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Article

On Transformation Involving Basic Analogue to the Aleph-Function of Two Variables

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Abstract: In our work, we derived the fractional order q-integrals and q-derivatives concerning a basic analogue to the Aleph-function of two variables (AFTV). We discussed a related application and the q-extension of the corresponding Leibniz rule. Finally, we presented two corollaries concerning the basic analogue to the I-function of two variables and the basic analogue to the Aleph-function of one variable.

Keywords: Fractional q-integral; q-derivative operators; basic analogue to the Aleph-function; basic analogue to the I-function; q-Leibniz rule

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1. Introduction

Fractional calculus represents an important part of mathematical analysis. The concept of fractional calculus was born from a famous correspondence between L'Hopital and Leibniz in 1695. In the last four decades, it has gained significant recognition and found many applications in diverse research fields (see [1–6]). The fractional basic (or $q-$) calculus is the extension of the ordinary fractional calculus in the q-theory (see [7–10]). We recall that basic series and basic polynomials, particularly the basic (or $q-$) hypergeometric functions and basic (or $q-$) hypergeometric polynomials, are particularly applicable in several fields, e.g., Finite Vector Spaces, Lie Theory, Combinatorial Analysis, Particle Physics, Mechanical Engineering, Theory of Heat Conduction, Non-Linear Electric Circuit Theory, Cosmology, Quantum Mechanics, and Statistics. In 1952, Al-Salam introduced the q-analogue to Cauchy's formula [11] (see also [12]). Agarwal [13] studied certain fractional q-integral and q-derivative operators. In addition, various researchers reported image formulas of various q-special functions under fractional q-calculus operators, for example, Kumar et al. [14], Sahni et al. [15], Yadav and Purohit [16], Yadav et al. [17,18], and maybe more. The q-extensions of the Saigo's fractional integral operators were defined by Purohit and Yadav [19]. Several authors utilised such operators to evaluate a general class of q-polynomials, the basic analogue to Fox's H-function, basic analogue to the I-function, fractional q-calculus formulas for various special functions, etc. The readers can see more related new details in [16–18,20] on fractional q-calculus.

The purpose of the present manuscript is to discuss expansion formulas, involving the basic analogue to AFTV [21]. The q-Leibniz formula is also provided.

We recall that q-shifted factorial $(a;q)_n$ has the following form [22,23]

$$(a;q)_n = \begin{cases} 1, & (n=0) \\ \prod_{i=0}^{n-1}(1-aq^i), & (n \in \mathbb{N}\cup\{\infty\}) \end{cases}, \tag{1}$$

such that $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$.

The expression of the q-shifted factorial for negative subscript is written by

$$(a;q)_{-n} = \frac{1}{(1-aq^{-1})\,(1-aq^{-2})\cdots(1-aq^{-n})} \quad (n \in \mathbb{N}_0). \tag{2}$$

Additionally, we have

$$(a;q)_\infty = \prod_{i=0}^{\infty}\left(1-aq^i\right) \quad (a, q \in \mathbb{C};\ |q| < 1). \tag{3}$$

Using (1)–(3), we conclude that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \qquad (n \in \mathbb{Z}), \tag{4}$$

its extension to $n = \alpha \in \mathbb{C}$ as:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha \in \mathbb{C};\ |q| < 1), \tag{5}$$

such that the principal value of q^α is considered.

We equivalently have a form of (1), given as

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)} \qquad (a \neq 0, -1, -2, \cdots), \tag{6}$$

where the q-gamma function is expressed as [8]:

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}}, \quad (a \neq 0, -1, -2, \cdots). \tag{7}$$

The expression of the q-analogue to the Riemann–Liouville fractional integral operator (RLI) of $f(x)$ has the following expression [12]:

$$I_q^\mu\{f(x)\} = \frac{1}{\Gamma_q(\mu)}\int_0^x (x-tq)_{\mu-1} f(t)\, \mathrm{d}_q t, \tag{8}$$

here, $\Re(\mu) > 0,\ |q| < 1$ and

$$[x-y]_v = x^v \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{n+v}}\right] = x^v\left(\frac{y}{x};q\right)_v \quad (x \neq 0). \tag{9}$$

The basic integral [8] is given by

$$\int_0^x f(t)\mathrm{d}_q t = x(1-q)\sum_{k=0}^{\infty} q^k f\left(xq^k\right). \tag{10}$$

Equation (8), in conjunction with (10); then, we have the series representation of (RLI), as follows

$$I_q^\mu f(x) = \frac{x^\mu(1-q)}{\Gamma_q(\mu)}\sum_{k=0}^{\infty} q^k\left[1-q^{k+1}\right]_{\mu-1} f\left(xq^k\right). \tag{11}$$

We mention that for $f(x) = x^{\lambda-1}$, the following can be written [16]

$$I_q^\mu\left(x^{\lambda-1}\right) = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\mu)} x^{\lambda+\mu-1} \qquad (\Re(\lambda+\mu) > 0). \tag{12}$$

2. Basic Analogue to Aleph-Function of Two Variables

We recall that AFTV [21] is an extension of the I-function possessing two variables [24]. Here, in the present article, we define a basic analogue to AFTV.

We record

$$G(q^a) = \left[\prod_{n=0}^{\infty}(1-q^{a+n})\right]^{-1} = \frac{1}{(q^a;q)_\infty}. \tag{13}$$

Next, we have

$$\begin{aligned}
&\aleph(z_1,z_2;q) \\
&= \aleph^{0,n_1;m_2,n_2:m_3,n_3}_{P_i,Q_i,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{matrix} z_1 \\ \\ z_2 \end{matrix} ;q \left| \begin{matrix} (a_j;\alpha_j,A_j)_{1,n_1}, [\tau_i(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i}; \\ \\ [\tau_i(b_{ji};\beta_{ji},B_{ji})]_{1,Q_i}; \end{matrix}\right.\right. \\
&\qquad \left.\begin{matrix} (c_j,\gamma_j)_{1,n_2}, \left[\tau_{i'}\left(c_{ji'},\gamma_{ji'}\right)\right]_{n_2+1,P_{i'}}; (e_j,E_j)_{1,n_3}, \left[\tau_{i''}\left(e_{ji''},\gamma_{ji''}\right)\right]_{n_3+1,P_{i''}} \\ \\ (d_j,\delta_j)_{1,m_2}, \left[\tau_{i'}\left(d_{ji'},\delta_{ji'}\right)\right]_{m_2+1,Q_{i'}}; (f_j,F_j)_{1,m_3}, \left[\tau_{i''}\left(f_{ji''},F_{ji''}\right)\right]_{m_3+1,Q_{i''}} \end{matrix}\right) \\
&= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s_1,s_2;q)z_1^{s_1}z_2^{s_2}\,\mathrm{d}_q s_1\,\mathrm{d}_q s_2,
\end{aligned} \tag{14}$$

where $\omega = \sqrt{-1}$, and

$$\begin{aligned}
\phi(s_1,s_2;q) &= \frac{\prod_{j=1}^{n_1} G\left(q^{1-a_j+\alpha_j s_1+A_j s_2}\right)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=1}^{Q_i} G\left(q^{1-b_{ji}+\beta_{ji}s_1+B_{ji}s_2}\right)\prod_{j=n_1+1}^{P_i} G\left(q^{a_{ji}-\alpha_{ji}s_1-A_{ji}s_2}\right)\right\}} \\
&\times \frac{\prod_{j=1}^{m_2} G\left(q^{d_j-\delta_j s_1}\right)\prod_{j=1}^{n_2} G\left(q^{1-c_j+\gamma_j s_1}\right)}{\sum_{i'=1}^{r'}\tau_{i'}\left\{\prod_{j=m_2+1}^{Q_{i'}} G\left(q^{1-d_{ji'}+\delta_{ji'}s_1}\right)\prod_{j=n_2+1}^{P_{i'}} G\left(q^{c_{ji'}-\gamma_{ji'}s_1}\right)\right\}G(q^{1-s_1})\sin\pi s_1} \\
&\times \frac{\prod_{j=1}^{m_3} G\left(q^{f_j-F_j s_2}\right)\prod_{j=1}^{n_3} G\left(q^{1-e_j+E_j s_2}\right)}{\sum_{i''=1}^{r''}\tau_{i''}\left\{\prod_{j=m_3+1}^{Q_{i''}} G\left(q^{1-f_{ji''}+F_{ji''}s_2}\right)\prod_{j=n_3+1}^{P_{i''}} G\left(q^{e_{ji''}-E_{ji''}s_2}\right)\right\}G(q^{1-s_2})\sin\pi s_2},
\end{aligned} \tag{15}$$

where $z_1, z_2 \neq 0$ and are real or complex. An empty product is elucidated as unity, and $P_i, P_{i'}, P_{i''}, Q_i, Q_{i'}, Q_{i''}, m_1, m_2, m_3, n_1, n_2, n_3$ are non-negative integers, such that $Q_i, Q_{i'}, Q_{i''} > 0; \tau_i, \tau_{i'}, \tau_{i''} > 0 (i = 1, \cdots, r;\ \ i' = 1, \cdots, r';\ i'' = 1, \cdots, r'')$. All the As, αs, γs, δs, Es, and Fs are presumed to be positive quantities for standardization intention, the as, bs, cs, ds, es, and fs are complex numbers. The definition of a basic analogue to AFTV will, however, make sense, even if some of these quantities are equal to zero. The contour L_1 is in the s_1-plane and goes from $-\omega\infty$ to $+\omega\infty$, with loops where necessary, to make sure that the poles of $G\left(q^{d_j-\delta_j s_1}\right)$ $(j = 1, \cdots, m_2)$ are to the right-hand side and all the poles of $G\left(q^{1-a_j+\alpha_j s_1+A_j s_2}\right)$ $(j = 1, \cdots, n_1)$, $G\left(q^{1-c_j+\gamma s_1}\right)$ $(j = 1, \cdots, n_2)$ lie to the left-hand side of L_1. The contour L_2 is in the s_2-plane and goes from $-\omega\infty$ to $+\omega\infty$, with loops where necessary, to ensure that the poles of $G\left(q^{f_j-F_j s_2}\right)$ $(j = 1, \cdots, m_3)$ are to the right-hand side and all the poles of $G\left(q^{1-a_j+\alpha_j s_1+A_j s_2}\right)$ $(j = 1, \cdots, n_1)$, $G\left(q^{1-e_j+E_j s_2}\right)$ $(j = 1, \cdots, n_2)$ lie to the left-hand side of L_2. For values of $|s_1|$ and $|s_2|$, the integrals converge, if $\Re(s_1\log(z_1) - \log\sin\pi s_1) < 0$ and $\Re(s_2\log(z_2) - \log\sin\pi s_2) < 0$.

3. Main Formulas

Here, we obtain fractional q-integral and q-derivative formulas concerning the basic analogue to AFTV. Here, we have the following notations:

$$A_1 = (a_j;\alpha_j, A_j)_{1,n_1}, [\tau_i(a_{ji};\alpha_{ji}, A_{ji})]_{n_1+1,P_i}; \; B_1 = [\tau_i(b_{ji};\beta_{ji}, B_{ji})]_{1,Q_i}. \tag{16}$$

$$C_1 = (c_j, \gamma_j)_{1,n_2}, \left[\tau_{i'}\left(c_{ji'}, \gamma_{ji'}\right)\right]_{n_2+1,P_{i'}}; (e_j, E_j)_{1,n_3}, \left[\tau_{i''}\left(e_{ji''}, \gamma_{ji''}\right)\right]_{n_3+1,P_{i''}}. \tag{17}$$

$$D_1 = (d_j, \delta_j)_{1,m_2}, \left[\tau_{i'}\left(d_{ji'}, \delta_{ji'}\right)\right]_{m_2+1,Q_{i'}}; (f_j, F_j)_{1,m_3}, \left[\tau_{i''}\left(f_{ji''}, F_{ji''}\right)\right]_{m_3+1,Q_{i''}}. \tag{18}$$

Theorem 1. *Let $\Re(\mu) > 0$, $\rho_i \in \mathbb{Z}^+$ $(i = 1, 2)$, and $|q| < 1$; then, the Riemann–Liouville fractional q-integral of (14) exists and is given as*

$$I_q^{\mu}\left\{x^{\lambda-1}\aleph^{0,n_1;m_2,n_2:m_3,n_3}_{P_i,Q_i,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{matrix} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{matrix}\; ; \, q \left| \begin{matrix} A_1;C_1 \\ \\ B_1;D_1\end{matrix}\right.\right)\right\} = (1-q)^{\mu}x^{\lambda+\mu-1}$$
$$\times\; \aleph^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{matrix} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{matrix}\; ; \, q \left| \begin{matrix} (1-\lambda;\rho_1,\rho_2), A_1;C_1 \\ \\ B_1, (1-\lambda-\mu;\rho_1,\rho_2);D_1\end{matrix}\right.\right), \tag{19}$$

where $\Re(s_i \log(z_i) - \log \sin \pi s_i) < 0$ $(i = 1, 2)$.

Proof. We apply the definitions (8) and (14) in the left-hand side of (19), we have (say $\mathcal{I}$)

$$\mathcal{I} = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-yq)_{\alpha-1}\left\{y^{\lambda-1}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s_1,s_2;q)\prod_{i=1}^{2}(z_iy^{\rho_i})^{s_i}\mathrm{d}_qs_1\mathrm{d}_qs_2\right\}\mathrm{d}_qy. \tag{20}$$

By using standard calculations, we arrive at

$$\mathcal{I} = \frac{y^{\lambda-1}}{\Gamma_q(\alpha)}\frac{1}{(2\pi\omega)^2}$$
$$\times \int_{L_1}\int_{L_2}\pi^2\phi(s_1,s_2;q)\prod_{i=1}^{2}z_i^{s_i}\left\{\int_0^x (x-yq)_{\alpha-1}y^{\lambda+\sum_{i=1}^2\rho_is_i-1}\mathrm{d}_qy\right\}\mathrm{d}_qs_1\,\mathrm{d}_qs_2$$
$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s_1,s_2;q)\prod_{i=1}^{2}z_i^{s_i}I_q^{\mu}\left\{x^{\lambda+\sum_{i=1}^2\rho_is_i-1}\right\}\mathrm{d}_qs_1\,\mathrm{d}_qs_2. \tag{21}$$

Next, we apply formula (12) to the equation above; then, we get

$$\mathcal{I} = \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s_1,s_2;q)\prod_{i=1}^{2}z_i^{s_i}\frac{\Gamma_q\left(\lambda+\sum_{i=1}^2\rho_is_i\right)}{\Gamma_q\left(\lambda+\mu+\sum_{i=1}^2\rho_is_i\right)}x^{\lambda+\mu+\sum_{i=1}^2\rho_is_i-1}\mathrm{d}_qs_1\mathrm{d}_qs_2. \tag{22}$$

Considering the above q-Mellin–Barnes double contour integrals in terms of the basic analogue to AFTV, we obtain (19). □

If we use a fractional q-derivative operator without initial values, defined as

$$I_q^{-\mu}\{f(x)\} = D_{x,q}^{\mu}\{f(x)\} = \frac{1}{\Gamma_q(-\mu)}\int_0^x (x-tq)_{-\mu-1}f(t)\,\mathrm{d}_qt, \tag{23}$$

where $\Re(\mu) < 0$; then, we yield the following result:

Theorem 2. *For $\Re(\mu) > 0$, $\rho_i \in \mathbb{Z}^+$ $(i = 1,2)$, and $|q| < 1$, the Riemann–Liouville fractional q-derivative of* (14) *exists and is given by*

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\aleph^{0,n_1;m_2,n_2:m_3,n_3}_{P_i,Q_i,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array} ;q \left| \begin{array}{c} A_1;C_1 \\ \\ B_1;D_1 \end{array}\right.\right)\right\} = (1-q)^{-\mu}x^{\lambda-\mu-1}$$
$$\times\ \aleph^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array} ;q \left| \begin{array}{c} (1-\lambda;\rho_1,\rho_2),A_1;C_1 \\ \\ B_1,(1-\lambda+\mu;\rho_1,\rho_2);D_1 \end{array}\right.\right), \tag{24}$$

where $\Re(s_i \log(z_i) - \log \sin \pi s_i) < 0$ $(i = 1,2)$.

Proof. If we replace μ by $-\mu$ in (19), and follow the proof of Theorem 1, then we can easily obtain (24). □

4. Leibniz's Formula

The q-expression of the Leibniz rule for the fractional q-derivatives [13] is written below

Lemma 1. *For regular functions $U(x)$ and $V(x)$, we have*

$$D^{\alpha}_{x,q}\{U(x)V(x)\} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}[q^{-\mu};q]_n}{(q;q)_n}\ D^{\mu-n}_{x,q}\{U(xq^n)\}D^{n}_{x,q}\{V(x)\}. \tag{25}$$

Next, we have the following formula:

Theorem 3. *For $\Re(\mu) < 0$, $\rho_i \in \mathbb{Z}^+$ $(i = 1,2)$, then the Riemann–Liouville fractional q-derivative of a product of two basic function is written as*

$$\aleph^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array} ;q \left| \begin{array}{c} (1-\lambda;\rho_1,\rho_2),A_1;C_1 \\ \\ B_1,(1-\lambda+\mu;\rho_1,\rho_2);D_1 \end{array}\right.\right)$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n\lambda+\frac{n(n-1)}{2}}[q^{-\mu};q]_n}{(q;q)_n\,(q^{\lambda};q)_{n-\mu}}$$
$$\times\ \aleph^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array} ;q \left| \begin{array}{c} (0;\rho_1,\rho_2),A_1;C_1 \\ \\ B_1,(n;\rho_1,\rho_2);D_1 \end{array}\right.\right), \tag{26}$$

where $\Re(s_i \log(z_i) - \log \sin \pi s_i) < 0$ $(i = 1,2)$.

Proof. To apply the q-Leibniz rule, we take

$$U(x) = x^{\lambda-1} \text{ and } V(x) = \aleph^{0,n_1;m_2,n_2:m_3,n_3}_{P_i,Q_i,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array} ;q \left| \begin{array}{c} A_1;C_1 \\ \\ B_1;D_1 \end{array}\right.\right).$$

By using Lemma 1, we obtain the following relation:

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\,\aleph^{0,n_1;m_2,n_2:m_3,n_3}_{P_i,Q_i,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array} ;q \left| \begin{array}{c} A_1;C_1 \\ \\ B_1;D_1 \end{array}\right.\right)\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}[q^{-\mu};q]_n}{(q;q)_n}\ D^{\mu-n}_{x,q}(xq^n)^{\lambda-1}D^{n}_{x,q}\{\aleph(z_1x^{\rho_1},z_2x^{\rho_2};q)\}. \tag{27}$$

Next, by using Theorem 2 and setting $\lambda = 1$, we obtain

$$\begin{aligned} &D^{n}_{x,q}\{\aleph(z_1x^{\rho_1}, z_2x^{\rho_2}; q)\} \\ &= (1-q)^{-n}x^{-n}\,\aleph^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array};q\,\middle|\,\begin{array}{c}(0;\rho_1,\rho_2),A_1;C_1 \\ \\ B_1,(n;\rho_1,\rho_2);D_1\end{array}\right). \end{aligned} \tag{28}$$

By using the above equation and the following result:

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\right\} = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda-\mu)}x^{\lambda-\mu-1}\ \ (\lambda \neq -1,-2,\cdots), \tag{29}$$

We can easily obtain the desired result (26) after several algebraic manipulations. □

5. Particular Cases

By setting $\tau_i, \tau_{i'}, \tau_{i''} \to 1$, the basic analogue to AFTV reduces to the basic analogue to the I-function of two variables [24].

Let

$$A'_1 = (a_j;\alpha_j,A_j)_{1,n_1}, (a_{ji};\alpha_{ji},A_{ji})_{n_1+1,P_i};\ B'_1 = (b_{ji};\beta_{ji},B_{ji})_{1,Q_i}. \tag{30}$$

$$C'_1 = (c_j,\gamma_j)_{1,n_2}, \left(c_{ji'},\gamma_{ji'}\right)_{n_2+1,P_{i'}}; (e_j,E_j)_{1,n_3}, \left(e_{ji''},\gamma_{ji''}\right)_{n_3+1,P_{i''}}. \tag{31}$$

$$D'_1 = (d_j,\delta_j)_{1,m_2}, \left(d_{ji'},\delta_{ji'}\right)_{m_2+1,Q_{i'}}; (f_j,F_j)_{1,m_3}, \left(f_{ji''},F_{ji''}\right)_{m_3+1,Q_{i''}}. \tag{32}$$

We have the following result:

Corollary 1.

$$\begin{aligned} &I^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array};q\,\middle|\,\begin{array}{c}(1-\lambda;\rho_1,\rho_2),A'_1;C'_1 \\ \\ B'_1,(1-\lambda+\mu;\rho_1,\rho_2);D'_1\end{array}\right) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n\lambda+\frac{n(n-1)}{2}}[q^{-\mu};q]_n}{(q;q)_n\,(q^{\lambda};q)_{n-\mu}} \\ &\times I^{0,n_1+1;m_2,n_2:m_3,n_3}_{P_i+1,Q_i+1;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}\left(\begin{array}{c} z_1x^{\rho_1} \\ \\ z_2x^{\rho_2}\end{array};q\,\middle|\,\begin{array}{c}(0;\rho_1,\rho_2),A'_1;C'_1 \\ \\ B'_1,(n;\rho_1,\rho_2);D'_1\end{array}\right), \end{aligned} \tag{33}$$

where $\Re(s_i\log(z_i) - \log\sin\pi s_i) < 0\ (i = 1,2)$.

Proof. By setting $\tau_i, \tau_{i'}, \tau_{i''} \to 1$ and following the proof of Theorem 3, we can easily obtain the desired result (33). □

Remark 1. *If the basic analogue to the I-function of two variables reduces to the basic analogue to the H-function of two variables [25], then we can obtain the result due to Yadav et al. [18].*

The basic analogue to AFTV reduces to the basic analogue to AFTV, defined by Ahmad et al. [26].

Let

$$A = (a_j,A_j)_{1,n},\cdots,[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}. \tag{34}$$

$$B = (b_j,B_j)_{1,m},\cdots,[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}. \tag{35}$$

Then, we have following relation:

Corollary 2.

$$\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(zx^{\rho};q\,\middle|\,\begin{matrix}(1-\lambda;\rho),A\\ B,(1-\lambda+\mu;\rho)\end{matrix}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n\lambda+\frac{n(n-1)}{2}}[q^{-\mu};q]_n}{(q;q)_n\,(q^{\lambda};q)_{n-\mu}}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(zx^{\rho};q\,\middle|\,\begin{matrix}(0;\rho),A\\ B,(n;\rho)\end{matrix}\right). \tag{36}$$

If we set $\tau_i \to 1$ in (36), then the basic analogue to AFTV reduces to the basic analogue to the I-function of one variable. We have

Corollary 3.

$$I^{m,n+1}_{p_i+1,q_i+1;r}\left(zx^{\rho};q\,\middle|\,\begin{matrix}(1-\lambda;\rho),(a_j,A_j)_{1,n},\cdots,(a_{ji},A_{ji})_{n+1,p_i}\\ \\ (b_j,B_j)_{1,m},\cdots,(b_{ji},B_{ji})_{m+1,q_i},(1-\lambda+\mu;\rho)\end{matrix}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n\lambda+\frac{n(n-1)}{2}}[q^{-\mu};q]_n}{(q;q)_n}$$
$$\times\; I^{m,n+1}_{p_i+1,q_i+1;r}\left(zx^{\rho};q\,\middle|\,\begin{matrix}(0;\rho),(a_j,A_j)_{1,n},\cdots,(a_{ji},A_{ji})_{n+1,p_i}\\ \\ (b_j,B_j)_{1,m},\cdots,(b_{ji},B_{ji})_{m+1,q_i},(n;\rho)\end{matrix}\right). \tag{37}$$

Remark 2. *If the basic analogue to AFTV reduces to the basic analogue to the H-function of one variable (see [27]), then we can report a similar expression.*

Remark 3. *We can generalize the q-extension of the Leibniz rule for the basic analogue to special multivariable functions; by this, we can obtain similar formulas by using similar methods.*

6. Conclusions

After the famous letter between L'Hopital and Leibniz from 1695, using integral transformations, we obtained a new field in mathematics, called fractional calculus. Among other things, there are fractional derivative and fractional integrals, as well as fractional differential equations. It is also well-known that fractional calculus operators and their basic (or $q-$) analogues have many applications, such as signal processing, bio-medical engineering, control systems, radars, sonars, to solve dual integral and series equations in elasticity, etc. In this article, we have proposed the fractional-order q-integrals and q-derivatives involving a basic analogue to AFTV [11,12,26,28]. Some remarkable applications of these integrals and derivatives have also been discussed. By specializing the various parameters as well as the variables in the basic analogue to AFTV, we can obtain a large number of q-extensions of the Leibniz rule, involving a large set of basic functions, that is, the product of such basic functions, which are describable in terms of the basic analogue to the H-function [25,27], the basic analogue to Meijer's G-function [27], the basic analogue to MacRobert's E-function [29], and the basic analogue to the hypergeometric function [10,16–18].

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