. Analytical solutions of a class of Fluids models with the Caputo fractional derivative. *Fractal Fract.* **2022**, *6*, 35. [CrossRef]