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Ergodic Stationary Distribution and Threshold Dynamics of a Stochastic Nonautonomous SIAM Epidemic Model with Media Coverage and Markov Chain

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Abstract: A stochastic nonautonomous SIAM (Susceptible individual–Infected individual–Aware individual–Media coverage) epidemic model with Markov chain and nonlinear noise perturbations has been constructed, which is used to research the hybrid dynamic impacts of media coverage and Lévy jumps on infectious disease transmission. The uniform upper bound and lower bound of the positive solution are studied. Based on defining suitable random Lyapunov functions, we researched the existence of a nontrival positive T-periodic solution. Sufficient conditions are derived to discuss the exponential ergodicity based on verifying a Foster–Lyapunov condition. Furthermore, the persistence in the average sense and extinction of infectious disease are investigated using stochastic analysis techniques. Finally, numerical simulations are utilized to provide evidence for the dynamical properties of the stochastic nonautonomous SIAM.

Keywords: media coverage; Lévy jumps; nontrival positive T-periodic solution; exponential ergodicity; persistence in mean; extinction

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1. Introduction

Recent studies have shown that public health alerts via social media exert a positive influence on usefully informing people of the prevalence about infectious disease [1]. Therefore, media coverage has effectively reduced the prevalence and shortened the duration of disease [2]. The influence of media message reminders on local behavioral response and public awareness response was studied in [3], and pharmaceutical interventions and the response of infected people to information have also been successful in controlling of the epidemic.

As the mass media has directed people's attention, it is often focused on infectious disease; thus, relying on the mass media to publicize the law of infectious disease transmission is extremely constructive for the effective treatment of the epidemic [4,5]. Assuming that the implementation of a public health alert program is proportional to the infected population, recent studies have shown that progress has been made in the social cost–benefit analysis of media campaigns for vaccination against infectious disease [6–9].

There is also a series of studies that specifically discussed the increased vaccination coverage of people due to social media advertising and television programs [10–12], which includes the example of discussing the function of media alerts to reduce the number of infected people. Particularly, in [12] a SIAM (Susceptible individual—Infected individual—Aware individual—Media coverage) epidemic model with media coverage and public health alerts was established as follows, and stability analysis around the endemic equilibrium was studied.

$$\begin{cases} \frac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta S(t)I(t) - \lambda S(t)\frac{M(t)}{M(t)+p} + \nu I(t) + \lambda_0 A(t) - hS(t), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t)I(t) - (\nu + \alpha + h)I(t), \\ \frac{\mathrm{d}A(t)}{\mathrm{d}t} = \lambda S(t)\frac{M(t)}{M(t)+p} - (\lambda_0 + h)A(t), \\ \frac{\mathrm{d}M(t)}{\mathrm{d}t} = r\Big(1 - \theta\frac{A(t)}{w+A(t)}\Big)I(t) - r_0(M(t) - M_0), \end{cases} \tag{1}$$

with the initial value for model (1) takes the following from:

$$S(0) \geq 0, I(0) \geq 0, A(0) \geq 0, M(0) \geq M_0,$$

$S(t), I(t), A(t)$ denotes the number of susceptible individuals, infected individuals and aware individuals, respectively. $M(t)$ is the cumulative number of TV programs and social media. Λ denotes the increase in the number of people who are susceptible. β stands for the rate of contact between susceptible individuals and infected individuals. ν, α and h denote the rate of recovery, disease-induced death and natural death, respectively. Furthermore, λ represents the rate of awareness among the susceptible, and λ_0 is the transfer rate of aware individuals to susceptible individuals. r is the growth rate in media coverage, and r_0 represents the diminution rate of advertisements [12].

It is well-known that dynamical effects of a periodically varying situation are different from those in a relatively stable situation [13]. Some parameters describing seasonal effects are affected by disturbances in time and usually exhibit [14]. Therefore, it is more accurate to assume periodicity of the surrounding situation and introduce time-varying periodic function parameters into the epidemic models, which can be found in [15–17] and the references therein.

Recently, there have been studies concentrated on discussing the spread dynamics of infectious disease using a stochastic mathematical model with Brownian motion [14–17]. Recently, some scholars found that, compared with Gaussian white noise and Brownian motion, Lévy jumps can more accurately describe the unexpected violent disturbances in the real situation [18,19]. Furthermore, Markov chain [20] is usually used to describe the vital transient transitions of important rates between two or more infectious states [21,22].

Taking the above mentioned content into account, media coverage, random perturbations and time-varying periodic function parameters are important disciplines in the modeling and dynamical analysis of infectious disease transmission. In this work, a random nonautonomous SIAM infectious disease model with Markov chain and nonlinear noise perturbations has been established as follows:

$$\begin{cases} \mathrm{d}S(t) = \Big[\Lambda(t) - \beta(t)S(t)I(t) - \lambda(t)S(t)\frac{M(t)}{M(t)+p(t)} + \nu(t)I(t) + \lambda_0(t)A(t) - h(t)S(t)\Big]\mathrm{d}t \\ \qquad +[\sigma_{11}(\gamma(t)) + \sigma_{12}(\gamma(t))S(t)]S(t)\mathrm{d}B_1(t) + \int_{\mathbb{Y}} c_1(u)S(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \\ \mathrm{d}I(t) = [\beta(t)S(t)I(t) - (\nu(t) + \alpha(t) + h(t))I(t)]\mathrm{d}t \\ \qquad +[\sigma_{21}(\gamma(t)) + \sigma_{22}(\gamma(t))I(t)]I(t)\mathrm{d}B_2(t) + \int_{\mathbb{Y}} c_2(u)I(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \\ \mathrm{d}A(t) = \Big[\lambda(t)S(t)\frac{M(t)}{M(t)+p(t)} - (\lambda_0(t) + h(t))A(t)\Big]\mathrm{d}t \\ \qquad +[\sigma_{31}(\gamma(t)) + \sigma_{32}(\gamma(t))A(t)]A(t)\mathrm{d}B_3(t) + \int_{\mathbb{Y}} c_3(u)A(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \\ \mathrm{d}M(t) = \Big[r(t)\Big(1 - \theta(t)\frac{A(t)}{w(t)+A(t)}\Big)I(t) - r_0(t)(M(t) - M_0(t))\Big]\mathrm{d}t \\ \qquad +[\sigma_{41}(\gamma(t)) + \sigma_{42}(\gamma(t))M(t)]M(t)\mathrm{d}B_4(t) + \int_{\mathbb{Y}} c_4(u)M(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t). \end{cases} \tag{2}$$

where $\Lambda(t), \beta(t), \lambda(t), p(t), \nu(t), \lambda_0(t), h(t), \alpha(t), r(t), \theta(t), w(t), r_0(t)$ are continuous T-periodic functions. $\sigma_{ij}^2(\cdot) > 0 (i = 1,2,3,4, j = 1,2)$ represent white noises. $\gamma(t)$ denotes a irreducible and continuous Markov chain, which is defined in $\mathbb{N} = \{1,2,3,\ldots,K\}$. $\gamma(t)$ is supposed to be generated by the following transition rate matrix $\Gamma = (\mu_{nj})_{K\times K}$,

$$\mathbb{P}\{\gamma(\tau + \triangle\tau) = j|\gamma(\tau) = n\} = \begin{cases} \mu_{nj}\triangle\tau + o(\triangle\tau), & n \neq j, \\ 1 + \mu_{nn}\triangle\tau + o(\triangle\tau), & n = j, \end{cases}$$

where $\mu_{nj} > 0$ is the transition rate from state n to state j, and $\mu_{nn} = -\Sigma_{n\neq j,n=1}^{K}\mu_{nj}$ holds for $n \neq j$.

Due primary to $\gamma(t)$ being an irreducible Markov procedure, it exists as a unique stationary probability distribution $\phi = (\phi_1, \phi_2, \cdots, \phi_K) \in \mathbb{R}^{1\times K}$ subject to $\sum_{n=1}^{K}\phi_n = 1$ and $\phi_n > 0$ hold for any $n \in \mathbb{N}$. $S(t-), I(t-), A(t-), M(t-)$ denotes left limit of $S(t), I(t), A(t), M(t)$, respectively. $\mathbb{Y}$ represents for a measurable subset of $\mathbb{R}_+$, X depicts an independent Poisson counting measure with Lévy measure ρ on $\mathbb{Y}$ with $\rho(\mathbb{Y}) < \infty$ such that $\widetilde{X}(\mathrm{d}t, \mathrm{d}u) = X(\mathrm{d}t, \mathrm{d}u) - \rho(\mathrm{d}u)\mathrm{d}t$. It is supposed that $c_i(u) > -1$, and there are four constants $\kappa_i > 0 (i = 1, 2, 3, 4)$ are constructed as below,

$$\max\{\int_{\mathbb{Y}}(\ln(1 + c_i(u))\rho\mathrm{d}u, \int_{\mathbb{Y}}(\ln(1 + c_i(u))^2\rho\mathrm{d}u\} \leq \kappa_i. \tag{3}$$

Based on the properties of Markov chain, we can regard system (2) as the subsystems defined as below:

$$\begin{cases} \mathrm{d}S(t) = \Big[\Lambda(t) - \beta(t)S(t)I(t) - \lambda(t)S(t)\frac{M(t)}{M(t)+p(t)} + \nu(t)I(t) + \lambda_0(t)A(t) - h(t)S(t)\Big]\mathrm{d}t \\ \qquad +[\sigma_{11}(n) + \sigma_{12}(n)S(t)]S(t)\mathrm{d}B_1(t) + \int_{\mathbb{Y}} c_1(u)S(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \\ \mathrm{d}I(t) = [\beta(t)S(t)I(t) - (\nu(t) + \alpha(t) + h(t))I(t)]\mathrm{d}t \\ \qquad +[\sigma_{21}(n) + \sigma_{22}(n)I(t)]I(t)\mathrm{d}B_2(t) + \int_{\mathbb{Y}} c_2(u)I(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \\ \mathrm{d}A(t) = \Big[\lambda(t)S(t)\frac{M(t)}{M(t)+p(t)} - (\lambda_0(t) + h(t))A(t)\Big]\mathrm{d}t \\ \qquad +[\sigma_{31}(n) + \sigma_{32}(n)A(t)]A(t)\mathrm{d}B_3(t) + \int_{\mathbb{Y}} c_3(u)A(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \\ \mathrm{d}M(t) = \Big[\, r(t)\Big(1 - \theta(t)\frac{A(t)}{w(t)+A(t)}\Big)I(t) - r_0(t)(M(t) - M_0(t))\Big]\mathrm{d}t \\ \qquad +[\sigma_{41}(n) + \sigma_{42}(n)M(t)]M(t)\mathrm{d}B_4(t) + \int_{\mathbb{Y}} c_4(u)M(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t). \end{cases} \tag{4}$$

Remark 1. *In recent related work, stochastic perturbations are usually represented by linear form perturbation of white noise, and the influences of linear noises perturbations on nonautonomous epidemic models were studied in [13–17]. However, in order to accurately depict some stochastic phenomena arising from infectious disease transmission in the real world, it is more constructive to introduce nonlinear noise perturbations into a nonautonomous epidemic model. Furthermore, some stochastic models have been established to discuss the prevalence mechanism of infectious diseases [23–31] without Lévy jumps.*

A SIS infectious disease system with regime-switching driven by Lévy jumps was investigated in [32], while the random dynamics for infectious disease system with hybrid dynamic impacts of Lévy jumps and media coverage are rarely reported. Taking the media coverage and random disturbance into dynamic impacts on threshold dynamics of random infectious disease model were investigated in [33–36], while Lévy jumps and periodic function parameters were not considered in [33–36].

The dynamic behavior of infectious disease systems in [37–39] were investigated under nonlinear noise perturbations and Lévy jumps, while all parameters were assumed to be constant values in [37–39], periodicity factors during transmission within the infectious disease regimes were not considered.

Although the stochastic infectious disease model and its dynamic analysis have attracted wide attention, as far as the authors know, the hybrid dynamic impacts of Lévy jumps and media coverage on random dynamics of the nonautonomous SIAM epidemic model with Markov chain and nonlinear noise perturbations have not been reported in previous related studies.

By incorporating Lévy jumps, nonlinear noise perturbations and periodic function parameters into the the epidemic system, we aim to study the hybrid dynamic impacts of media coverage on infectious disease transmission driven by Lévy jumps. For the rest of this work, we will make some arrangements as below: In the next section, the uniform upper bound and lower bound of the solution for stochastic nonautonomous system will be investigated. Based on constructing certain appropriate stochastic Lyapunov functions, sufficient conditions for existence of a nontrival positive T-periodic solution will be discussed.

Based on verifying a Foster–Lyapunov criterion, sufficient conditions for the exponential ergodicity are discussed. Furthermore, some sufficient conditions are derived to discuss the persistence in the mean and extinction of the infectious disease. In the third section, numerical simulations are used to prove the accuracy of the theoretical derivation. Lastly, section four is the conclusion of this paper.

2. Qualitative Analysis

For the sake of the narrative, we define the following mathematical symbols,

$$g^u = \sup_{t\in\mathbb{R}_+} g(t), \quad g^l = \inf_{t\in\mathbb{R}_+} g(t), \quad \langle g(t)\rangle = \frac{1}{t}\int_0^t g(t)\mathrm{d}t.$$

Lemma 1. *For any initial value $(S(0), I(0), A(0), M(0), n) \in \mathbb{R}_+^4 \times \mathbb{N}$, when the sufficient condition (3) holds, then there exists a uniform upper bound and a uniform lower bound for the solution of system (4).*

Proof. Let $y_1(t) = S^\eta(t)$, $\eta \in (0,1)$. Utilizing the Itô's formula to $e^t y_1(t)$ and, integrating both sides from 0 to t, the following results can be obtained.

$$\begin{aligned}
&\mathbb{E}(e^t y_1(t)) \\
= \;& \mathbb{E}\int_0^t e^s\left[1+\eta\left(\frac{\Lambda(t)}{S(t)} - \beta(t)I(t) - \frac{\lambda(t)M(t)}{M(t)+p(t)} + \frac{\nu(t)I(t)}{S(t)} + \frac{\lambda_0(t)A(t)}{S(t)} - h(t)\right)\right]y_1(t)\mathrm{d}s \\
&+y_1(0) + \mathbb{E}\int_0^t e^s\left[\int_{\mathbb{Y}}((1+c_1(u))^\eta - 1)\widetilde{X}(\mathrm{d}u,\mathrm{d}t)\right]y_1(t)\mathrm{d}s \\
&-\mathbb{E}\int_0^t e^s\left[\frac{\eta(1-\eta)}{2}(\sigma_{11}(n)+\sigma_{12}(n)S(t))^2\right]y_1(t)\mathrm{d}s \\
\leq \;& \mathbb{E}\int_0^t e^s\left[1+\eta\left(\frac{\Lambda(t)}{S(t)} - \beta(t)I(t) - \frac{\lambda(t)M(t)}{M(t)+p(t)} + \frac{\nu(t)I(t)}{S(t)} + \frac{\lambda_0(t)A(t)}{S(t)} - h(t)\right)\right]y_1(t)\mathrm{d}s \\
&+y_1(0) - \mathbb{E}\int_0^t e^s\left[\frac{\eta(1-\eta)}{2}(\sigma_{11}(n)+\sigma_{12}(n)S(t))^2\right]y_1(t)\mathrm{d}s \\
&+\mathbb{E}\int_0^t e^s\left[\int_{\mathbb{Y}}((1+c_1(u))^\eta - 1 - \eta c_1(u))\rho\mathrm{d}u\right]y_1(t)\mathrm{d}s.
\end{aligned}$$

When $S(t) \geq 0$ and $0 < \eta < 1$, based on the inequality $S^\eta(t) \leq 1 + \eta(S(t)-1)$, if sufficient condition (3) holds, we can obtain the following results

$$\begin{aligned}
&y_1(t)\left[1+\eta\left(\frac{\Lambda(t)}{S(t)} + \frac{\nu(t)I(t)}{S(t)} + \frac{\lambda_0(t)A(t)}{S(t)} - \beta(t)I(t) - \frac{\lambda(t)M(t)}{M(t)+p(t)} - h(t)\right)\right] \\
&-y_1(t)\left[\frac{\eta(1-\eta)}{2}(\sigma_{11}(n)+\sigma_{12}(n)S(t))^2 - \int_{\mathbb{Y}}((1+c_1(u))^\eta - 1 - \eta c_1(u))\rho\mathrm{d}u\right] \\
\leq \;& \left[1+\eta\left(\frac{\Lambda(t)}{S(t)} - \beta(t)I(t) - \frac{\lambda(t)M(t)}{M(t)+p(t)} + \frac{\nu(t)I(t)}{S(t)} + \frac{\lambda_0(t)A(t)}{S(t)} - h(t)\right)\right]S^\eta(t) \\
\leq \;& G_1(\eta),
\end{aligned}$$

where $G_1(\eta)$ is a positive definite function associated with η.

Hence, we can reach the following conclusion

$$\mathbb{E}(e^t y_1(t)) \leq y_1(0) + \mathbb{E}\int_0^t e^s G_1(\eta)\mathrm{d}s,$$

which reveals that $\limsup_{t\to\infty}\mathbb{E}(S(t)^\eta) \leq G_1(\eta)$.

Basing on utilizing the similar arguments, one can find that

$$\begin{cases} \limsup_{t\to\infty} \mathbb{E}(I(t)^\eta) \le G_2(\eta), \\ \limsup_{t\to\infty} \mathbb{E}(A(t)^\eta) \le G_3(\eta), \\ \limsup_{t\to\infty} \mathbb{E}(M(t)^\eta) \le G_4(\eta). \end{cases}$$

Let $(\chi(t), n) = (S(t), I(t), A(t), M(t), n) \in \mathbb{R}_+^4 \times \mathbb{N}$, it yields that

$$2^{(1-\frac{\eta}{2})\wedge 0} \mid \chi(t) \mid^\eta \le S^\eta(t) + I^\eta(t) + A^\eta(t) + M^\eta(t),$$

which follows that

$$\begin{aligned} \limsup_{t\to\infty} \mathbb{E} \mid \chi(t) \mid^\eta &\le 0.5^{(1-\frac{\eta}{2})\wedge 0} \limsup_{t\to\infty} \mathbb{E}[S^\eta(t) + I^\eta(t) + A^\eta(t) + M^\eta(t)] \\ &\le 0.5^{(1-\frac{\eta}{2})\wedge 0}[G_1(\eta) + G_2(\eta) + G_3(\eta) + G_4(\eta)] := G(\eta). \end{aligned}$$

For any $\eta \in (0,1)$, let $G(\varepsilon) = \left(\frac{G(\eta)}{\varepsilon}\right)^{\frac{1}{\eta}}$, by applying the Chebyshev's inequality, we will obtain the following results

$$\begin{cases} \mathbb{P}[\chi(t) < G(\varepsilon)] \le G(\varepsilon)^\eta \mathbb{P}[\chi(t)^{-\eta}(t)], \\ \\ \liminf_{t\to\infty}[\chi(t) \le G(\varepsilon)] \ge 1 - \varepsilon. \end{cases}$$

Nextly, based on using Chebyshev's inequality and similar arguments, one can find a constant $Q(\varepsilon) > 0$ subject to

$$\liminf_{t\to\infty}[\chi(t) \ge Q(\varepsilon)] \ge 1 - \varepsilon.$$

Taking the above mentioned discussions into consideration, one can draw a conclusion that there exists a uniform upper bound and a uniform lower bound for the solution of system (4) with any initial value $(S(0), I(0), A(0), M(0), n)$. □

Lemma 2. *For every initial value $(S(0), I(0), A(0), M(0), n) \in \mathbb{R}_+^4 \times \mathbb{N}$. When sufficient condition (3) holds, then system (4) exists a unique positive solution $(S(t), I(t), A(t), M(t), n)$ that remains in $\mathbb{R}_+^4 \times \mathbb{N}$ with probability one.*

Proof. First, based on some standard arguments and analysis, it is not difficult to show that system (4) meets the local Lipschitz conditions. Thus, system (4) exists with a unique local positive solution on $t \in [0, \tau_e)$ most likely for any initial value $(S(0), I(0), A(0), M(0), n)$, where τ_e represents the explosion time. For the sake of proving the positive solution is global, next, we will show that $\tau_e = \infty$.

Secondly, it is assumed that there exists a sufficiently large integer $N_0^* \ge 0$ subject to $(S(0), I(0), A(0), M(0))$ all on the interval $\left[\frac{1}{N_0^*}, N_0^*\right]$. For any positive integer $n \ge N_0^*$, we can construct the stopping time as below,

$$\tau_s = \inf\left\{ t \in [0, \tau_e) \middle| \begin{array}{c} \min\{S(t), I(t), A(t), M(t)\} \le \frac{1}{n}, \text{or} \\ \max\{S(t), I(t), A(t), M(t)\} \ge n \end{array} \right\}.$$

According to the mathematical properties of τ_s, it is clear that τ_e increases as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty} \tau_s$, and then we can obtain that $\tau_\infty \le \tau_e$ most likely. If $\tau_\infty = \infty$ holds most likely, it can be obtained that $\tau_e = \infty$ most likely holds, which obtains that $(S(t), I(t), A(t), M(t), n) \in \mathbb{R}_+^4 \times \mathbb{N}$ holds for all $t \ge 0$.

If $\tau_\infty = \infty$ most likely does not hold, then we can find two positive constants $\tilde{N}_0^* > 0$ and $\epsilon < 1$ subject to $\mathbb{P}\{\tau_\infty \le \tilde{N}_0^*\} \ge \epsilon$. Therefore, we can find a positive integer $N_1 > \tilde{N}_0^*$ subject to $\mathbb{P}\{\tau_s \le \tilde{N}_0^*\} \ge \epsilon$ holds for any $n > N_1$.

By utilizing a C^4- function $V : \mathbb{R}_+^4 \to \mathbb{R}_+ \cup \{0\}$ as below,

$$\begin{aligned} V(S(t), I(t), A(t), M(t)) &= S(t) - a_1 - a_1 \ln \frac{S(t)}{a_1} + I(t) - 1 - \ln I(t) \\ &\quad + A(t) - 1 + \ln A(t) + a_1(M(t) - 1 - \ln M(t)), \end{aligned}$$

where $a_1 = \frac{\alpha^l + h^l}{r^u + \beta^u}$.

Based primary on utilizing Itô's formula, we can find the following results

$$\begin{aligned} & \mathrm{d}V(S(t), I(t), A(t), M(t)) \\ = \; & \left(1 - \frac{a_1}{S(t)}\right)\left[\Lambda(t) - \beta(t)S(t)I(t) - \frac{\lambda(t)S(t)M(t)}{M(t) + p(t)}\right]\mathrm{d}t \\ & + \left(1 - \frac{a_1}{S(t)}\right)[\nu(t)I(t) + \lambda_0(t)A(t) - h(t)S(t)]\mathrm{d}t \\ & + \left(1 - \frac{1}{I(t)}\right)[(\beta(t)S(t)I(t) - (\nu(t) + \alpha(t) + h(t))I(t))]\mathrm{d}t \\ & + \left(1 - \frac{1}{A(t)}\right)\left[\lambda(t)S(t)\frac{M(t)}{M(t) + p(t)} - (\lambda_0(t) + h(t))A(t)\right]\mathrm{d}t \\ & + \left(a_1 - \frac{a_1}{M(t)}\right)\left[r(t)\left(1 - \theta\frac{A(t)}{w(t) + A(t)}\right)I(t) - r_0(t)(M(t) - M_0)\right]\mathrm{d}t \\ & + \left[\frac{a_1}{2}(\sigma_{11}(n)S(t) + \sigma_{12}(n))^2 + \frac{1}{2}(\sigma_{21}(n)I(t) + \sigma_{22}(n))^2\right]\mathrm{d}t \\ & + \left[\frac{1}{2}(\sigma_{31}(n)A(t) + \sigma_{32}(n))^2 + \frac{a_1}{2}(\sigma_{41}(n)M(t) + \sigma_{42}(n))^2\right]\mathrm{d}t \\ & + \left[a_1\int_{\mathbb{Y}}(c_1(u) - \ln(1 + c_1(u)))\rho\mathrm{d}u + \int_{\mathbb{Y}}(c_2(u) - \ln(1 + c_2(u)))\rho\mathrm{d}u\right]\mathrm{d}t \\ & + \left[\int_{\mathbb{Y}}(c_3(u) - \ln(1 + c_3(u)))\rho\mathrm{d}u + a_1\int_{\mathbb{Y}}(c_4(u) - \ln(1 + c_4(u)))\rho\mathrm{d}u\right]\mathrm{d}t \\ & + (\sigma_{11}(n)S(t) + \sigma_{12}(n))(S(t) - a_1)\mathrm{d}B_1(t) + (\sigma_{21}(n)I(t) + \sigma_{22}(n))(I(t) - 1)\mathrm{d}B_2(t) \\ & + (\sigma_{31}(n)A(t) + \sigma_{32}(n))(A(t) - 1)\mathrm{d}B_3(t) + a_1(\sigma_{41}(n)M(t) + \sigma_{42}(n))(M(t) - 1)\mathrm{d}B_4(t) \\ & + a_1\int_{\mathbb{Y}}[c_1(u)S(t) - \ln(1 + c_1(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) + \int_{\mathbb{Y}}[c_2(u)I(t) - \ln(1 + c_2(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) \\ & + \int_{\mathbb{Y}}[c_3(u)A(t) - \ln(1 + c_3(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) + a_1\int_{\mathbb{Y}}[c_4(u)M(t) - \ln(1 + c_4(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) \end{aligned}$$

Furthermore, it follows from simple computations that

$$\begin{aligned} & \mathrm{d}V(S(t), I(t), A(t), M(t)) \\ = \; & \mathcal{L}V\mathrm{d}t + (\sigma_{11}(n)S(t) + \sigma_{12}(n))(S(t) - a_1)\mathrm{d}B_1(t) + (\sigma_{21}(n)I(t) + \sigma_{22}(n))(I(t) - 1)\mathrm{d}B_2(t) \\ & + (\sigma_{31}(n)A(t) + \sigma_{32}(n))(A(t) - 1)\mathrm{d}B_3(t) + a_1(\sigma_{41}(n)S(t) + \sigma_{42}(n))(M(t) - 1)\mathrm{d}B_4(t) \\ & + a_1\int_{\mathbb{Y}}[c_1(u)S(t) - \ln(1 + c_1(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) + \int_{\mathbb{Y}}[c_2(u)I(t) - \ln(1 + c_2(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) \\ & + \int_{\mathbb{Y}}[c_3(u)A(t) - \ln(1 + c_3(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t) + a_1\int_{\mathbb{Y}}[c_4(u)M(t) - \ln(1 + c_4(u))]\widetilde{X}(\mathrm{d}u, \mathrm{d}t), \end{aligned}$$

where $\mathcal{L}V$ is defined as follows,

$$
\begin{aligned}
\mathcal{L}V \quad = \quad & \Lambda(t) - h(t)S(t) - (\alpha(t) + h(t))I(t) - h(t)A(t) + a_1 r(t)\left(1 - \theta(t)\frac{A(t)}{\omega(t) + A(t)}\right)I(t) \\
& + a_1\left[r_0(t)(M_0(t) - M(t)) - \frac{\Lambda(t)}{S(t)} + \beta(t)I(t)\right] + (a_1 - S(t))\frac{\lambda(t)M(t)}{M(t) + p(t)} \\
& - a_1\left[\frac{\nu(t)I(t)}{S(t)} + a_1\frac{\lambda_0(t)A(t)}{S(t)} - a_1 h(t)\right] - \beta(t)S(t) + \nu(t) + \alpha(t) + h(t) \\
& - a_1\left[\frac{r(t)I(t)}{M(t)} - \frac{\theta(t)r(t)I(t)}{M(t)}\frac{A(t)}{\omega(t) + A(t)} - r_0(t) + r_0(t)\frac{M_0(t)}{M(t)}\right] \\
& + \frac{a_1}{2}(\sigma_{11}(n)S(t) + \sigma_{12}(n))^2 + \frac{1}{2}(\sigma_{21}(n)I(t) + \sigma_{22}(n))^2 \\
& + \frac{1}{2}(\sigma_{31}(n)A(t) + \sigma_{32}(n))^2 + \frac{a_1}{2}(\sigma_{41}(n)M(t) + \sigma_{42}(n))^2 \\
& + \int_{\mathbb{Y}}(c_1(u) - \ln(1 + c_1(u)))\rho \mathrm{d}u + a_1\int_{\mathbb{Y}}(c_2(u) - \ln(1 + c_2(u)))\rho \mathrm{d}u \\
& + \int_{\mathbb{Y}}(c_3(u) - \ln(1 + c_3(u)))\rho \mathrm{d}u + a_1\int_{\mathbb{Y}}(c_4(u) - \ln(1 + c_4(u)))\rho \mathrm{d}u.
\end{aligned}
$$

When the condition (3) are met, we can obtain the following results based on simple computations

$$
\begin{aligned}
\mathcal{L}V \quad \leq \quad & \Lambda(t) + [a_1(r(t) + \beta(t)) - (\alpha(t) + h(t))]I(t) + a_1 r_0(t)M_0(t) + a_1\lambda(t) \\
& + a_1(h(t) + r_0(t)) + \nu(t) + \alpha(t) + 2h(t) + \lambda_0(t) + a_1\frac{r(t)\theta(t)I(t)A(t)}{M(t)(\omega(t) + A(t))} \\
& + \frac{a_1}{2}(\sigma_{11}(n)S(t) + \sigma_{12}(n))^2 + \frac{1}{2}(\sigma_{21}(n)I(t) + \sigma_{22}(n))^2 + \frac{1}{2}(\sigma_{31}(n)A(t) + \sigma_{32}(n))^2 \\
& + \frac{a_1}{2}(\sigma_{41}(n)M(t) + \sigma_{42}(n))^2 + a_1(\kappa_1 + \kappa_4) + \kappa_2 + \kappa_3.
\end{aligned}
$$

Based on the properties of parametric function and Lemma 1 of this paper, one can find that

$$
\begin{aligned}
\mathcal{L}V \quad \leq \quad & \Lambda^u + [a_1(r^u + \beta^u) - (\alpha^l + h^l)]I(t) + a_1(r_0^u M_0^u + \lambda^u + h^u) + \nu^u + \alpha^u + h^u \\
& + (\lambda_0 + h)^u + a_1 r_0^u + a_1\frac{r^u\theta^u G^2(\varepsilon)}{Q(\varepsilon)(\omega^l + Q(\varepsilon))} + \frac{a_1}{2}(\sigma_{11}G(\varepsilon) + \sigma_{12})^2 + \frac{1}{2}(\sigma_{21}G(\varepsilon) + \sigma_{22})^2 \\
& + \frac{1}{2}(\sigma_{31}G(\varepsilon) + \sigma_{32})^2 + \frac{a_1}{2}(\sigma_{41}G(\varepsilon) + \sigma_{42})^2 + a_1(\kappa_1 + \kappa_4) + \kappa_2 + \kappa_3,
\end{aligned}
$$

where $a_1 = \frac{\alpha^l + h^l}{r^u + \beta^u}$.

The rest of the discussions resemble those in [16,20]; thus, we omitted them. One can find that $\tau_\infty = \infty$, which means that the solution of (4) will not explosion in a finite time most likely. □

Lemma 3. *If a sufficient condition (3) holds, the following properties holds for the positive solution of (4) with every initial value* $(S(0), I(0), A(0), M(0), n) \in \mathbb{R}_+^4 \times \mathbb{N}$,

$$
\begin{cases}
\limsup_{t\to\infty}\langle S(t)\rangle \leq \frac{\Lambda^u}{h^l}, \quad \limsup_{t\to\infty}\langle I(t)\rangle \leq \frac{\Lambda^u}{\alpha^l + h^l}, \\
\limsup_{t\to\infty}\langle A(t)\rangle \leq \frac{\Lambda^u}{h^l}, \quad \limsup_{t\to\infty}\langle M(t)\rangle \leq \frac{r^u\Lambda^u}{r_0^l\alpha^l} + M_0^u.
\end{cases}
$$

Proof. Based on the first three formulas of system (4), we can find the results as below

$$\begin{aligned}
&\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}+\frac{A(t)-A(0)}{t}\\
\leq\ &\Lambda^u-h^l\langle S(t)\rangle-(\alpha^l+h^l)\langle I(t)\rangle-h^l\langle A(t)\rangle\\
&+\frac{1}{t}\sum_{i=1}^{3}\left[\int_0^t(\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i(t)+\int_0^t\int_{\mathbb{Y}}c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u,\mathrm{d}s)\right],
\end{aligned}$$

which reveals that

$$\begin{aligned}
h^l\langle S(t)\rangle\ \leq\ &\Lambda^u+\frac{S(0)+I(0)+A(0)-S(t)-I(t)-A(t)}{t}\\
&+\frac{1}{t}\sum_{i=1}^{3}\left[\int_0^t(\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i(t)+\int_0^t\int_{\mathbb{Y}}c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u,\mathrm{d}s)\right].
\end{aligned}$$

Hence, it is not difficult to show that

$$\begin{aligned}
\langle S(t)\rangle\ \leq\ &\frac{\Lambda^u}{h^l}+\frac{S(0)+I(0)+A(0)-S(t)-I(t)-A(t)}{h^lt}\\
&+\frac{1}{h^lt}\sum_{i=1}^{3}\left[\int_0^t(\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i(t)+\int_0^t\int_{\mathbb{Y}}c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u,\mathrm{d}s)\right],
\end{aligned}$$

which follows that

$$\langle S(t)\rangle\leq\frac{\Lambda^u}{h^l}+\frac{S(0)+I(0)+A(0)}{h^lt}+\sum_{i=1}^{3}\left(\frac{\psi_{1i}}{h^lt}+\frac{\psi_{2i}}{h^lt}\right),$$

where ψ_{1i} and ψ_{2i} $(i=1,2,3)$ will be defined as follows,

$$\begin{cases}\psi_{1i}=\int_0^t(\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i(t),\\ \psi_{2i}=\int_0^t\int_{\mathbb{Y}}c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u,\mathrm{d}s).\end{cases}$$

By using Lemma 1, Lemma 2 and exponential martingale inequalities, it can be obtained that

$$\langle\psi_{1i},\psi_{1i}\rangle=\int_0^t(\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))^2\chi_i^2(t)\mathrm{d}s,$$

and

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\langle\psi_{1i},\psi_{1i}\rangle}{t}\ &=\ \limsup_{t\to\infty}\frac{1}{t}\int_0^t(\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))^2\chi_i^2(t)\mathrm{d}s\\
&\leq\ (\sigma_{i1}^u+\sigma_{i2}^uG(\varepsilon))^2G^2(\varepsilon)\\
&<\ \infty,
\end{aligned}$$

which follows that

$$\limsup_{t\to\infty}\frac{\psi_{1i}}{t}=0, \tag{5}$$

holds for $i=1,2,3$.

Further computations show that

$$\mathbb{P}\left\{\sup_{0\leq i\leq j}\left[\psi_{2i}-\frac{1}{2}\langle\psi_{2i},\psi_{2i}\rangle\right]>2\ln j\right\}\leq\frac{1}{j^2}. \tag{6}$$

It is easy to show that we can find a random integer $j_0 = j_0(\omega)$ holds with almost all $\omega \in \Omega$. Hence, it can be concluded that

$$\sup_{0 \le i \le j} \left[\psi_{2i} - \frac{1}{2} \langle \psi_{2i}, \psi_{2i} \rangle \right] \le 2 \ln j_0 \tag{7}$$

holds for $\omega \in \Omega$ most likely, which reveals that

$$\psi_{2i} \le 2 \ln j_0 + \frac{1}{2} \langle \psi_{2i}, \psi_{2i} \rangle \tag{8}$$

holds for $i = 1, 2, 3$ and all $0 \le t \le j_0$.

Consequently, we have

$$\begin{aligned} \langle S(t) \rangle &\le \frac{\Lambda^u}{h^l} + \frac{S(0) + I(0) + A(0)}{h^l t} + \frac{2 \ln j_0}{h^l t} + \sum_{i=1}^{3} \frac{\psi_{1i}}{h^l t} \\ &\le \frac{\Lambda^u}{h^l} + \frac{S(0) + I(0) + A(0)}{h^l t} + \frac{2 \ln j_0}{h^l (j_0 - 1)} + \sum_{i=1}^{3} \frac{\psi_{1i}}{h^l t}. \end{aligned}$$

By taking the superior limit of $\langle S(t) \rangle$, for all $0 \le t \le j_0$, it yields

$$\begin{aligned} \limsup_{t \to \infty} \langle S(t) \rangle &\le \limsup_{t \to \infty} \frac{\Lambda^u}{h^l} + \limsup_{t \to \infty} \frac{S(0) + I(0) + A(0)}{h^l t} \\ &\quad + \limsup_{t \to \infty} \frac{2 \ln j_0}{h^l t} + \limsup_{t \to \infty} \sum_{i=1}^{3} \frac{\psi_{1i}}{h^l t} \\ &\le \limsup_{t \to \infty} \frac{\Lambda^u}{h^l} + \limsup_{t \to \infty} \frac{S(0) + I(0) + A(0)}{h^l t} \\ &\quad + \limsup_{t \to \infty} \frac{2 \ln j_0}{h^l (j_0 - 1)} + \limsup_{t \to \infty} \sum_{i=1}^{3} \frac{\psi_{1i}}{h^l t} \\ &\le \frac{\Lambda^u}{h^l}. \end{aligned}$$

Based on using the similar arguments and discussions mentioned above, one can find

$$\limsup_{t \to \infty} \langle I(t) \rangle < \frac{\Lambda^u}{\alpha^l + h^l}, \quad \limsup_{t \to \infty} \langle A(t) \rangle < \frac{\Lambda^u}{h^l}, \quad \limsup_{t \to \infty} \langle M(t) \rangle < \frac{r^u \Lambda^u}{r_0^l \alpha^l} + M_0^u,$$

and the proofs are omitted here. Hence, we can draw the next conclusions

$$\begin{cases} \limsup_{t \to \infty} \langle S(t) \rangle \le \frac{\Lambda^u}{h^l}, & \limsup_{t \to \infty} \langle I(t) \rangle \le \frac{\Lambda^u}{\alpha^l + h^l}, \\ \limsup_{t \to \infty} \langle A(t) \rangle \le \frac{\Lambda^u}{h^l}, & \limsup_{t \to \infty} \langle M(t) \rangle \le \frac{r^u \Lambda^u}{r_0^l \alpha^l} + M_0^u. \end{cases}$$

This proof is ending. □

Theorem 1. *When $R_1 > 0$ and $R_2 > 0$ holds, there exists a nontrival positive T-periodic solution of system (4), where R_i $(i = 1, 2)$ will be constructed as below:*

$$\begin{cases} R_1 = \left\langle \frac{\beta(t)\Lambda(t)}{\nu(t) + \alpha(t) + h(t) + \zeta_2(u) + \frac{1}{2} \sum_{n=1}^{K} \phi_n \sigma_{21}^2(n)} - \frac{2(\lambda(t) + h(t) + l_2(t)\Lambda(t) + \zeta_1(u)) + \sum_{n=1}^{K} \phi_n \sigma_{11}^2(n)}{2} \right\rangle, \\ R_2 = h(t) - (\sigma_{12}^2(n) \vee \sigma_{22}^2(n) \vee \sigma_{32}^2(n)). \end{cases} \tag{9}$$

and $\zeta_i(u)$, $l_i(t)$ $(i = 1, 2)$ will be constructed as below,

$$\begin{cases} \zeta_i(u) = \int_{\mathbb{Y}} (c_i(u) - \ln(1 + c_i(u)))\rho du, \\ l_1(t) = \frac{\beta(t)\Lambda(t)}{[\nu(t)+\alpha(t)+h(t)+\zeta_1(u)+\frac{1}{2}\sum_{n=1}^{K}\phi_n\sigma_{21}^2(n)]^2}, \\ l_2(t) = \frac{\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)\}}{h(t)}. \end{cases} \tag{10}$$

Furthermore, $\vec{\omega}(n)$ is assumed to be a twice continuously differentiable function that characterizes a Markov process and its Itô's derivative is defined as follows:

$$\mathcal{L}\vec{\omega}(n) = \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n). \tag{11}$$

Proof. First, we define $U_1(t)$ as follows

$$U_1(t) = -\ln S(t) - l_1(t)\ln I(t) + l_2(t)(S(t) + I(t) + A(t)) + \vec{\omega}(n).$$

Based primary on utilizing Itô's formula, we can find the results as below

$$\begin{aligned} \mathcal{L}U_1(t) \quad = \quad & -\frac{\Lambda(t) + \nu(t)I(t) + \lambda_0(t)A(t)}{S(t)} + \beta(t)I(t) + \frac{\lambda(t)M(t)}{M(t) + p(t)} \\ & + h(t) + l_1(t)[-\beta(t)S(t) + \nu(t) + \alpha(t) + h(t)] \\ & + \frac{1}{2}[\sigma_{11}(n) + \sigma_{12}(n)S(t)]^2 + \frac{l_1(t)}{2}[\sigma_{21}(n) + \sigma_{22}(n)I(t)]^2 \\ & + \int_{\mathbb{Y}} (c_1(u) - \ln(1 + c_1(u))\rho \mathrm{d}u + l_1(t)\int_{\mathbb{Y}} (c_2(u) - \ln(1 + c_2(u)))\rho \mathrm{d}u \\ & + \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n) + l_2(t)\Lambda(t) - l_2(t)[h(t)(S(t) + I(t) + A(t)) - \alpha(t)I(t)] \\ \leq \quad & -2\sqrt{l_1(t)\beta(t)\Lambda(t)} + l_1(t)\left(\nu(t) + \alpha(t) + h(t) + \frac{1}{2}\sigma_{21}^2(n) + \zeta_2(u)\right) \\ & + \lambda(t) + h(t) + \frac{1}{2}\sigma_{11}^2(n) + \zeta_1(u) + l_2(t)[\Lambda(t) - h(t)(S(t) + I(t))] \\ & + \max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)\}(S(t) + I(t)) + \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n) \\ & + \beta(t)I(t) + \max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t) + I^2(t)). \end{aligned} \tag{12}$$

According to irreducibility property of n, for $(\sigma_{11}^2(1), \sigma_{11}^2(2), \ldots, \sigma_{11}^2(K))$, there exists a functional vector $\vec{\omega}(n) = (\omega(1), \omega(2), \ldots, \omega(K))$, and $\vec{\omega}(n)$ has been mentioned in (11),

$$\frac{1}{2}\sigma_{11}^2(n) + \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n) = \frac{1}{2}\sum_{n=1}^{K} \phi_n\sigma_{11}^2(n).$$

By using similar arguments, for $(\sigma_{21}^2(1), \sigma_{21}^2(2), \ldots, \sigma_{21}^2(K))$, we have

$$\frac{1}{2}\sigma_{21}^2(n) + \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n) = \frac{1}{2}\sum_{n=1}^{K} \phi_n\sigma_{21}^2(n),$$

where $n \in \mathbb{N}$ and $\mu_{nj} > 0$ depicts the rate that switch from state n to state j.

Based on the above analysis, $R^s(t)$ is constructed as below,

$$\begin{aligned} R^s(t) \quad := \quad & \frac{\beta(t)\Lambda(t)}{\nu(t)+\alpha(t)+h(t)+\zeta_2(u)+\frac{1}{2}\sum_{n=1}^{K}\phi_n\sigma_{21}^2(n)} \\ & -\lambda(t)-h(t)-\zeta_1(u)-l_2(t)\Lambda(t)-\frac{1}{2}\sum_{n=1}^{K}\phi_n\sigma_{11}^2(n), \end{aligned}$$

one can obtain the following results

$$\begin{aligned} \mathcal{L}U_1(t) \quad \leq \quad & -R^s(t)+\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) \\ & +\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)). \end{aligned} \tag{13}$$

In the following part, we construct a T-periodic function as follows,

$$\Phi(t) = -\int_0^t (R_1 - R^s(\tau))\mathrm{d}\tau,$$

where $R_1 = \langle R^s \rangle_T$ is construct as below

$$\begin{aligned} R_1 = \langle R^s \rangle_T \quad = \quad & \Bigg\langle \frac{\beta(t)\Lambda(t)}{\nu(t)+\alpha(t)+h(t)+\zeta_2(u)+\frac{1}{2}\sum_{n=1}^{K}\phi_n\sigma_{21}^2(n)} \\ & -\lambda(t)-h(t)-\zeta_1(u)-l_2(t)\Lambda(t)-\frac{1}{2}\sum_{n=1}^{K}\phi_n\sigma_{11}^2(n)\Bigg\rangle. \end{aligned}$$

Based on some simple computations, we can find the following results

$$\begin{aligned} \mathcal{L}(U_1(t)+\Phi(t)) \quad \leq \quad & -R_1+\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) \\ & +\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)). \end{aligned} \tag{14}$$

Secondly, we define $U_2(t)$ as follows

$$U_2(t) = S(t)+I(t)+A(t)+\frac{1}{S(t)+I(t)+A(t)}.$$

By using Itô formula, it yields that

$$\begin{aligned} \mathcal{L}U_2(t) \quad = \quad & \Lambda(t)-h(t)U_1(t)-\alpha I(t)-\frac{\Lambda(t)-h(t)(S(t)+I(t)+A(t))-\alpha I(t)}{(S(t)+I(t)+A(t))^2} \\ & +\frac{S^2(t)(\sigma_{11}(n)+\sigma_{12}(n)S(t))^2+I^2(t)(\sigma_{21}(n)+\sigma_{22}(n)I(t))^2}{(S(t)+I(t)+A(t))^3} \\ & +\frac{A^2(t)(\sigma_{31}(n)+\sigma_{32}(n)A(t))^2}{(S(t)+I(t)+A(t))^3}+\frac{\int_{\mathbb{Y}}\left(\frac{1}{c_{\min}(u)}-1+c_{\max}(u)\right)\rho\mathrm{d}u}{S(t)+I(t)+A(t)} \\ \leq \quad & -\Big[h(t)-(\sigma_{12}^2(n)\vee\sigma_{22}^2(n)\vee\sigma_{32}^2(n))\Big]\left[S(t)+I(t)+A(t)+\frac{1}{S(t)+I(t)+A(t)}\right] \\ & +\Lambda(t)-\frac{\Lambda(t)}{(S(t)+I(t)+A(t))^2}+\frac{2h(t)+\alpha(t)+\int_{\mathbb{Y}}\left(\frac{1}{c_{\min}(u)}-1+c_{\max}(u)\right)\rho\mathrm{d}u}{S(t)+I(t)+A(t)} \\ & +(\sigma_{11}^2(n)\vee\sigma_{21}^2(n)\vee\sigma_{31}^2(n))\frac{1}{S(t)+I(t)+A(t)} \\ & +2[\sigma_{11}(n)\sigma_{12}(n)\vee\sigma_{21}(n)\sigma_{22}(n)\vee\sigma_{31}(n)\sigma_{32}(n)]. \end{aligned}$$

Hence, one can find the following results

$$\mathcal{L}U_2(t) \leq -R_2\left[S(t)+I(t)+A(t)+\frac{1}{S(t)+I(t)+A(t)}\right]+W_1(t), \tag{15}$$

where R_2 is defined in (9) and

$$\begin{cases} c_{\min}(u)=\min\{c_1(u),c_2(u),c_3(u)\}, \\ c_{\max}(u)=\max\{c_1(u),c_2(u),c_3(u)\}, \end{cases} \tag{16}$$

and

$$\begin{aligned} W_1(t) &= \Lambda(t)+\frac{\left[2h(t)+\alpha(t)+\int_{\mathbb{Y}}\left(\frac{1}{c_{\min}(u)}-1+c_{\max}(u)\right)\rho\mathrm{d}u\right]^2}{2\Lambda(t)} \\ &\quad +2(\sigma_{11}(n)\sigma_{12}(n)\vee\sigma_{21}(n)\sigma_{22}(n)\vee\sigma_{31}(n)\sigma_{32}(n)) \\ &\quad +\frac{\left[(\sigma_{11}^2(n)\vee\sigma_{21}^2(n)\vee\sigma_{31}^2(n))-(\sigma_{12}^2(n)\vee\sigma_{22}^2(n)\vee\sigma_{32}^2(n))\right]^2}{2\Lambda(t)}. \end{aligned}$$

Based on the boundedness of the parametric functions, the following results can be obtained that

$$\begin{aligned} W_1(t)\leq\overline{W}_1 &= \Lambda^u+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u) \\ &\quad +\frac{\left[2h^u+\alpha^u+\int_{\mathbb{Y}}\left(\frac{1}{c_{\min}^l(u)}-1+c_{\max}^u(u)\right)\rho\mathrm{d}u\right]^2}{2\Lambda^l} \\ &\quad +\frac{\left[((\sigma_{11}^2)^u\vee(\sigma_{21}^2)^u\vee(\sigma_{31}^2)^u)-((\sigma_{12}^2)^l\vee(\sigma_{22}^2)^l\vee(\sigma_{32}^2)^l)\right]^2}{2\Lambda^l}, \end{aligned} \tag{17}$$

where $\overline{W}_1$ represents the supreme of $W_1(t)$.

Thirdly, for any constant $\xi\in(0,1)$, $U_3(t)$ is defined as below,

$$U_3(t) = \Sigma_{i=1}^4(1+\chi_i(t))^\xi+l_3(t)(\Sigma_{i=1}^3\chi_i(t)),$$

For the convenience of proof, $F_i(t)$ $(i=1,\cdots,4)$ are constructed as below,

$$F_i(t)=-\frac{\xi(1-\xi)(1+\chi_i(t))^\xi\chi_i^2(t)}{2}(\sigma_{i1}(n)\wedge\sigma_{i2}(n))^2-\int_{\mathbb{Y}}(1+\chi_i(t))^\xi\rho\mathrm{d}u.$$

where $\chi(t)=(\chi_1(t),\chi_2(t),\chi_3(t),\chi_4(t))=(S(t),I(t),A(t),M(t))$.

Based on utilizing the simple computations, one can be yield the results as below

$$\begin{aligned} \mathcal{L}U_3(t) &\leq [\max\{\xi\nu^u,\xi\lambda_0^u+\xi\lambda^u,\xi r^u\}-l_3(t)h(t)](\Sigma_{i=1}^3\chi_i(t)) \\ &\quad +W_2(t)+\Sigma_{i=1}^4F_i(t), \end{aligned} \tag{18}$$

where $l_3(t)$ and $W_2(t)$ are defined as follows,

$$l_3(t)=\frac{\max\{\xi\nu^u,\xi\lambda_0^u+\xi\lambda^u,\xi r^u\}}{h(t)},$$

$$\begin{aligned} W_2(t) &= \xi\Lambda^u + l_3(t)\Lambda^u - \xi(1+S(t))^{\xi-1}\left[\beta^l S(t)I(t) + \lambda^l \frac{S(t)M(t)}{M(t)+p^u} + h^l S(t)\right]. \\ &\quad -\xi(1+I(t))^{\xi-1}(\nu^l+\alpha^l+h^l)I(t) - \xi(1+A(t))^{\xi-1}(\lambda_0^l+h^l)A(t) \\ &\quad +\xi\beta^u S(t)I(t) - \xi(1+M(t))^{\xi-1}r_0^l M(t) + \xi r_0^u M_0^u \\ &\quad + \sum_{i=1}^{4}\int_{\mathbb{Y}}\left[(1+c_i(u)\chi_i(t)+\chi_i(t))^{\xi} - \xi(1+\chi_i(t))^{\xi-1}c_i(u)\chi_i(t)\right]\rho \mathrm{d}u. \end{aligned}$$

It is not difficult to show $W_2(t)$ is continuous in $(0,+\infty)$ and it follows from Lemma 1 that

$$\begin{aligned} W_2(t) \le \overline{W}_2 &= (\xi + l_3^u)\Lambda^u - \xi(1+G(\varepsilon))^{\xi-1}\left[\beta^l Q(\varepsilon) + \lambda^l \frac{Q(\varepsilon)}{G(\varepsilon)+p^u} + h^l Q(\varepsilon)\right] \\ &\quad -\xi(1+G(\varepsilon))^{\xi-1}(\nu^l+\alpha^l+\lambda_0^l+r_0^l+2h^l)Q(\varepsilon) + \xi(\beta^u G(\varepsilon) + r_0^u M_0^u) \\ &\quad + \sum_{i=1}^{4}\int_{\mathbb{Y}}\left[(1+c_i(u)G(\varepsilon)+G(\varepsilon))^{\xi} - \xi(1+Q(\varepsilon))^{\xi-1}c_i(u)Q(\varepsilon)\right]\rho \mathrm{d}u, \end{aligned} \tag{19}$$

where $\overline{W}_2$ represents the supreme of $W_2(t)$.

Finally, we define $U(t)$ as follows,

$$U(t) = \Theta(U_1(t) + \Phi(t)) + U_2(t) + U_3(t),$$

where Θ is a sufficient large positive constant such that for $\chi_i(t) \to 0^+$ $(i = 1, \cdots, 4)$

$$-\Theta R_1 + \overline{W}_1 + \overline{W}_2 + \sup\sum_{i=1}^{4} F_i(t) < -2. \tag{20}$$

A continuous function $U(t)$ will be defined as below, and there exists a minimum $U(S_0, I_0, A_0, M_0, n_0)$ around $(S_0, I_0, A_0, M_0, n_0)$ when $U(t)$ tends to ∞.

Hence, we formulate a non-negative function as follows,

$$\widetilde{U}(t) = U(S(t), I(t), A(t), M(t), n) - U_0(S_0, I_0, A_0, M_0, n_0).$$

By using (14), (15) and (18), one can be yielded that

$$\begin{aligned} \mathcal{L}\widetilde{U}(t) &= \mathcal{L}U(S(t), I(t), A(t), M(t), n) - \mathcal{L}U(S_0, I_0, A_0, M_0, n_0) \\ &\le -\Theta R_1 + \Theta\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) \\ &\quad +\Theta\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)) \\ &\quad -\frac{R_2[1+(\Sigma_{i=1}^3\chi_i(t))^2]}{\Sigma_{i=1}^3\chi_i(t)} + \overline{W}_1 + \overline{W}_2 + \Sigma_{i=1}^4 F_i(t), \end{aligned}$$

where $\overline{W}_1$ and $\overline{W}_2$ have been defined in (17) and (19).

When $\chi_i(t) \to 0$ or $\chi_i(t) \to \infty$, if $R_2 > 0$, one can find that

$$-\frac{R_2[1+(\Sigma_{i=1}^3\chi_i(t))^2]}{\Sigma_{i=1}^3\chi_i(t)} \to -\infty. \tag{21}$$

Based on Lemma 1, when $\chi_i(t) \to 0$, it yields that

$$\begin{aligned} &\Theta\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) \\ &+\Theta\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)) \\ \le\ &2\Theta\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}G^2(\varepsilon) \\ &+2\Theta\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}G(\varepsilon). \end{aligned} \tag{22}$$

By using (20)–(22), if $R_1 > 0$ hold, when $\chi_i(t) \to 0$, it gives that

$$\begin{aligned}\mathcal{L}\widetilde{U}(t) &\leq -\frac{R_2[1+(\Sigma_{i=1}^3\chi_i(t))^2]}{\Sigma_{i=1}^3\chi_i(t)} - \Theta R_1 + \overline{W}_1 + \overline{W}_2 \\ &\quad +\Theta\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) + \sup\Sigma_{i=1}^4 F_i(t) \\ &\quad +\Theta\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)) \\ &\leq -1. \end{aligned} \tag{23}$$

When $\chi_i(t) \to \infty$ $(i = 1, \cdots, 4)$, it is easy to show that

$$\widetilde{\Theta} \to -\infty, \tag{24}$$

where $\widetilde{\Theta}$ is constructed as follows,

$$\begin{aligned}\widetilde{\Theta} &= \Theta\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)) \\ &\quad +\Theta\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) - \int_{\mathbb{Y}}(1+\chi_i(t))^{\xi}\rho \mathrm{d}u \\ &\quad -\frac{\xi(1-\xi)(1+\chi_i(t))^{\xi}\chi_i^2(t)}{2}(\sigma_{i1}(n)\wedge\sigma_{i2}(n))^2. \end{aligned} \tag{25}$$

Furthermore, if $R_1 > 0$ and $R_2 > 0$ hold, it follows from (21) and (24), it yields that

$$\begin{aligned}\mathcal{L}\widetilde{U}(t) &\leq -\frac{R_2[1+(\Sigma_{i=1}^3\chi_i(t))^2]}{\Sigma_{i=1}^3\chi_i(t)} - \Theta R_1 + \overline{W}_1 + \overline{W}_2 \\ &\quad -\frac{\xi(1-\xi)(1+\chi_i(t))^{\xi}\chi_i^2(t)}{2}(\sigma_{i1}(n)\wedge\sigma_{i2}(n))^2 - \int_{\mathbb{Y}}(1+\chi_i(t))^{\xi}\rho \mathrm{d}u \\ &\quad +\Theta\max\{\sigma_{11}(n)\sigma_{12}(n), l_1(t)\sigma_{21}(n)\sigma_{22}(n)+\beta(t)\}(S(t)+I(t)) \\ &\quad +\Theta\max\{\sigma_{12}^2(n), l_1(t)\sigma_{22}^2(n)\}(S^2(t)+I^2(t)) \\ &\leq -\infty - \Theta R_1 + \overline{W}_1 + \overline{W}_2 - \infty \\ &< -1, \end{aligned} \tag{26}$$

where $\overline{W}_1$ and $\overline{W}_2$ have been defined in (17) and (19).

According to Lemma 1 and (26), we can find following results

(i) system (4) exists a unique global solution;

(ii) we can find a T-periodic function $\widetilde{U}(t) \in C^1 \times \mathbb{N}$ and $\mathcal{L}\widetilde{U}(t) < -1$ on the outside of some compact set.

Hence, sufficient condition (i) and condition (ii) in Theorem 3.8 [40] all hold, which means that system (4) exists a nontrival positive T-periodic solution.

The proof is ending. □

Theorem 2. *When $\widetilde{R}^s > 0$ holds, the solution of system (4) is f-exponentially ergodic, where $\widetilde{R}^s = \sum_{n=1}^K \phi_n \widetilde{R}_n$, and $\widetilde{R}_n$ $(n = 1, 2, \cdots, K)$ are defined as follows,*

$$\begin{aligned}\widetilde{R}_n &= \beta^l Q(\varepsilon) - (\nu^u + \alpha^u + h^u) - \frac{(\vartheta+1)(\sigma_{21}^u(n)+\sigma_{22}^u(n)G(\varepsilon))^2}{2} \\ &\quad -\int_{\mathbb{Y}}\left[\frac{(1+c_2(u))^{-\vartheta}-1}{\vartheta} - c_2(u)\right]\rho du, \quad (0<\vartheta<1). \end{aligned} \tag{27}$$

Proof. For the diffusion matrix form of system (4), we have

$$\begin{cases} D_{\min} \parallel \chi(t) \parallel^2 \leq \sum_{i=1}^4 \left[\sigma_{i1}(n)\chi_i(t) + \sigma_{i2}(n)\chi_i^2(t)\right]^2 \chi_i^2(t), \\ \sum_{i=1}^4 \left[\sigma_{i1}(n)\chi_i(t) + \sigma_{i2}(n)\chi_i^2(t)\right]^2 \chi_i^2(t) \leq D_{\max} \parallel \chi(t) \parallel^2, \end{cases} \tag{28}$$

where $D_{\min}$ and $D_{\max}$ are defined as follows,

$$\begin{cases} D_{\min} = \min\left\{\sum_{i=1}^{4}\left[\sigma_{i1}(n)\chi_i(t)+\sigma_{i2}(n)\chi_i^2(t)\right]^2\chi_i^2(t)\right\}, \\ D_{\max} = \max\left\{\sum_{i=1}^{4}\left[\sigma_{i1}(n)\chi_i(t)+\sigma_{i2}(n)\chi_i^2(t)\right]^2\chi_i^2(t)\right\}. \end{cases}$$

and $\chi(t) = (\chi_1(t), \cdots, \chi_4(t)) = (S(t), I(t), A(t), M(t))$.

It follows from (28) that uniform elliptic conditions hold for the diffusion matrix of system (4). Furthermore, the diffusion of initial value $(S(t), I(0), A(0), M(0), n)$ transition probability exists a positive smooth density on $\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{N}$.

Furthermore, according to Lemma 2 in [41], for the following linear equation,

$$(\widetilde{R}_1, \cdots, \widetilde{R}_K)^T - \Gamma\chi(t) = (\widetilde{R}^s, \cdots, \widetilde{R}^s)^T, \tag{29}$$

where $\widetilde{R}^s = \sum_{n=1}^{K}\phi_n\widetilde{R}_n$. It follows from simple computations, we can find a unique positive solution $(\varpi_1, \cdots, \varpi_K)^T$ of Equation (29).

If $\widetilde{R}_n > 0$ $(n = 1, 2, \cdots, K)$ hold, it is easy to show that $\widetilde{R}^s = \sum_{n=1}^{K}\phi_n\widetilde{R}_n > 0$, where $\sum_{n=1}^{K}\phi_n = 1$ and $\phi_n > 0$ hold for any $n \in \mathbb{N}$.

Based on the above analysis, we define $U_4(t)$ and $U_5(t)$ as follows,

$$\begin{cases} U_4(t) = \Sigma_{i=1}^{3}\chi_i(t) + \frac{\alpha^l}{r^u}M(t), \\ U_5(t) = (1+\vartheta\varpi_n)I(t)^{-\vartheta}. \end{cases}$$

Based primary on utilizing the Itô formula to system (4), it yields that

$$\begin{aligned} \mathcal{L}U_4(t) &= \Lambda^u - h^l(\Sigma_{i=1}^{3}\chi_i(t)) - \alpha^l I(t) \\ &\quad + \frac{\alpha^l}{r^u}\left[r^u I(t)\left(1-\theta^l\frac{A(t)}{\omega^u + A(t)}\right) - r_0^l M(t) + r_0^u M_0^u\right]. \end{aligned} \tag{30}$$

and

$$\begin{aligned} \mathcal{L}U_5(t) &= \vartheta(1+\vartheta\varpi_n)I(t)^{-\vartheta}[\nu(t)+\alpha(t)+h(t)-\beta(t)S(t)] \\ &\quad + \frac{\vartheta(1+\vartheta\varpi_n)I(t)^{-\vartheta}(\vartheta+1)(\sigma_{21}(n)+\sigma_{22}(n)I(t))^2}{2} \\ &\quad + I(t)^{-\vartheta}\left\{\vartheta\sum_{n=1,j=1}\mu_{nj}\varpi_n + (1+\vartheta\varpi_n)\int_{\mathbb{Y}}[(1+c_2(u))^{-\vartheta} - 1 + \vartheta c_2(u)]\rho\mathrm{d}u\right\} \\ &= \vartheta I(t)^{-\vartheta}(1+\vartheta\varpi_n)[-\beta(t)S(t)+\nu(t)+\alpha(t)+h(t)] \\ &\quad + \vartheta I(t)^{-\vartheta}(1+\vartheta\varpi_n)\frac{(\vartheta+1)(\sigma_{21}(n)+\sigma_{22}(n)I(t))^2}{2} \\ &\quad + \vartheta I(t)^{-\vartheta}\sum_{n=1,j=1}\mu_{nj}\varpi_n + \int_{\mathbb{Y}}\left[\frac{(1+c_2(u))^{-\vartheta}-1}{\vartheta} + c_2(u)\right]\rho\mathrm{d}u \\ &= \vartheta I(t)^{-\vartheta}\sum_{n=1,j=1}\mu_{nj}\varpi_n - (1+\vartheta\varpi_n)[\beta(t)S(t) - (\nu(t)+\alpha(t)+h(t)] \\ &\quad - \vartheta I(t)^{-\vartheta}\left\{\frac{\vartheta+1}{2}[\sigma_{21}(n)+\sigma_{22}(n)I(t)]^2 + \int_{\mathbb{Y}}\left[\frac{(1+c_2(u))^{-\vartheta}-1}{\vartheta} + c_2(u)\right]\rho\mathrm{d}u\right\}. \end{aligned}$$

If (29) holds, then it is easy to show that $\sum_{n=1,j=1}\mu_{nj}\varpi_n = \widetilde{R_n} - \widetilde{R^s}$, which yields that

$$
\begin{aligned}
\mathcal{L}U_5(t) &= \vartheta I(t)^{-\vartheta}\left\{\sum_{n=1,j=1}\mu_{nj}\varpi_n-(1+\vartheta\varpi_n)[\beta(t)S(t)-(\nu(t)+\alpha(t)+h(t))]\right\}\\
&\quad -\vartheta I(t)^{-\vartheta}\left\{\frac{\vartheta+1}{2}[\sigma_{21}(n)+\sigma_{22}(n)I(t)]^2+\int_{\mathbb{Y}}\left[\frac{(1+c_2(u))^{-\vartheta}-1}{\vartheta}+c_2(u)\right]\rho\mathrm{d}u\right\}\\
&\le \vartheta I(t)^{-\vartheta}\left\{\widetilde{R_n}-\widetilde{R^s}-(1+\vartheta\varpi_n)\left[\beta^l Q(\varepsilon)-(\nu^u+\alpha^u+h^u)\right]\right\}\\
&\quad +\frac{(\vartheta+1)\vartheta I(t)^{-\vartheta}(1+\vartheta\varpi_n)}{2}[\sigma_{21}(n)+\sigma_{22}(n)G(\varepsilon)]^2\\
&\quad +\vartheta I(t)^{-\vartheta}(1+\vartheta\varpi_n)\int_{\mathbb{Y}}\left[\frac{(1+c_2(u))^{-\vartheta}-1}{\vartheta}+c_2(u)\right]\rho\mathrm{d}u\\
&\le \vartheta I(t)^{-\vartheta}\left[\widetilde{R_n}-\widetilde{R^s}-(1+\vartheta\varpi_n)\widetilde{R_n}\right]\\
&= \vartheta I(t)^{-\vartheta}[-\vartheta\varpi_n\widetilde{R_n}-\widetilde{R^s}].
\end{aligned}
\tag{31}
$$

By using (30) and (31), it can be obtained that

$$
\begin{aligned}
\mathcal{L}[U_4(t)+U_5(t)] &= \mathcal{L}\left[\Sigma_{i=1}^3\chi_i(t)+\frac{\alpha^l}{r^u}M(t)+(1+\vartheta\varpi_n)I(t)^{-\vartheta}\right]\\
&\le -\widetilde{M}\left[\Sigma_{i=1}^3\chi_i(t)+\frac{\alpha^l}{r^u}M(t)+(1+\vartheta\varpi_n)I(t)^{-\vartheta}\right]+\Lambda^u+\frac{r_0^u\alpha^l}{r^u}\\
&= -\widetilde{M}[U_4(t)+U_5(t)]+\Lambda^u+\frac{r_0^u\alpha^l}{r^u},
\end{aligned}
$$

where $\widetilde{M}=\min\left\{h^l,r_0^l,\frac{K(K\varpi_n\widetilde{R}_n+\widetilde{R^s})}{1+K\varpi_n}\right\}$ $(n=1,2,\cdots,K)$.

According to Theorem 6.1 in [42] and Theorem 6.3 in [43], all the sufficient conditions for existence of exponential ergodicity hold.

Hence, based on the above analysis, if $\widetilde{R}^s>0$, the positive solution of system (4) is f-exponentially ergodic.

The proof is ending. □

Remark 2. *Let $\mathbb{P}(t,(\chi(t),n),\cdot)$ depict the transition probability of $(\chi(t),n)$. According to Theorem 2 of this paper, for some positive constant $\delta\in(0,1)$, it can be found that $(\chi(t),n)$ is considered to be f-exponentially ergodic if there exists a probability measure $\pi(\cdot)$ and a finite-valued function $v(\chi(t),n)$ such that*

$$\|\mathbb{P}(t,(\chi(t),n),\cdot)-\pi(\cdot)\|\le v(\chi(t),n)\delta^t,$$

holds for all $t\ge 0$ and $(\chi(t),n)\in\mathbb{R}_+^4\times\mathbb{N}$.

In the next part, we will concentrate on hybrid dynamic impacts of random perturbations and media coverage on the variations of epidemic transmission.

Theorem 3. *For the infected individual $I(t)$ of system (4),*

(i) if $R_I<1$ and R_I is defined in (32),

$$R_I=\frac{\beta^u\Lambda^u}{h^l(\nu^l+\alpha^l+h^l+\zeta_2^l)}+\frac{\sigma_2^u\Lambda^u}{(\alpha^l+h^l)(\nu^l+\alpha^l+h^l+\zeta_2^l)},\tag{32}$$

then the number of infected individual $I(t)$ of system (4) satisfies

$$\lim_{t\to\infty}I(t)=0,$$

which means infected individual tends to zero exponentially;

(ii) *if $R_E > 0$ and R_E is defined in (33),*

$$R_E = \frac{\Lambda^u(\beta^l + h^l)}{\lambda^u + h^u + \zeta_1^u + \frac{1}{2}\sum_{n=1}^{K}\phi_n(\sigma_{11}^u + \sigma_{12}^u G(\varepsilon))^2} - \left[r_0^u M_0^u + \nu^u + \alpha^u + h^u + \frac{1}{2}(\sigma_{21}^u + \sigma_{22}^u G(\varepsilon))^2 + \zeta_2^u\right], \tag{33}$$

then the number of infected individual $I(t)$ of system (4) meets

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds > 0,$$

which means infected individual will be persistent in the average sense.

Proof. (i) Based on applying Itô's formula to system (4), we can obtain the results as below

$$\begin{aligned} \mathrm{d}\ln I(t) &= [\beta(t)S(t) - (\nu(t) + \alpha(t) + h(t))]\mathrm{d}t - \frac{(\sigma_{21}(n) + \sigma_{22}(n)I(t))^2}{2}\mathrm{d}t \\ &\quad + \left[\int_{\mathbb{Y}}[\ln(1 + c_2(u)) - c_2(u)]\rho\mathrm{d}u\right]\mathrm{d}t + (\sigma_{21}(n) + \sigma_{22}(n)I(t))\mathrm{d}B_2(t) \\ &\quad + \int_{\mathbb{Y}} \ln(1 + c_2(u))\widetilde{X}(\mathrm{d}t, \mathrm{d}u). \end{aligned}$$

Based primary on integrating from 0 to t among both sides of the above equation, the following results can be yielded

$$\begin{aligned} &\ln I(t) - \ln I(0) \\ = &\int_0^t [\beta(s)S(s) - (\nu(s) + \alpha(s) + h(s))]\mathrm{d}s \\ &- \int_0^t \frac{(\sigma_{21}(n) + \sigma_{22}(n)I(s))^2}{2}\mathrm{d}s + \int_0^t \left[\int_{\mathbb{Y}}(\ln(1 + c_2(u)) - c_2(u))\rho\mathrm{d}u\right]\mathrm{d}s \\ &+ \int_0^t (\sigma_{21}(n) + \sigma_{22}(n)I(s))\mathrm{d}B_2(t) + \int_0^t\int_{\mathbb{Y}} \ln(1 + c_2(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}s). \end{aligned}$$

Further computations show

$$\begin{aligned} &\frac{\ln I(t) - \ln I(0)}{t} \\ \le\ &\beta^u\langle S(t)\rangle - (\nu^l + \alpha^l + h^l) - \sigma_{21}(n)\sigma_{22}(n)\langle I(t)\rangle - \zeta_2^l \\ &+ \frac{\int_0^t(\sigma_{21}(n) + \sigma_{22}(n)I(t))\mathrm{d}B_2(t)}{t} + \frac{\int_0^t\int_{\mathbb{Y}}\ln(1 + c_2(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}s)}{t} \\ \le\ &\beta^u\frac{\Lambda^u}{h^l} + \widetilde{\sigma}^*\frac{\Lambda^u}{(\alpha^l + h^l)} - (\nu^l + \alpha^l + h^l) - \zeta_2^l \\ &+ \frac{\int_0^t(\sigma_{21}(n) + \sigma_{22}(n)I(t))\mathrm{d}B_2(t)}{t} + \frac{\int_0^t\int_{\mathbb{Y}}\ln(1 + c_2(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}s)}{t}, \end{aligned} \tag{34}$$

where $\widetilde{\sigma}^* = \max\{-\sigma_{21}(n)\sigma_{22}(n)\}$.

By using the mathematical properties of white noise, it is not difficult to show that $\widetilde{\sigma}^* \geq 0$. For $\int_0^t(\sigma_{21}(n)+\sigma_{22}(n)I(t))\mathrm{d}B_2(t)$, based on Lemma 1 (the boundedness of $I(t)$) and exponential martingale inequality from Lemma 3 that

$$\begin{aligned}
& \limsup_{t\to\infty} \frac{\left\langle \int_0^t(\sigma_{21}(n)+\sigma_{22}(n)I(t))\mathrm{d}B_2(t), \int_0^t(\sigma_{21}(n)+\sigma_{22}(n)I(t))\mathrm{d}B_2(t)\right\rangle}{t} \\
= & \limsup_{t\to\infty} \frac{1}{t}\int_0^t [(\sigma_{21}(n)+\sigma_{22}(n)I(t))I(t)]^2\mathrm{d}s \\
< & (\sigma_{21}^u+\sigma_{22}^u G(\varepsilon))^2 G(\varepsilon)^2 \\
< & \infty.
\end{aligned} \tag{35}$$

Hence, it can be concluded that

$$\lim_{t\to\infty} \frac{\int_0^t(\sigma_{21}(n)+\sigma_{22}(n)I(t))\mathrm{d}B_2(t)}{t} = 0. \tag{36}$$

Let $\psi_3(t) = \int_0^t\int_{\mathbb{Y}} \ln(1+c_2(u))\widetilde{X}(\mathrm{d}u,\mathrm{d}s)$, by applying the exponential martingales inequality, it follows from similar arguments in Lemma 3 that

$$\mathbb{P}\left\{\sup_{0\leq i\leq j}\left[\psi_3 - \frac{1}{2}\langle\psi_3,\psi_3\rangle\right] > 2\ln j\right\} \leq \frac{1}{j^2}. \tag{37}$$

We can easily to find a random integer $j_0^* = j_0^*(\omega)$ holds for the almost whole $\omega\in\Omega$, and it can be obtained that

$$\sup_{0\leq i\leq j_0^*}\left[\psi_3 - \frac{1}{2}\langle\psi_3,\psi_3\rangle\right] \leq 2\ln j_0^* \tag{38}$$

holds for $\omega\in\Omega$ most likely.

Hence, it can be obtained that

$$\psi_3 \leq 2\ln j_0^* + \frac{1}{2}\langle\psi_3,\psi_3\rangle \tag{39}$$

holds for all $0\leq t\leq j_0^*$.

By taking the superior limit for (34), if $R_I<1$ holds, then it yields that

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\ln I(t)-\ln I(0)}{t} \leq & \beta^u\frac{\Lambda^u}{h^l} + \widetilde{\sigma}^*\frac{\Lambda^u}{(\alpha^l+h^l)} - (\nu^l+\alpha^l+h^l+\zeta_2^l) \\
& + \limsup_{t\to\infty}\frac{\int_0^t\int_{\mathbb{Y}}\ln(1+c_2(u))\widetilde{X}(\mathrm{d}u,\mathrm{d}s)}{t} \\
& + \limsup_{t\to\infty}\frac{\int_0^t(\sigma_{21}(n)+\sigma_{22}(n)I(t))\mathrm{d}B_2(t)}{t} \\
\leq & \beta^u\frac{\Lambda^u}{h^l} + \widetilde{\sigma}^*\frac{\Lambda^u}{(\alpha^l+h^l)} - (\nu^l+\alpha^l+h^l+\zeta_2^l) + \limsup_{t\to\infty}\frac{2\ln j_0^*}{j_0^*-1} \\
& + \limsup_{t\to\infty}\frac{\int_0^t(\sigma_{21}+\sigma_{22}(n)I(t))\mathrm{d}B_2(t)}{t}.
\end{aligned} \tag{40}$$

holds for all $t\leq j_0^*$. By utilizing the above equation, we can find the following results

$$\limsup_{t\to\infty}\frac{\ln I(t)-\ln I(0)}{t} < 0,$$

which derives that $\lim_{t\to\infty} I(t) = 0$.

The proof of (i) of Theorem 3 is ending.

(ii) First, we construct $U_6(t)$ as follows,

$$U_6(t) = S(t) - l_4 - l_4 \ln \frac{S(t)}{l_4} + I(t) - 1 - \ln I(t) + A(t) + M(t) + \vec{\omega}(n),$$

where l_4 is defined as follows: $l_4 = \frac{\Lambda^u(\beta^l + d^l)}{\left[\lambda^u + h^u + \zeta_1^u + \sum_{n=1}^K \frac{\phi_n}{2}\left(\sigma_{11}^u + \sigma_{12}^u G(\varepsilon)\right)^2\right]^2}$.

Based primary on utilizing Itô's formula and simple computations, one can find that

$$\begin{aligned}
\mathcal{L}U_6(t) \leq\ & \Lambda^u - h^l S(t) + r^u I(t) + r_0^u M_0^u - l_4\left[\frac{\Lambda^l}{S(t)} - \beta^u - \lambda^u - h^u\right] \\
& -\beta^l S(t) + \frac{1}{2}\left[l_4(\sigma_{11}^u + \sigma_{21}^u S(t))^2 + (\sigma_{21}^u + \sigma_{22}^u I(t))^2\right] \\
& +\nu^u + \alpha^u + h^u + l_4\zeta_1^u + \zeta_2^u + \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n) \\
\leq\ & -2\sqrt{l_4\Lambda^u(\beta^l + d^l)} + l_4\left[\lambda^u + h^u + \frac{1}{2}(\sigma_{11}^u + \sigma_{12}^u G(\varepsilon))^2 + \zeta_1^u\right] \\
& +r_0^u M_0^u + \nu^u + \alpha^u + h^u + \frac{1}{2}[\sigma_{21}^u + \sigma_{22}^u G(\varepsilon)]^2 \\
& +(l_4\beta^u + r^u)I(t) + \zeta_2^u + \sum_{n=1,j=1}^{K} \mu_{nj}\vec{\omega}(n),
\end{aligned}$$

which derives that

$$\begin{aligned}
\mathrm{d}U_6(t) \leq\ & -\frac{\Lambda^u(\beta^l + d^l)}{\lambda^u + h^u + \zeta_1^u + \frac{1}{2}\sum_{n=1}^K \phi_n[\sigma_{11}^u + \sigma_{12}^u G(\varepsilon)]^2} \\
& +r_0^u M_0^u + (\nu^u + \alpha^u + h^u) + \frac{1}{2}(\sigma_{21}^u + \sigma_{22}^u G(\varepsilon))^2 + \zeta_2^u + (l_1\beta^u + r^u)I(t) \\
& + \sum_{i=1}^{4}\left[(\sigma_{i1}(n) + \sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i(t) + \int_{\mathbb{Y}} c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}t)\right] \\
& +l_4[\sigma_{11}(n) + \sigma_{12}(n)S(t)]\mathrm{d}B_1(t) + [\sigma_{21}(n) + \sigma_{22}(n)I(t)]\mathrm{d}B_2(t) \\
& +l_4\int_{\mathbb{Y}} \ln(1 + c_1(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}t) + \int_{\mathbb{Y}} \ln(1 + c_2(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}t),
\end{aligned}$$

holds for $i = 1, 2, 3, 4$.

By integrating both sides of above equation from 0 to t and dividing by t, one can yields that

$$\begin{aligned}
\frac{U_6(t) - U_6(0)}{t} \leq\ & -R_E + \frac{l_4\beta^u + r^u}{t}\int_0^t I(s)\mathrm{d}s \\
& + \sum_{i=1}^{4}\left[\frac{1}{t}\int_0^t (\sigma_{i1}(n) + \sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i + \frac{1}{t}\int_0^t\int_{\mathbb{Y}} c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u, \mathrm{d}s)\right] \\
& +\frac{l_4}{t}\int_0^t (\sigma_{11}(n) + \sigma_{12}(n)S(t))\mathrm{d}B_1 + \frac{1}{t}\int_0^t (\sigma_{21}(n) + \sigma_{22}(n)I(t))\mathrm{d}B_2 \\
& +\frac{l_4}{t}\int_0^t\int_{\mathbb{Y}} \ln(1 + c_1(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}s) + \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \ln(1 + c_2(u))\widetilde{X}(\mathrm{d}u, \mathrm{d}s),
\end{aligned} \tag{41}$$

where R_E has been defined in (33).

Based on similar arguments utilized in Lemma 3 of this paper, it gives that

$$\begin{cases} \lim_{t\to\infty} \frac{1}{t}\int_0^t (\sigma_{i1}(n)+\sigma_{i2}(n)\chi_i(t))\chi_i(t)\mathrm{d}B_i(t)=0, \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{Y}} c_i(u)\chi_i(t-)\widetilde{X}(\mathrm{d}u,\mathrm{d}s)=0, \\ \lim_{t\to\infty} \frac{l_4}{t}\int_0^t \int_{\mathbb{Y}} \ln(1+c_1(u))\widetilde{X}(\mathrm{d}u,\mathrm{d}s)=0, \\ \lim_{t\to\infty} \frac{l_4}{t}\int_0^t (\sigma_{11}(n)+\sigma_{12}(n)S(t))\mathrm{d}B_1(t)=0, \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t (\sigma_{21}(n)+\sigma_{22}(n)I(t))\mathrm{d}B_2(t)=0, \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{Y}} \ln(1+c_2(u))\widetilde{X}(\mathrm{d}u,\mathrm{d}s)=0, \end{cases}$$

holds for $i=1,2,3,4$.

Based on taking the inferior limit on the both sides of (41), if $R_E>0$ holds, then one can be obtained that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)\mathrm{d}s \geq \frac{R_E}{l_4\beta^u+r^u}>0,$$

which means infected individual is persistent in mean.

This proof of (ii) of Theorem 3 is ending. □

3. Numerical Simulations

In this chapter, we will prove the results obtained above through numerical simulation, which are utilized to show hybrid dynamic impacts of media coverage and nonlinear perturbations on random dynamics of system (4). The parameter functions utilized in this section are as follows,

$$\begin{cases} \Lambda(t)=5+0.5\sin t, & \beta(t)=3+0.3\sin t \\ \lambda(t)=0.12+0.05\sin t, & \lambda_0(t)=0.08+0.01\sin t, \\ \nu(t)=0.02+0.01\sin t, & \alpha(t)=1\times10^{-3}+5\times10^{-4}\sin t, \\ h(t)=4\times10^{-3}+8\times10^{-4}\sin t, & r(t)=0.006+0.003\sin t, \\ r_0(t)=0.05+0.01\sin t, & \theta(t)=5\times10^{-3}+1\times10^{-3}\sin t, \\ \omega(t)=0.06+0.01\sin t, & p(t)=1.2+0.6\sin t, \\ M_0(t)=5+0.02\times\sin t. \end{cases}$$

It is assumed that $n\in\mathbb{N}=\{1,2,3,4\}$, $\mathbb{Y}=\{1,2,3,4,5\}$ and the transition matrix is given as follows:

$$\Gamma=\begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 4 & -4 \end{pmatrix}.$$

Hence, it follows from simple algebraic computations that $(\phi_1,\phi_2,\phi_3,\phi_4)=\frac{1}{\sqrt{30}}(2,1,4,3)$.

3.1. Numerical Simulation I

It is assumed that $\sigma_{i1}(n)=0.04+0.01\sin n, \sigma_{i2}(n)=\sqrt{0.1+0.05\sin n}$ $(i=1,2,3,4)$, if $c_1(u)=0.5, c_2(u)=0.03, c_3(u)=0.02, c_4(u)=0.05, \vartheta=0.5$, then it is easy to show that $\widetilde{R}_1=0.0349, \widetilde{R}_2=-0.0252, \widetilde{R}_3=0.0445, \widetilde{R}_4=-0.0273$, which follows that $\widetilde{R}_s=0.0257>0$. Based on Theorem 2, one can be concluded that $(S(t),I(t),A(t),M(t),n)$ of system (4) is f-exponentially ergodic. The dynamical responses $S(t),I(t),A(t),M(t)$ of system (4) with initial value $(0.4,0.1,0.05,5)$ are plotted in Figure 1a, Figure 1b, Figure 1c, Figure 1d, respectively, which indicates an exponential convergence.

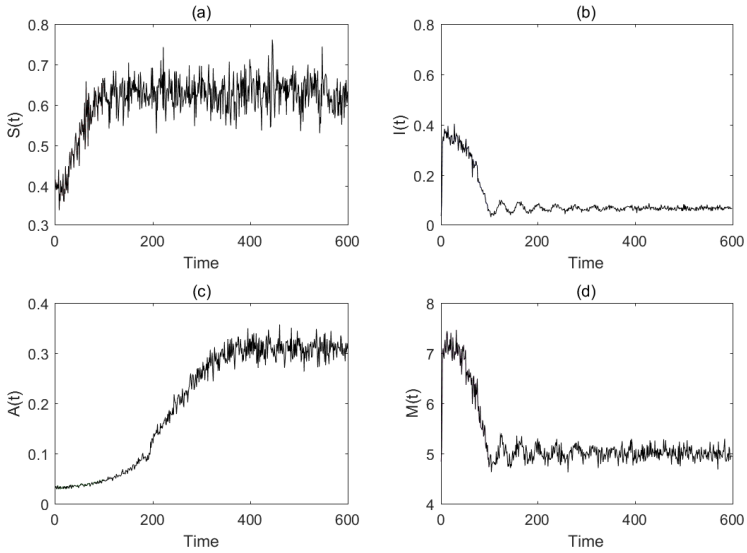

Figure 1. If $\sigma_{i1}(n) = 0.04 + 0.01\sin n, \sigma_{i2}(n) = \sqrt{0.1 + 0.05\sin n}$ $(i = 1,2,3,4)$, $c_1(u) = 0.5, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05, \vartheta = 0.5$, then it is easy to show that $\widetilde{R}_1 = 0.0349, \widetilde{R}_2 = -0.0252, \widetilde{R}_3 = 0.0445, \widetilde{R}_4 = -0.0273$, which follows that $\widetilde{R}_s = 0.0257 > 0$. The dynamical responses $S(t), I(t), A(t), M(t)$ of system (4) with initial value $(0.4, 0.1, 0.05, 5)$ are plotted in (**a**–**d**), respectively, which indicates an exponential convergence.

If $c_1(u) = 0.4, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05, \vartheta = 0.5$, then it is easy to show that $\widetilde{R}_1 = 0.0426, \widetilde{R}_2 = -0.0849, \widetilde{R}_3 = 0.0551, \widetilde{R}_4 = -0.0415$, which follows that $\widetilde{R}_s = 0.0176 > 0$. Based on Theorem 2, one can be concluded that $(S(t), I(t), A(t), M(t), n)$ of system (4) is f-exponentially ergodic. The dynamical responses $S(t), I(t), A(t), M(t)$ of system (4) with initial value $(0.35, 0.15, 0.04, 5)$ are plotted in Figure 2a, Figure 2b, Figure 2c, Figure 2d, respectively, which indicates an exponential convergence.

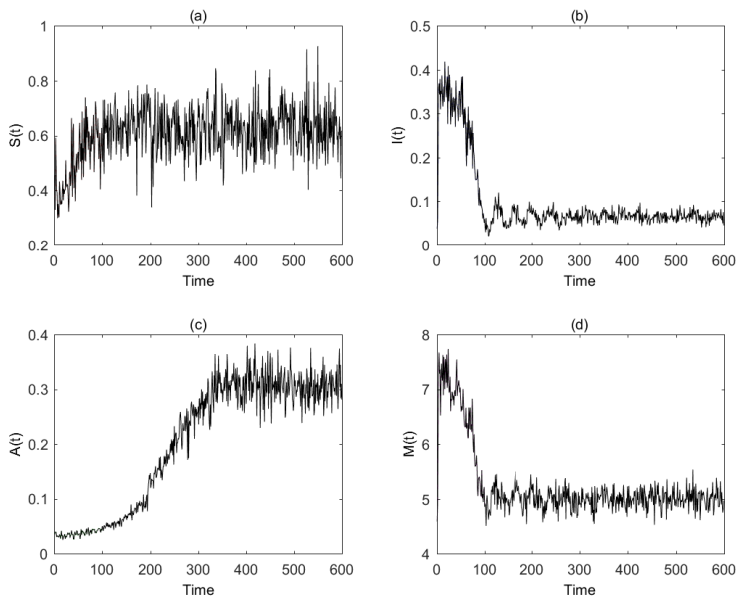

Figure 2. If $\sigma_{i1}(n) = 0.04 + 0.01\sin n, \sigma_{i2}(n) = \sqrt{0.1 + 0.05\sin n}$ $(i = 1,2,3,4)$, $c_1(u) = 0.4, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05, \vartheta = 0.5$, then it is easy to show that $\widetilde{R}_1 = 0.0426, \widetilde{R}_2 = -0.0849, \widetilde{R}_3 = 0.0551, \widetilde{R}_4 = -0.0415$, which follows that $\widetilde{R}_s = 0.0176 > 0$. The dynamical responses $S(t), I(t), A(t), M(t)$ of system (4) with initial value $(0.35, 0.15, 0.04, 5)$ are plotted in (**a**–**d**), respectively, which indicates an exponential convergence.

In order to show the dynamic effects of Lévy jumps, values of $\widetilde{R}_s$ are given in Table 1 under four different $c_i(u)$ $(i = 1, 2, 3, 4)$, and the detailed values can be found in Table 1. Furthermore, total variation norms $\|\mathbb{P}(t, (\chi(t), n), \cdot) - \pi(\cdot)\|$ are plotted in Figure 3 due to variations of c_i $(i = 1, 2, 3, 4)$ under four different cases.

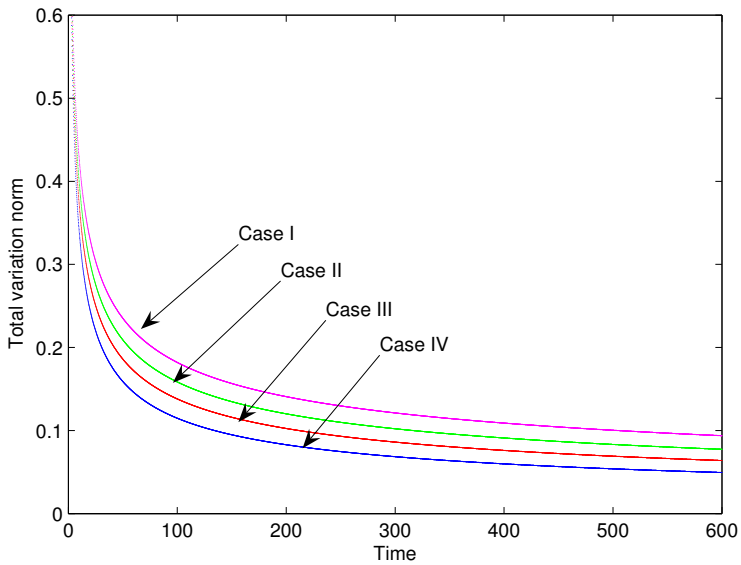

Figure 3. Total variation norms $\|\mathbb{P}(t, (\chi(t), n), \cdot) - \pi(\cdot)\|$ are plotted due to variations of $c_i(u)$ $(i = 1, 2, 3, 4)$ under four different cases corresponding to Table 1.

Table 1. When $\sigma_{i1}(n) = 0.04 + 0.01 \sin n, \sigma_{i2}(n) = \sqrt{0.1 + 0.05 \sin n}$ $(i = 1, 2, 3, 4)$, values of $\widetilde{R}_s$ are given under four different values of c_i $(i = 1, 2, 3, 4)$.

	Values of $c_i(u)$ $(i = 1, 2, 3, 4)$	$\widetilde{R}_s$
Case I	$c_1(u) = 0.5, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05$	0.0257
Case II	$c_1(u) = 0.4, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05$	0.0176
Case III	$c_1(u) = 0.3, c_2(u) = 0.02, c_3(u) = 0.01, c_4(u) = 0.03$	0.0159
Case IV	$c_1(u) = 0.2, c_2(u) = 0.02, c_3(u) = 0.01, c_4(u) = 0.02$	0.0113

3.2. Numerical Simulation II

It is assumed that $\sigma_{i1}(n) = 0.04 + 0.01 \sin n, \sigma_{i2}(n) = \sqrt{0.1 + 0.05 \sin n}$ $(i = 1, 2, 3, 4)$. If $c_1(u) = 0.5, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05$ and initial value $(0.4, 0.1, 0.05, 5)$, then it can be obtained that $R_I = 0.7143 < 1$, which follows from Theorem 3-(i) that the infected individual $I(t)$ of system (4) tends to zero exponentially. On the other hand, if $c_1(u) = 0.5, c_2(u) = 0.02, c_3(u) = 0.01, c_4(u) = 0.03$, then it can be obtained that $R_E = 0.2683 > 0$, which follows from Theorem 3-(ii) that the number of infected individual $I(t)$ of system (4) is persistent in average sense. Dynamical responses of the number of infected individual $I(t)$ are shown in Figure 4a and Figure 4b, respectively.

It is assumed that $\sigma_{i1}(n) = 0.09 + 0.04 \sin n, \sigma_{i2}(n) = \sqrt{0.2 + 0.09 \sin n}$ $(i = 1, 2, 3, 4)$. If $c_1(u) = 0.04, c_2(u) = 0.3, c_3(u) = 0.06, c_4(u) = 0.03$ and initial value $(0.35, 0.15, 0.04, 5)$, then it can be obtained that $R_I = 0.8926 < 1$, which follows from Theorem 3-(i) that the number of infected individual $I(t)$ of system (4) tends to zero exponentially. On the other hand, if $c_1(u) = 0.4, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05$, then it can be obtained that $R_E = 0.1972 > 0$, which follows from Theorem 3-(ii) that the number of infected individual $I(t)$ of system (4) is persistent in average sense. Dynamical responses of the infected individual $I(t)$ are shown in Figure 5a and Figure 5b, respectively.

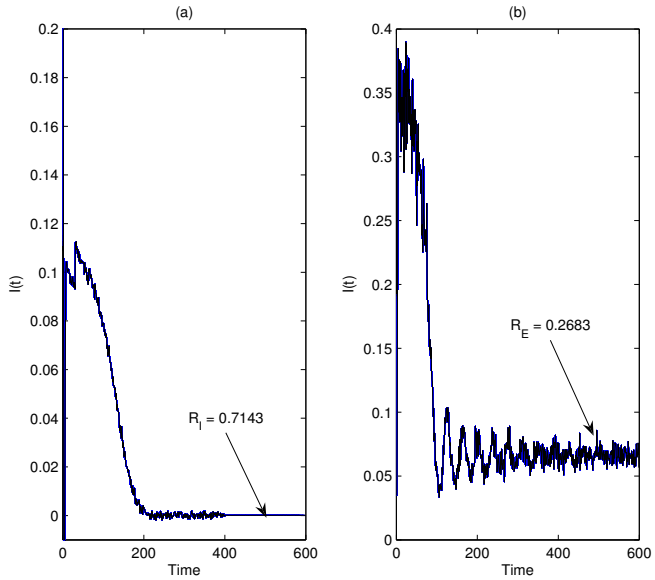

Figure 4. If $\sigma_{i1}(n) = 0.04 + 0.01 \sin n, \sigma_{i2}(n) = \sqrt{0.1 + 0.05 \sin n}$ $(i = 1, 2, 3, 4)$, $c_1(u) = 0.5, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05$ and initial value $(0.4, 0.1, 0.05, 5)$, dynamical responses of the infected individual $I(t)$ are shown in (**a**,**b**), respectively. It can be obtained that $R_I = 0.7143 < 1$, which follows that infectious disease becomes extinct exponentially. In other words, if $c_1(u) = 0.5, c_2(u) = 0.02, c_3(u) = 0.01, c_4(u) = 0.03$, then it can be obtained that $R_E = 0.2683 > 0$, which follows that infectious disease persists in mean.

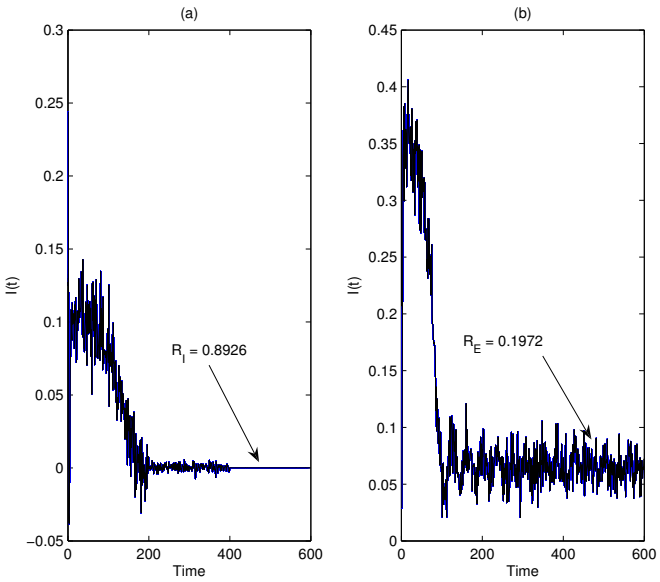

Figure 5. If $\sigma_{i1}(n) = 0.09 + 0.04 \sin n, \sigma_{i2}(n) = \sqrt{0.2 + 0.09 \sin n}$ $(i = 1, 2, 3, 4)$, $c_1(u) = 0.04, c_2(u) = 0.3, c_3(u) = 0.06, c_4(u) = 0.03$ and initial value $(0.35, 0.15, 0.04, 5)$, dynamical responses of the infected individual $I(t)$ are shown in (**a**,**b**), respectively. It can be obtained that $R_I = 0.8926 < 1$, which follows that infectious disease becomes extinct exponentially. In other words, if $c_1(u) = 0.4, c_2(u) = 0.03, c_3(u) = 0.02, c_4(u) = 0.05$, then it can be obtained that $R_E = 0.1972 > 0$, which follows that infectious disease persists in mean.

Remark 3. *From the above two numerical experiments under two different values of $\sigma_{i1}(n), \sigma_{i2}(n)$ $(i = 1, 2, 3, 4)$, it reveals that the Gaussian white noises performing on $I(t)$ play sufficiently effective roles in reducing the spread of infectious disease. For finite state spaces, although the infectious*

disease only persists within one certain state, there still exists opportunity for infectious disease to persist eventually. Furthermore, Lévy jumps may act double roles in directly controlling the infectious disease based on the values of $c_i(u)$ $(i = 1, 2, 3, 4)$.

Based on the numerical simulations in Figures 1 and 2, the dynamical responses fluctuate with larger amplitudes under comparatively strong disturbances depicting by Lévy jumps. In the real world, the strong disturbances usually lead to oscillations in the real world, which highly relevant to the vivid phenonomena, i.e., contemporary controlled state—recurrence—re-controlled state within transmission of epidemics.

When the amplitudes of white noises maintain some certain levels, the dynamic changes of total variation norm due to variations of Lévy jumps are discussed, which are indicated in Table 1 and Figure 3. It reveals that the transmission of infectious disease becomes severe with large stochastic fluctuations from surrounding environment in the real world. However, it follows from Figures 4 and 5 that infectious disease may tend to extinction when the stochastic fluctuations from surrounding environment decrease, and the transmission will be controlled within certain duration.

4. Conclusions

Media coverage, random disturbances and time-varying periodic function parameters are important disciplines in the modeling and dynamical analysis of infectious disease transmission. One of the key themes in epidemiology is the study of the stochastic dynamics of infectious disease system. Current field observations of the public health alerts and stochastic perturbations in stochastic nonautonomous infectious disease dynamics has highlighted the necessity of improving related systems that do not consider the joint dynamic impacts of Lévy jumps and media coverage.

In the last few years, scholars have introduced a media coverage feedback mechanism in mathematical model formulation to account for the constructive effects of public health alerts. Stochastic perturbations are usually represented by linear form perturbation of white noise, and the influences of linear noises perturbations on nonautonomous epidemic models were studied in [13–17]. However, in order to accurately depict some stochastic phenomena arising from infectious disease transmission in the real world, it is more constructive to introduce nonlinear noise perturbations into nonautonomous epidemic model.

Furthermore, stochastic models have been established to discuss the prevalence mechanism of infectious disease [23–31] without Lévy jumps. A SIS infectious disease model with regime-switching and driven by Lévy jumps was investigated in [32], while combined dynamic impacts of media coverage and Lévy jumps on random dynamics of infectious disease system are rarely reported.

Hybrid dynamic effects of media coverage and stochastic perturbations in the threshold dynamics of random epidemic system have been investigated in [33–36], while Lévy jumps and periodic function parameters were not considered in [33–36]. The dynamic behavior of infectious disease systems in [37–39] were investigated under nonlinear noise perturbations and Lévy jumps, while all parameters were assumed to be constant values in [37–39], periodicity factors during transmission within the infectious disease regimes were not considered.

Although the stochastic infectious disease model and its dynamic analysis have attracted wide attention, as far as the authors know, the hybrid dynamic impacts of Lévy jumps and media coverage on random dynamics of the nonautonomous SIAM epidemic model with Markov chain and nonlinear noise perturbations have not been reported in previous related studies.

In order to depict the impact of public health alerts and stochastic dynamics of nonautonomous SIAM epidemic model, we extend the work done in [12] by incorporating Lévy jumps, nonlinear noise perturbations and periodic function parameters into the epidemic model. The existence of a stochastically ultimate upper bound and a uniform lower bound of a positive solution of the proposed SIAM epidemic model was studied in Lemma 1.

The existence and uniqueness of globally positive solution to the proposed SIAM epidemic model was studied in Lemma 2. Based on defining certain fitted stochastic

Lyapunov functions, sufficient conditions for existence of a nontrival positive T-periodic solution were discussed in Theorem 1. By verifying a Foster–Lyapunov condition, some sufficient conditions for the exponential ergodicity were investigated in Theorem 2. Furthermore, several conditions were derived in Theorem 3, which were utilized to discuss the persistence in an average sense and the extinction of the epidemic system.

Finally, numerical simulations were provided to support the theoretical findings. The main analytical findings are theoretically beneficial to reveal the transmission mechanism of infectious disease under a stochastic surrounding environment. Furthermore, by utilizing the findings associated with the elimination mechanism of infectious disease, it is also constructive for agencies to formulate policies and measures to control the spread of infectious disease.

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