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New Class Up and Down λ-Convex Fuzzy-Number Valued Mappings and Related Fuzzy Fractional Inequalities

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Abstract: The fuzzy-number valued up and down λ-convex mapping is originally proposed as an intriguing generalization of the convex mappings. The newly suggested mappings are then used to create certain Hermite–Hadamard- and Pachpatte-type integral fuzzy inclusion relations in fuzzy fractional calculus. It is also suggested to revise the Hermite–Hadamard integral fuzzy inclusions with regard to the up and down λ-convex fuzzy-number valued mappings ($U{\cdot}D$ λ-convex $F{\cdot}N{\cdot}V{\cdot}Ms$). Moreover, Hermite–Hadamard–Fejér has been proven, and some examples are given to demonstrate the validation of our main results. The new and exceptional cases are presented in terms of the change of the parameters "i" and "α" in order to assess the accuracy of the obtained fuzzy inclusion relations in this study.

Keywords: fuzzy-number valued mappings; fuzzy integrals; up and down λ-convex fuzzy-number valued mappings; fuzzy Hermite–Hadamard type inequalities

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1. Introduction

The generalized convexity of mappings offers a pretty potent principle and tool, which is frequently employed in a variety of mathematical physics issues in addition to applied analysis and nonlinear analysis. See the published publications [1–13] and the references therein for a large number of scholars' recent efforts to investigate several intriguing integral inequalities that are due to generalized convexity from various angles. One of the most important mathematical inequalities associated with convex mappings, and one that is frequently employed in many other areas of the mathematical sciences, notably in optimization analysis, is the Hermite-integral Hadamard's inequality. This inequality stands out in particular because it provides an approximation of the mean value's error bound in relation to the integrable convex mapping, which has drawn academic interest and research from a large number of academics in the field of mathematical analysis. Significant works published recently relate various families of convex mappings to the Hermite–Hadamard-type (H·H-type) integral inequalities. For example, we can refer to Szostok [14] for higher-order convex mappings, to Korus [15] for s-convex mappings, to Andric and Pecaric [16] for (h, g, m)-convex mappings, to Latif [17] for GA-convex and geometrically quasiconvex mappings, to Niezgoda [18] for G-symmetrized convex mappings, to Demir et al. [19] for trigonometrically convex mappings, and so on. For more information, see [20–34].

It has been demonstrated that fractional calculus, as a rather robust technique, is an essential foundational element not only in the mathematical sciences but also in the applied sciences. The field has drawn a lot of research interest in order to answer the important topic. Therefore, many authors have discovered some important integral inequalities by effectively combining different fractional integral techniques. For example, Ahmad et al. [35] studied four different inequalities for convex mappings involving fractional integrals with exponential kernels, Mohammed and Sarikaya [36] studied some inequalities involving Sarikaya fractional integrals for twice differentiable mappings, and Set et al. [37] studied the Hermetic inequalities involving Atangana–Baleanu fractional integral operators, Khan et al. [38] and Meftah et al. [39] proposed H·H-type inequalities for conformable fractional integral operators, and Dragomir [40] obtained H·H-type inequalities for generalized conformable Riemann–Liouville fractional integrals. We recommend [41–56] to readers who are interested in learning additional significant results relating to fractional integrals.

The study of the characteristics and uses of interval-valued mappings (I·V·Ms) is the focus of the branch of set value analysis known as interval analysis [57], which now has a major impact on both the pure and practical sciences. The error boundaries of a numerical solution for a finite state machine were first determined using interval analysis. Interval analysis has grown rapidly over the last several decades and has significant implications for many disciplines of applied sciences, including neural network output optimization [58], computer graphics [59], and automatic error analysis [60]. Numerous academics have researched the current hot topics of various interval analysis theories up until this point in time. For instance, Budak et al. [61] expanded the interval-valued mapping $Y(\varkappa)$ defined on $\mathbb{R}$ to the interval-valued mapping $Y(\varkappa, y)$ defined on $\mathbb{R}^2$ via Riemann–Liouville fractional integrals. For interval-valued coordinated convex mappings, they derived a few fractional integral inequalities of the H·H-type. For interval-valued mappings, Costa et al. [62] created certain inequalities based on the Kulisch–Miranker order relation. They were able to obtain the Gauss inequalities for interval mappings by utilizing Aumann's and Kaleva's improper integrals. A family of log-s-convex fuzzy-interval-valued mappings was explored by Liu et al. [63]. With the use of this sort of mapping, they were able to obtain a few Jensen- and H·H-Fejer-type inequalities. Interval-valued preinvex mappings are a notion first developed by Srivastava et al. [64]. The authors also provided the Riemann–Liouville fractional integrals-specific modifications of the H·H-type inequalities. The concept of interval-valued harmonical h-convex mapping was introduced by the authors in [65]. They obtained many H·H-type inequalities for the interval Riemann integrals using this idea. Additionally, [66] and [67] address a few applications of interval-valued mappings in optimization theory. The reader who is interested in recent advancements in interval-valued mappings can consult [68–76] and the references they cite.

Recently, Khan et al., inspired by the following research articles, introduced different classes of convexity and nonconvexity in the fuzzy environment, see [77–80]. Moreover, with the help of new classes, some new versions of fuzzy H·H- and Pachpatte-type integral inequalities were obtained with the fuzzy Riemann and fuzzy fractional integral by using fuzzy order relation. Recently, Khan et al. [81] discussed the level-wise characterization of fuzzy inclusion relation and then acquired the new versions of fuzzy H·H-type inequalities for $U{\cdot}D$ convex fuzzy mappings and products of $U{\cdot}D$ convex fuzzy mappings with support of fuzzy inclusion relation. Some new classes were also introduced by applying some mild restrictions on $U{\cdot}D$ convex fuzzy mappings to achieve new and classical exceptional cases. For more information related to $F\text{-}N{\cdot}V{\cdot}M$ and fuzzy-related concepts, see [82–95] and the references therein

The present study is dedicated to resolving various fuzzy inclusion relations relating to fuzzy fractional integrals and is motivated and inspired by the aforementioned studies, particularly the findings studied in [54,71,85]. We offer a family of fuzzy-number valued λ-convex mappings to accomplish this goal. It allows us to create certain fuzz fractional integral inclusion relations for the exceptional H·H- and Pachpatte-type integral inequalities with the help of fuzzy-number valued λ-convex mappings, respectively.

2. Preliminaries

Let $\mathcal{X}_I$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\mathfrak{w} \in \mathcal{X}_I$ be defined by

$$\mathfrak{w} = [\mathfrak{w}_*,\ \mathfrak{w}^*] = \{\varkappa \in \mathbb{R}|\ \mathfrak{w}_* \le \varkappa \le \mathfrak{w}^*\},(\mathfrak{w}_*,\ \mathfrak{w}^* \in \mathbb{R}). \tag{1}$$

If $\mathfrak{w}_* = \mathfrak{w}^*$, then $\mathfrak{w}$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\mathfrak{w}_* \ge 0$, then $[\mathfrak{w}_*,\ \mathfrak{w}^*]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{X}_I^+$ and defined as $\mathcal{X}_I^+ = \{[\mathfrak{w}_*,\ \mathfrak{w}^*] : [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{X}_I \text{ and } \mathfrak{w}_* \ge 0\}$.

Let $m \in \mathbb{R}$ and $m \cdot \mathfrak{w}$ be defined by

$$m \cdot \mathfrak{w} = \begin{cases} [m\mathfrak{w}_*,\ m\mathfrak{w}^*] \text{ if } m > 0, \\ \quad \{0\} \qquad \text{if } m = 0, \\ [m\mathfrak{w}^*, m\mathfrak{w}_*] \ \text{ if } m < 0. \end{cases} \tag{2}$$

Then the Minkowski difference $\mathfrak{x} - \mathfrak{w}$, addition $\mathfrak{w} + \mathfrak{x}$ and $\mathfrak{w} \times \mathfrak{x}$ for $\mathfrak{w}, \mathfrak{x} \in \mathcal{X}_I$ are defined by

$$[\mathfrak{x}_*,\ \mathfrak{x}^*] + [\mathfrak{w}_*,\ \mathfrak{w}^*] = [\mathfrak{x}_* + \mathfrak{w}_*,\ \ \mathfrak{x}^* + \mathfrak{w}^*], \tag{3}$$

$$[\mathfrak{x}_*,\ \mathfrak{x}^*] \times [\mathfrak{w}_*,\ \mathfrak{w}^*] = [\min\{\mathfrak{x}_*\mathfrak{w}_*,\ \mathfrak{x}^*\mathfrak{w}_*,\ \mathfrak{x}_*\mathfrak{w}^*,\ \mathfrak{x}^*\mathfrak{w}^*\},\ \max\{\mathfrak{x}_*\mathfrak{w}_*,\ \mathfrak{x}^*\mathfrak{w}_*,\ \mathfrak{x}_*\mathfrak{w}^*,\ \mathfrak{x}^*\mathfrak{w}^*\}] \tag{4}$$

$$[\mathfrak{x}_*,\ \mathfrak{x}^*] - [\mathfrak{w}_*,\ \mathfrak{w}^*] = [\mathfrak{x}_* - \mathfrak{w}^*,\ \ \mathfrak{x}^* - \mathfrak{w}_*], \tag{5}$$

Remark 1. *(i) For given* $[\mathfrak{x}_*,\ \mathfrak{x}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{X}_I$*, the relation "* $\supseteq_I$ *" defined on* $\mathcal{X}_I$ *by*

$$[\mathfrak{w}_*,\ \mathfrak{w}^*] \supseteq_I [\mathfrak{x}_*,\ \mathfrak{x}^*] \text{ if and only if } \mathfrak{w}_* \le \mathfrak{x}_*,\ \ \mathfrak{x}^* \le \mathfrak{w}^*, \tag{6}$$

for all $[\mathfrak{x}_*,\ \mathfrak{x}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{X}_I$*, it is a partial interval inclusion relation. The relation* $[\mathfrak{w}_*,\ \mathfrak{w}^*] \supseteq_I [\mathfrak{x}_*,\ \mathfrak{x}^*]$ *coincident to* $[\mathfrak{w}_*,\ \mathfrak{w}^*] \supseteq [\mathfrak{x}_*,\ \mathfrak{x}^*]$ *on* $\mathcal{X}_I$*. It can be easily seen that "*$\supseteq_I$*" looks like "up and down" on the real line* $\mathbb{R}$*, so we call "* $\supseteq_I$ *" is "up and down" (or "U·D" order, in short)* [85].

(ii) For given $[\mathfrak{x}_*,\ \mathfrak{x}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{X}_I$*, we say that* $[\mathfrak{x}_*,\ \mathfrak{x}^*] \le_I [\mathfrak{w}_*,\ \mathfrak{w}^*]$ *if and only if* $\mathfrak{x}_* \le \mathfrak{w}_*,\ \mathfrak{x}^* \le \mathfrak{w}^*$ *or* $\mathfrak{x}_* \le \mathfrak{w}_*,\ \mathfrak{x}^* < \mathfrak{w}^*$*, it is an partial interval order relation. The relation* $[\mathfrak{x}_*,\ \mathfrak{x}^*] \le_I [\mathfrak{w}_*,\ \mathfrak{w}^*]$ *coincident to* $[\mathfrak{x}_*,\ \mathfrak{x}^*] \le [\mathfrak{w}_*,\ \mathfrak{w}^*]$ *on* $\mathcal{X}_I$*. It can be easily seen that "* $\le_I$ *" looks like "left and right" on the real line* $\mathbb{R}$*, so we call "* $\le_I$ *" is "left and right" (or "LR" order, in short)* [77,85].

For$[\mathfrak{x}_*,\ \mathfrak{x}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{X}_I$*, the Hausdorff-Pompeiu distance between intervals* $[\mathfrak{x}_*,\ \mathfrak{x}^*]$ *and* $[\mathfrak{w}_*,\ \mathfrak{w}^*]$*is defined by*

$$d_H([\mathfrak{x}_*,\ \mathfrak{x}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*]) = max\{|\mathfrak{x}_* - \mathfrak{w}_*|,\ |\mathfrak{x}^* - \mathfrak{w}^*|\}. \tag{7}$$

It is familiar fact that $(\mathcal{X}_I, d_H)$ *is a complete metric space* [83].

Definition 1. *A fuzzy subset L of* $\mathbb{R}$ *is distinguished by a mapping* $\widetilde{\mathfrak{w}} : \mathbb{R} \to [0,1]$ *called the membership mapping of L. That is, a fuzzy subset L of* $\mathbb{R}$ *is a mapping* $\widetilde{\mathfrak{w}} : \mathbb{R} \to [0,1]$*. So, for further study, we have chosen this notation. We appoint* $\mathbb{E}$ *to denote the set of all fuzzy subsets of* $\mathbb{R}$ [82,83].

Let $\widetilde{\mathfrak{w}} \in \mathbb{E}$. Then, $\widetilde{\mathfrak{w}}$ is known as a fuzzy-number if the following properties are satisfied by $\widetilde{\mathfrak{w}}$:

(1) $\widetilde{\mathfrak{w}}$ should be normal if there exists $\varkappa \in \mathbb{R}$ and $\widetilde{\mathfrak{w}}(\varkappa) = 1$;
(2) $\widetilde{\mathfrak{w}}$ should be upper semi continuous on $\mathbb{R}$ if for given $\varkappa \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\widetilde{\mathfrak{w}}(\varkappa) - \widetilde{\mathfrak{w}}(s) < \varepsilon$ for all $\varkappa,\ s \in \mathbb{R}$ with $|\varkappa - s| < \delta$;
(3) $\widetilde{\mathfrak{w}}$ should be fuzzy convex that is $\widetilde{\mathfrak{w}}((1-m)\varkappa + ms) \ge min(\widetilde{\mathfrak{w}}(\varkappa),\ \widetilde{\mathfrak{w}}(s))$, for all $\varkappa, s \in \mathbb{R}$ and $m \in [0,\ 1]$;
(4) $\widetilde{\mathfrak{w}}$ should be compactly supported that is $cl\{m \in \mathbb{R}|\ \widetilde{\mathfrak{w}}(\varkappa) > 0\}$ is compact. We appoint $\mathbb{E}_C$ to denote the set of all fuzzy-numbers of $\mathbb{R}$.

Definition 2. [82,83] *Given $\widetilde{\mathfrak{w}} \in \mathbb{E}_C$, the level sets or cut sets are given by $[\widetilde{\mathfrak{w}}]^i = \{\varkappa \in \mathbb{R} | \ \widetilde{\mathfrak{w}}(\varkappa) > i\}$ for all $i \in [0,\ 1]$ and by $[\widetilde{\mathfrak{w}}]^0 = \{\varkappa \in \mathbb{R} | \ \widetilde{\mathfrak{w}}(\varkappa) > 0\}$. These sets are known as i-level sets or i-cut sets of $\widetilde{\mathfrak{w}}$.*

Proposition 1. [78] *Let $\widetilde{\mathfrak{w}}, \widetilde{\mathfrak{x}} \in \mathbb{E}_C$. Then, relation " $\leq_{\mathbb{F}}$ " given on $\mathbb{E}_C$ by*

$$\widetilde{\mathfrak{w}} \leq_{\mathbb{F}} \widetilde{\mathfrak{x}} \text{ when and only when, } [\widetilde{\mathfrak{w}}]^i \leq_I [\widetilde{\mathfrak{x}}]^i, \text{ for every } i \in [0,\ 1], \tag{8}$$

it is left and right order relation.

Proposition 2. [81] *Let $\widetilde{\mathfrak{w}}, \widetilde{\mathfrak{x}} \in \mathbb{E}_C$. Then, relation " $\supseteq_{\mathbb{F}}$ " given on $\mathbb{E}_C$ by*

$$\widetilde{\mathfrak{w}} \supseteq_{\mathbb{F}} \widetilde{\mathfrak{x}} \text{ when and only when, } [\widetilde{\mathfrak{w}}]^i \supseteq_I [\widetilde{\mathfrak{x}}]^i, \text{ for every } i \in [0,\ 1], \tag{9}$$

it is U·D order relation on $\mathbb{E}_C$.

Proof: The proof follows directly from the $U{\cdot}D$ relation $\supseteq_I$ defined on $\mathcal{X}_I$. □

Remember the approaching notions, which are offered in the literature. If $\widetilde{\mathfrak{w}}, \widetilde{\mathfrak{x}} \in \mathbb{E}_C$ and $i \in \mathbb{R}$, then, for every $i \in [0,\ 1]$, the arithmetic operations are defined by

$$[\widetilde{\mathfrak{w}} \oplus \widetilde{\mathfrak{x}}]^i = [\widetilde{\mathfrak{w}}]^i + [\widetilde{\mathfrak{x}}]^i, \tag{10}$$

$$[\widetilde{\mathfrak{w}} \otimes \widetilde{\mathfrak{x}}]^i = [\widetilde{\mathfrak{w}}]^i \times [\widetilde{\mathfrak{x}}]^i, \tag{11}$$

$$[m \odot \widetilde{\mathfrak{w}}]^i = m.[\widetilde{\mathfrak{w}}]^i. \tag{12}$$

These operations follow directly from the Equations (4)–(6), respectively.

Theorem 1. *The space $\mathbb{E}_C$ dealing with a supremum metric, i.e., for $\widetilde{\mathfrak{w}},\ \widetilde{\mathfrak{x}} \in \mathbb{E}_C$* [83]

$$d_\infty(\widetilde{\mathfrak{w}},\ \widetilde{\mathfrak{x}}) = \sup_{0 \leq i \leq 1} d_H\left([\widetilde{\mathfrak{w}}]^i,\ [\widetilde{\mathfrak{x}}]^i\right), \tag{13}$$

is a complete metric space, where H denote the well-known Hausdorff metric on space of intervals.

Riemann Integral Operators for Interval and Fuzzy-Number Valued Mappings

Now we define and discuss some properties of fractional integral operators of interval and fuzzy-number valued mappings.

Theorem 2. *If $Y : [u, z] \subset \mathbb{R} \to \mathcal{X}_I$ is an interval-valued mapping (I·V·M) satisfying that $Y(\varkappa) = [Y_*(\varkappa),\ Y^*(\varkappa)]$, then Y is Aumann integrable (IA-integrable) over $[u, z]$ when and only when, $Y_*(\varkappa)$ and $Y^*(\varkappa)$ both are integrable over $[u, z]$ such that* [83,84]

$$(IA)\int_u^z Y(\varkappa)d\varkappa = \left[\int_u^z Y_*(\varkappa)d\varkappa,\ \int_u^z Y^*(\varkappa)d\varkappa\right]. \tag{14}$$

Definition 3. *Let $\widetilde{Y} : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ is called F-N·V·M. Then, for every $i \in [0,\ 1]$ as well as i-levels define the family of I·V·Ms $Y_i : \mathbb{I} \subset \mathbb{R} \to \mathcal{X}_I$ satisfying that $Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)]$ for every $\varkappa \in \mathbb{I}$. Here, for every $i \in [0,\ 1]$, the endpoint real-valued mappings $Y_*(\cdot, i),\ Y^*(\cdot, i) : \mathbb{I} \to \mathbb{R}$ are called lower and upper mappings of Y* [77].

Definition 4. *Let* $\widetilde{Y} : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ *be a F-N·V·M. Then* $\widetilde{Y}(\varkappa)$ *is said to be continuous at* $\varkappa \in \mathbb{I}$, *if for every* $i \in [0,\ 1]$, $Y_i(\varkappa)$ *is continuous when and only when both endpoint mappings* $Y_*(\varkappa, i)$ *and* $Y^*(\varkappa, i)$ *are continuous at* $\varkappa \in \mathbb{I}$ [77].

Definition 5. *Let* $\widetilde{Y} : [u,\ z] \subset \mathbb{R} \to \mathbb{E}_C$ *is F-N·V·M. The fuzzy Aumann integral ((FA)-integral) of* $\widetilde{Y}$ *over* $[u,\ z]$, *denoted by* $(FA)\int_u^z \widetilde{Y}(\varkappa)d\varkappa$, *is defined level-wise by* [84]

$$\left[(FA)\int_u^z \widetilde{Y}(\varkappa)d\varkappa\right]^i = (IA)\int_u^z Y_i(\varkappa)d\varkappa = \left\{\int_u^z Y(\varkappa, i)d\varkappa : Y(\varkappa, i) \in S(Y_i)\right\}, \tag{15}$$

where $S(Y_i) = \{Y(., i) \to \mathbb{R} : Y(., i)$ *is integrable and* $Y(\varkappa, i) \in Y_i(\varkappa)\}$, *for every* $i \in [0,\ 1]$. $\widetilde{Y}$ *is (FA)-integrable over* $[u,\ z]$ *if* $(FA)\int_u^z \widetilde{Y}(\varkappa)d\varkappa \in \mathbb{E}_C$.

Theorem 3. *Let* $\widetilde{Y} : [u,\ z] \subset \mathbb{R} \to \mathbb{E}_C$ *be a F-N·V·M as well as i-levels define the family of I·V·Ms* $Y_i : [u,\ z] \subset \mathbb{R} \to \mathcal{X}_I$ *satisfying that* $Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)]$ *for every* $\varkappa \in [u,\ z]$ *and for every* $i \in [0,\ 1]$. *Then* $\widetilde{Y}$ *is (FA)-integrable over* $[u,\ z]$ *when and only when,* $Y_*(\varkappa, i)$ *and* $Y^*(\varkappa, i)$ *both are integrable over* $[u,\ z]$. *Moreover, if* $\widetilde{Y}$ *is (FA)-integrable over* $[u,\ z]$, *then* [78]

$$\left[(FA)\int_u^z \widetilde{Y}(\varkappa)d\varkappa\right]^i = \left[\int_u^z Y_*(\varkappa, i)d\varkappa,\ \int_u^z Y^*(\varkappa, i)d\varkappa\right] = (IA)\int_u^z Y_i(\varkappa)d\varkappa, \tag{16}$$

for every $i \in [0,\ 1]$.

The family of all (FA)-integrable *F-N·V·Ms* over $[u,\ z]$, are denoted by $\mathcal{FA}_{([u,\ z],\ i)}$.

Allahviranloo et al. [86] introduced the following fuzzy Riemann–Liouville fractional integral operators:

Definition 6. *Let* $\alpha > 0$ *and* $L([u,\ z], \mathbb{E}_C)$ *be the collection of all Lebesgue measurable F-N·V·M on* $[u, z]$. *Then the fuzzy left and right Riemann–Liouville fractional integral of* $Y \in L([u,\ z], \mathbb{E}_C)$ *with order* $\alpha > 0$ *are defined by*

$$\mathcal{I}_{u^+}^{\alpha}\ Y(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_u^{\varkappa}(\varkappa - \varphi)^{\alpha-1}Y(\varphi)d\varphi, \quad (\varkappa > u), \tag{17}$$

and

$$\mathcal{I}_{z^-}^{\alpha}\ Y(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{z}(\varphi - \varkappa)^{\alpha-1}Y(\varphi)d\varphi, \quad (\varkappa < z), \tag{18}$$

respectively, where $\Gamma(\varkappa) = \int_0^{\infty}\varphi^{\varkappa-1}e^{-\varphi}d\varphi$ is the Euler gamma mapping. The fuzzy left and right Riemann–Liouville fractional integral $\varkappa$ based on left and right endpoint mappings can be defined, that is

$$\left[\mathcal{I}_{u^+}^{\alpha}\ Y(\varkappa)\right]^{\lambda} = \frac{1}{\Gamma(\alpha)}\int_u^{\varkappa}(\varkappa - \varphi)^{\alpha-1}Y_{\lambda}(\varphi)d\varphi$$

$$= \frac{1}{\Gamma(\alpha)}\int_u^{\varkappa}(\varkappa - \varphi)^{\alpha-1}[Y_*(\varphi,\ \lambda), Y^*(\varphi,\ \lambda)]d\varphi,\ (\varkappa > u), \tag{19}$$

where

$$\mathcal{I}_{u^+}^{\alpha}\ Y_*(\varkappa,\ \lambda) = \frac{1}{\Gamma(\alpha)}\int_u^{\varkappa}(\varkappa - \varphi)^{\alpha-1}Y_*(\varphi,\ \lambda)d\varphi, \quad (\varkappa > u), \tag{20}$$

and

$$\mathcal{I}_{u^+}^{\alpha}\ Y^*(\varkappa,\ \lambda) = \frac{1}{\Gamma(\alpha)}\int_u^{\varkappa}(\varkappa - \varphi)^{\alpha-1}Y^*(\varphi,\ \lambda)d\varphi, \quad (\varkappa > u), \tag{21}$$

Similarly, we can define right Riemann–Liouville fractional integral Y of $\varkappa$ based on left and right endpoint mappings.

Breckner discussed the coming emerging idea of interval-valued convexity in [79].

A I·V·M $Y : \mathbb{I} = [u,\ z] \to \mathcal{X}_I$ is called convex I·V·M if

$$Y(m\varkappa + (1-m)s) \supseteq mY(\varkappa) + (1-m)Y(s), \tag{22}$$

for all $\varkappa,\ y \in [u,\ z],\ m \in [0,\ 1]$, where $\mathcal{X}_I$ is the collection of all real-valued intervals. If (17) is reversed, then Y is called concave.

Definition 7. *The F-N·V·M* $\widetilde{Y} : [u,\ z] \to \mathbb{E}_C$ *is called convex F-N·V·M on* $[u,\ z]$ *if* [80]

$$\widetilde{Y}(m\varkappa + (1-m)s\ ) \leq_{\mathbb{F}} m \odot \widetilde{Y}(\varkappa) \oplus (1-m) \odot \widetilde{Y}(s), \tag{23}$$

for all $\varkappa,\ s \in [\mathrm{u},\ \mathrm{z}],\ \mathrm{m} \in [0,\ 1]$, where $\widetilde{\mathrm{Y}}(\varkappa) \geq_{\mathbb{F}} \widetilde{0}$ for all $\varkappa \in [\mathrm{u},\ \mathrm{z}]$. If (18) is reversed then, $\widetilde{\mathrm{Y}}$ is called concave F-N·V·M on $[\mathrm{u},\ \mathrm{z}]$. $\widetilde{\mathrm{Y}}$ is affine if and only if it is both convex and concave F-N·V·M.

Definition 8. *The F-N·V·M* $\widetilde{Y} : [u,\ z] \to \mathbb{E}_C$ *is called U·D convex F-N·V·M on* $[u,\ z]$ *if* [85]

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} m \odot \widetilde{Y}(\varkappa) \oplus (1-m) \odot \widetilde{Y}(s), \tag{24}$$

for all $\varkappa,\ s \in [u,\ z],\ m \in [0,\ 1]$, where $\widetilde{Y}(\varkappa) \geq_{\mathbb{F}} \widetilde{0}$ for all $\varkappa \in [u,\ z]$. If (19) is reversed then, $\widetilde{Y}$ is called *U·D* concave *F-N·V·M* on $[u,\ z]$. $\widetilde{Y}$ is *U·D* affine *F-N·V·M* if and only if it is both *U·D* convex and, *U·D* concave *F-N·V·M*.

Definition 9. *Let K be convex set and* $\lambda : [0,\ 1] \subseteq K \to \mathbb{R}^+$ *such that* $\lambda \not\equiv 0$. *Then F-N·V·M* $\widetilde{Y} : K \to \mathbb{E}_C$ *is said to be U·D λ-convex on K if*

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} \lambda(m) \odot \widetilde{Y}(\varkappa) \oplus \lambda(1-m) \odot \widetilde{Y}(s), \tag{25}$$

for all $\varkappa,\ s \in K,\ m \in [0,\ 1]$, where $\widetilde{Y}(\varkappa) \geq_{\mathbb{F}} \widetilde{0}$. The *F-N·V·M* $\widetilde{Y} : K \to \mathbb{E}_C$ is said to be *U·D* λ-concave on K if inequality (25) is reversed. Moreover, $\widetilde{Y}$ is known as affine *U·D* λ-convex *F-N·V·M* on K if

$$\widetilde{Y}(m\varkappa + (1-m)s) = \lambda(m) \odot \widetilde{Y}(\varkappa) \oplus \lambda(1-m) \odot \widetilde{Y}(s), \tag{26}$$

for all $\varkappa,\ s \in K,\ m \in [0,\ 1]$, where $\widetilde{Y}(\varkappa) \geq_{\mathbb{F}} \widetilde{0}$.

Remark 2. *The U·D λ-convex F-N·V·Ms have some very nice properties similar to convex F-N·V·M,*

1) if $\widetilde{Y}$ is *U·D* λ-convex *F-N·V·M*, then $\alpha\widetilde{Y}$ is also *U·D* λ-convex for $\alpha \geq 0$.
2) if $\widetilde{Y}$ and $\widetilde{\mathcal{T}}$ both are *U·D* λ-convex *F-N·V·Ms*, then $\max\left(\widetilde{Y}(\varkappa), \widetilde{\mathcal{T}}(\varkappa)\right)$ is also *U·D* λ-convex *F-N·V·M*.

Here, we will go through a few unique exceptional cases of *U·D* λ-convex *F-N·V·Ms*:

(i) If $\lambda(m) = m^s$, then *U·D* λ-convex *F-N·V·M* becomes *U·D s*-convex *F-N·V·M*, that is

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} m^s \odot \widetilde{Y}(\varkappa) \oplus (1-m)^s \odot \widetilde{Y}(s),\ \forall\, \varkappa,\ s \in K,\ m \in [0,\ 1]. \tag{27}$$

(ii) If $\lambda(m) = m$, then *U·D* λ-convex *F-N·V·M* becomes *U·D* convex *F-N·V·M*, see [84], that is

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} m \odot \widetilde{Y}(\varkappa) \oplus (1-m) \odot \widetilde{Y}(s),\ \forall\, \varkappa,\ s \in K,\ m \in [0,\ 1]. \tag{28}$$

(iii) If $\lambda(m) \equiv 1$, then *U·D* λ-convex *F-N·V·M* becomes *U·D P*-convex *F-N·V·M*, that is

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} \widetilde{Y}(\varkappa) \oplus \widetilde{Y}(s),\ \forall\ \varkappa,\ s \in K,\ m \in [0,\ 1]. \tag{29}$$

Note that there are also new special cases (i) and (iii) as well.

Theorem 4. *Let K be convex set, non-negative real-valued mapping $\lambda : [0,\ 1] \subseteq K \to \mathbb{R}$ such that $\lambda \not\equiv 0$ and let $\widetilde{Y} : K \to \mathbb{E}_C$ be a F-N·V·M, whose i-levels define the family of I·V·Ms $Y_i : K \subset \mathbb{R} \to \mathcal{X}_I^+ \subset \mathcal{X}_I$ are given by*

$$Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)], \tag{30}$$

for all $\varkappa \in K$ and for all $i \in [0,\ 1]$. Then $\widetilde{Y}$ is $U{\cdot}D$ λ-convex on K, if and only if, for all $i \in [0,\ 1]$, $Y_*(\varkappa,\ i)$ and $Y^*(\varkappa,\ i)$ are λ-convex.

Proof. Assume that for each $i \in [0,\ 1]$, $Y_*(\varkappa,\ i)$ and $Y^*(\varkappa,\ i)$ are λ-convex and λ-concave on K. Then, we have

$$Y_*(m\varkappa + (1-m)s,\ i) \le \lambda(m) Y_*(\varkappa,\ i) + \lambda(1-m) Y_*(s,\ i),\ \forall\, \varkappa, s \in K,\ m \in [0,\ 1],$$

and

$$Y^*(m\varkappa + (1-m)s,\ i) \ge \lambda(m) Y^*(\varkappa,\ i) + \lambda(1-m) Y^*(s,\ i), \forall\, \varkappa, s \in K,\ m \in [0,\ 1].$$

Then by (30), (10) and (12), we obtain

$$Y_i(m\varkappa + (1-m)s) = [Y_*(m\varkappa + (1-m)s,\ i),\ Y^*(m\varkappa + (1-m)s,\ i)],$$

$$\supseteq_I [\lambda(m) Y_*(\varkappa,\ i),\ \lambda(m) Y^*(\varkappa,\ i)] + [\lambda(1-m) Y_*(s,\ i),\ \lambda(1-m) Y^*(s,\ i)],$$

that is

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} \lambda(m) \odot \widetilde{Y}(\varkappa) \oplus \lambda(1-m) \odot \widetilde{Y}(s), \forall\, \varkappa, s \in K,\ m \in [0,\ 1].$$

Hence, $\widetilde{Y}$ is $U{\cdot}D$ λ-convex F-$N{\cdot}V{\cdot}M$ on K.

Conversely, let $\widetilde{Y}$ is $U{\cdot}D$ λ-convex F-$N{\cdot}V{\cdot}M$ on K. Then, for all $\varkappa, s \in K$ and $m \in [0,\ 1]$, we have

$$\widetilde{Y}(m\varkappa + (1-m)s) \supseteq_{\mathbb{F}} \lambda(m) \odot \widetilde{Y}(\varkappa) \oplus \lambda(1-m) \odot \widetilde{Y}(s).$$

Therefore, from (30), we have

$$Y_i(m\varkappa + (1-m)s) = [Y_*(m\varkappa + (1-m)s,\ i),\ Y^*(m\varkappa + (1-m)s,\ i)].$$

Again, from (30), (10) and (12), we obtain

$$\lambda(m) Y_i(\varkappa) + \lambda(1-m) Y_i(\varkappa)$$

$$= [\lambda(m) Y_*(\varkappa,\ i),\ \lambda(m) Y^*(\varkappa,\ i)] + [\lambda(1-m) Y_*(s,\ i),\ \lambda(1-m) Y^*(s,\ i)],$$

for all $\varkappa, s \in K$ and $m \in [0,\ 1]$. Then by $U{\cdot}D$ λ-convexity of $\widetilde{Y}$, we have for all $\varkappa, s \in K$ and $m \in [0,\ 1]$. such that

$$Y_*(m\varkappa + (1-m)s,\ i) \le \lambda(m) Y_*(\varkappa,\ i) + \lambda(1-m) Y_*(s,\ i),$$

and

$$Y^*(m\varkappa + (1-m)s,\ i) \ge \lambda(m) Y^*(\varkappa,\ i) + \lambda(1-m) Y^*(s,\ i),$$

for each $i \in [0,\ 1]$. Hence, the result follows. □

Remark 3. *If $Y_*(\varkappa,\ i) = Y^*(\varkappa,\ i)$ with $i = 1$, then $U{\cdot}D$ λ-convex F-N·V·M reduces to the λ-convex mapping.*

If $Y_*(\varkappa,\ i) = Y^*(\varkappa,\ i)$ with $i = 1$ and $\lambda(m) = m^s$ with $s \in (0,\ 1)$, then U·D λ-convex F-N·V·M reduces to the s-convex mapping.

If $Y_*(\varkappa,\ i) = Y^*(\varkappa,\ i)$ with $i = 1$ and $\lambda(m) = m$ with $s \in (0,\ 1)$, then *U·D* λ-convex *F-N·V·M* reduces to the convex mapping.

If $Y_*(\varkappa,\ i) = Y^*(\varkappa,\ i)$ with $i = 1$ and $\lambda(m) = 1$, then *U·D* λ-convex *F-N·V·M* reduces to the *P*-convex mapping.

Example 1. *We consider* $\lambda(m) = m$, *for* $m \in [0,\ 1]$ *and the F-N·V·M* $\widetilde{Y} : [0,\ 1] \to \mathbb{E}_C$ *defined by*

$$\widetilde{Y}(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{2\varkappa^2} & \sigma \in \left[0,\ 2\varkappa^2\right] \\ \frac{4\varkappa^2-\sigma}{2\varkappa^2} & \sigma \in \left(2\varkappa^2,\ 4\varkappa^2\right] \\ 0 & otherwise, \end{cases} \tag{31}$$

Then, for each $i \in [0,\ 1]$, we have $Y_i(\varkappa) = \left[2i\varkappa^2, (4-2i)\varkappa^2\right]$. Since end point mappings $Y_*(\varkappa, i)$, $Y^*(\varkappa, i)$ are λ-convex and λ-concave mappings for each $i \in [0,\ 1]$, respectively. Hence $\widetilde{Y}(\varkappa)$ is *U·D* λ-convex *F-N·V·M*.

Definition 10. *Let* $Y : [u,\ z] \to \mathbb{E}_C$ *be a F-N·V·M, whose i-levels define the family of I·V·Ms* $Y_i : [u,\ z] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ *are given by*

$$Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)], \tag{32}$$

for all $\varkappa \in [u,\ z]$ and for all $i \in [0,\ 1]$. Then, Y is lower *U·D* λ-convex (λ-concave) *F-N·V·M* on $[u,\ z]$, if and only if, for all $i \in [0,\ 1]$, $Y_*(\varkappa,\ i)$ is a λ-convex (λ-concave) mapping and $Y^*(\varkappa,\ \lambda)$ is a λ-affine mapping.

Definition 11. *Let* $Y : [u,\ z] \to \mathbb{E}_C$ *be a F-N·V·M, whose i-levels define the family of I·V·Ms* $Y_i : [u,\ z] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ *are given by*

$$Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)], \tag{33}$$

for all $\varkappa \in [u,\ z]$ and for all $i \in [0,\ 1]$. Then, Y is an upper *U·D* λ-convex (λ-concave) *F-N·V·M* on $[u,\ z]$, if and only if, for all $i \in [0,\ 1]$, $Y_*(\varkappa,\ i)$ is an λ-affine mapping and $Y^*(\varkappa,\ i)$ is a λ-convex (λ-concave) mapping.

Remark 4. *If* $\lambda(m) = m$, *then both concepts "U·D* λ*-convex F-N·V·M" and classical "convex F-N·V·M, see [80]" are behave alike when* Y *is lower U·D convex F-N·V·M.*

Both concepts "convex interval-valued mapping, see [79]"and "left and right λ-convex interval-valued mapping, see [85]" are coincident when Y is lower *U·D* λ-convex *F-N·V·M* with $i = 1$.

3. Main Results

The following is a proposal for our first primary result based on the newly presented fuzzy-number valued *U·D* λ-convex mappings and the H·H-type integral inequalities.

Theorem 5. *Let* $\widetilde{Y} : [u,\ z] \to \mathbb{E}_C$ *be a U·D* λ*-convex F-N·V·M on* $[u,\ z]$, *whose i-levels define the family of I·V·Ms* $Y_i : [u,\ z] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are given by* $Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)]$ *for all* $\varkappa \in [u,\ z]$ *and for all* $i \in [0,\ 1]$. *If* $\widetilde{Y} \in L([u,\ z], \mathbb{E}_C)$, *then*

$$\begin{aligned} &\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)} \odot \widetilde{Y}\left(\frac{u+z}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\alpha)}{(z-u)^{\alpha}} \odot \left[\mathcal{I}_{u^+}^{\alpha}\ \widetilde{Y}(z) \oplus \mathcal{I}_{z^-}^{\alpha}\ \widetilde{Y}(u)\right] \\ &\supseteq_{\mathbb{F}} \left[\widetilde{Y}(u) \oplus \widetilde{Y}(z)\right] \odot \int_0^1 m^{\alpha-1}[\lambda(m) + \lambda(1-m)]dm. \end{aligned} \tag{34}$$

If $\widetilde{Y}(\varkappa)$ is $U{\cdot}D$ λ-concave $F{\cdot}N{\cdot}V{\cdot}M$, then

$$\begin{gathered}\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)}\odot\widetilde{Y}\left(\frac{u+z}{2}\right)\subseteq_{\mathbb{F}}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\odot\left[\mathcal{I}_{u^{+}}^{\alpha}\ \widetilde{Y}(z)\oplus\mathcal{I}_{z^{-}}^{\alpha}\ \widetilde{Y}(u)\right]\\ \subseteq_{\mathbb{F}}\left[\widetilde{Y}(u)\oplus\widetilde{Y}(z)\right]\odot\int_{0}^{1}m^{\alpha-1}[\lambda(m)+\lambda(1-m)]dm.\end{gathered} \tag{35}$$

Proof. Let $\widetilde{Y}:[u,\ z]\to\mathbb{E}_{C}$ be a $U{\cdot}D$ λ-convex $F{\cdot}N{\cdot}V{\cdot}M$. Then, by hypothesis, we have

$$\frac{1}{\lambda\left(\frac{1}{2}\right)}\odot\widetilde{Y}\left(\frac{u+z}{2}\right)\supseteq_{\mathbb{F}}\widetilde{Y}(mu+(1-m)z)\oplus\widetilde{Y}((1-m)u+mz).$$

Therefore, for every $i\in[0,\ 1]$, we have

$$\begin{gathered}\frac{1}{\lambda\left(\frac{1}{2}\right)}Y_{*}\left(\frac{u+z}{2},\ i\right)\leq Y_{*}(mu+(1-m)z,\ i)+Y_{*}((1-m)u+mz,\ i),\\ \frac{1}{\lambda\left(\frac{1}{2}\right)}Y^{*}\left(\frac{u+z}{2},\ i\right)\geq Y^{*}(mu+(1-m)z,\ i)+Y^{*}((1-m)u+mz,i).\end{gathered}$$

Multiplying both sides by $m^{\alpha-1}$ and integrating the obtained result with respect to m over $(0,1)$, we have

$$\begin{gathered}\frac{1}{\lambda\left(\frac{1}{2}\right)}\int_{0}^{1}m^{\alpha-1}Y_{*}\left(\frac{u+z}{2},\ i\right)dm\\ \leq\int_{0}^{1}m^{\alpha-1}Y_{*}(mu+(1-m)z,i)dm+\int_{0}^{1}m^{\alpha-1}Y_{*}((1-m)u+mz,\ i)dm,\\ \frac{1}{\lambda\left(\frac{1}{2}\right)}\int_{0}^{1}m^{\alpha-1}Y^{*}\left(\frac{u+z}{2},i\right)dm\\ \geq\int_{0}^{1}m^{\alpha-1}Y^{*}(mu+(1-m)z,\ i)dm+\int_{0}^{1}m^{\alpha-1}Y^{*}((1-m)u+mz,i)dm.\end{gathered}$$

Let $x=mu+(1-m)z$ and $z=(1-m)u+mz$. Then, we have

$$\begin{gathered}\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)}Y_{*}\left(\frac{u+z}{2},\ i\right)\leq\frac{1}{(z-u)^{\alpha}}\ \int_{u}^{z}(z-x)^{\alpha-1}Y_{*}(x,i)dx+\frac{1}{(z-u)^{\alpha}}\int_{u}^{z}(z-u)^{\alpha-1}Y_{*}(z,i)dz\\ \frac{1}{\alpha\lambda\left(\frac{1}{2}\right)}Y_{*}\left(\frac{u+z}{2},\ i\right)\geq\frac{1}{(z-u)^{\alpha}}\ \int_{u}^{z}(z-x)^{\alpha-1}Y^{*}(x,i)dx+\frac{1}{(z-u)^{\alpha}}\int_{u}^{z}(z-u)^{\alpha-1}Y^{*}(z,i)dz,\end{gathered}$$

$$\begin{gathered}\leq\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^{+}}^{\alpha}\ Y_{*}(z,\ i)+\mathcal{I}_{z^{-}}^{\alpha}\ Y_{*}(u,\ i)\right]\\ \geq\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^{+}}^{\alpha}\ Y^{*}(z,\ i)+\mathcal{I}_{z^{-}}^{\alpha}\ Y^{*}(u,\ i)\right],\end{gathered}$$

that is

$$\begin{gathered}\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)}\left[Y_{*}\left(\frac{u+z}{2},\ i\right),\ Y^{*}\left(\frac{u+z}{2},\ i\right)\right]\\ \supseteq_{I}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\left[\mathcal{I}_{u^{+}}^{\alpha}\ Y_{*}(z,\ i)+\mathcal{I}_{z^{-}}^{\alpha}\ Y_{*}(u,\ i)\right],\ \left[\mathcal{I}_{u^{+}}^{\alpha}\ Y^{*}(z,\ i)+\mathcal{I}_{z^{-}}^{\alpha}\ Y^{*}(u,\ i)\right]\right],\end{gathered}$$

thus,

$$\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)}\ Y_{i}\left(\frac{u+z}{2}\right)\supseteq_{I}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^{+}}^{\alpha}\ Y_{i}(z)+\mathcal{I}_{z^{-}}^{\alpha}\ Y_{i}(u)\right]. \tag{36}$$

In a similar way as above, we have

$$\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^{+}}^{\alpha}\ Y_{i}(z)+\mathcal{I}_{z^{-}}^{\alpha}\ Y_{i}(u)\right]\supseteq_{I}[Y_{i}(u)+Y_{i}(z)]\int_{0}^{1}m^{\alpha-1}[\lambda(m)-\lambda(1-m)]dm. \tag{37}$$

Combining (36) and (37), we have

$$\begin{gathered}\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)}Y_{i}\left(\frac{u+z}{2}\right)\supseteq_{I}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^{+}}^{\alpha}\ Y_{i}(z)+\mathcal{I}_{z^{-}}^{\alpha}\ Y_{i}(u)\right]\\ \supseteq_{I}[Y_{i}(u)+Y_{i}(z)]\int_{0}^{1}m^{\alpha-1}[\lambda(m)-\lambda(1-m)]dm,\end{gathered}$$

that is

$$\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)} \odot \widetilde{Y}\left(\frac{u+z}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\alpha)}{(z-u)^{\alpha}} \odot \left[\mathcal{I}_{u+}^{\alpha}\ \widetilde{Y}(z) \oplus \mathcal{I}_{z-}^{\alpha}\ \widetilde{Y}(u)\right]$$
$$\supseteq_{\mathbb{F}} \left[\widetilde{Y}(u) \oplus \widetilde{Y}(z)\right] \odot \int_0^1 m^{\alpha-1}[\lambda(m) - \lambda(1-m)]dm.$$

Hence, the required result. □

Remark 5. *From Theorem 5 we clearly see that:*

If $\lambda(m) = m$, then Theorem 5 reduces to the result for λ-convex *F-N·V·M*, see [55]:

$$\widetilde{Y}\left(\frac{\tau+z}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\alpha+1)}{2(z-\tau)^{\alpha}} \odot \left[\mathcal{I}_{\tau+}^{\alpha}\ \widetilde{Y}(z) \oplus \mathcal{I}_{z-}^{\alpha}\ \widetilde{Y}(\tau)\right] \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\tau) \oplus \widetilde{Y}(z)}{2}. \tag{38}$$

If $\alpha = 1$, then Theorem 5 reduces to the result for λ-convex *F-N·V·M*, see [55]:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{Y}\left(\frac{u+z}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{z-u} \odot (FA) \int_u^z \widetilde{Y}(\varkappa)d\varkappa \supseteq_{\mathbb{F}} \left[\widetilde{Y}(u) \oplus \widetilde{Y}(z)\right] \odot \int_0^1 \lambda(m)dm. \tag{39}$$

If $\alpha = 1$ and $\widetilde{Y}$ is lower *U·Dλ*-convex *F-N·V·M*, then Theorem 5 reduces to the result for λ-convex *F-N·V·M*, see [54]:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{Y}\left(\frac{u+z}{2}\right) \leq_{\mathbb{F}} \frac{1}{z-u} \odot (FA) \int_u^z \widetilde{Y}(\varkappa)d\varkappa \leq_{\mathbb{F}} \left[\widetilde{Y}(u) \oplus \widetilde{Y}(z)\right] \odot \int_0^1 \lambda(m)dm. \tag{40}$$

If $\lambda(m) = m$ and $\widetilde{Y}$ is lower *U·D* λ-convex *F-N·V·M*, then Theorem 5 reduces to the result for convex *F-N·V·M*, see [53]:

$$\widetilde{Y}\left(\frac{u+z}{2}\right) \leq_{\mathbb{F}} \frac{\Gamma(\alpha+1)}{2(z-u)^{\alpha}} \odot \left[\mathcal{I}_{u+}^{\alpha}\ \widetilde{Y}(z) \oplus \mathcal{I}_{z-}^{\alpha}\ \widetilde{Y}(u)\right] \leq_{\mathbb{F}} \frac{\widetilde{Y}(u) \oplus \widetilde{Y}(z)}{2}. \tag{41}$$

Let $\alpha = 1$ and $\widetilde{Y}$ is lower *U·D* λ-convex *F-N·V·M* with $\lambda(m) = m$. Then, Theorem 5 reduces to the result for convex-I·V·M given in [52]:

$$\widetilde{Y}\left(\frac{u+z}{2}\right) \leq_{\mathbb{F}} \frac{1}{z-u} \odot (FA) \int_u^z \widetilde{Y}(\varkappa)d\varkappa \leq_{\mathbb{F}} \frac{\widetilde{Y}(u) \oplus \widetilde{Y}(z)}{2}. \tag{42}$$

Let $\alpha = 1 = i$ and $Y_*(\varkappa,\ i) \neq Y^*(\varkappa,\ i)$. Then, from Theorem 5 we obtain the following inequality given in [65]:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} Y\left(\frac{u+z}{2}\right) \supseteq \frac{1}{z-u} (IA) \int_u^z Y(\varkappa)d\varkappa \supseteq [Y(u) +\ Y(z)] \int_0^1 \lambda(m)dm. \tag{43}$$

If $Y_*(\varkappa,\ i) = Y^*(\varkappa,\ i)$ and $i = 1$, then, from Theorem 5 we obtain the following inequality given in [68]:

$$\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)} Y\left(\frac{u+z}{2}\right) \leq \frac{\Gamma(\alpha)}{(z-u)^{\alpha}} \left[\mathcal{I}_{u+}^{\alpha}\ Y(z) + \mathcal{I}_{z-}^{\alpha}\ Y(u)\right] \leq [Y(u) + Y(z)] \int_0^1 m^{\alpha-1}[\lambda(m) - \lambda(1-m)]dm. \tag{44}$$

Let $\alpha = 1 = i$ and $Y_*(\varkappa,\ i) = Y^*(\varkappa,\ i)$. Then, from Theorem 5 we obtain the following inequality given in [68]:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} Y\left(\frac{u+z}{2}\right) \leq \frac{1}{z-u} (R) \int_u^z Y(\varkappa)d\varkappa \leq [Y(u) + Y(z)] \int_0^1 \lambda(m)dm. \tag{45}$$

Example 2. *Let* $\alpha = \frac{1}{2}$, $\varkappa \in [2,3]$, *and the F-N·V·M* $Y : [u,\ z] = [2,\ 3] \to \mathbb{E}_C$, *defined by*

$$Y(\varkappa)(\theta) = \begin{cases} \frac{\theta - 2 + \varkappa^{\frac{1}{2}}}{1 - \varkappa^{\frac{1}{2}}} & \theta \in \left[2 - \varkappa^{\frac{1}{2}},\ 3\right] \\ \frac{2 + \varkappa^{\frac{1}{2}} - \theta}{\varkappa^{\frac{1}{2}} - 1} & \theta \in \left(3,\ 2 + \varkappa^{\frac{1}{2}}\right] \\ 0 & \textit{otherwise}, \end{cases} \tag{46}$$

Then, for each $i \in [0,\ 1]$, we have $Y_i(\varkappa) = \left[(1-i)\left(2 - \varkappa^{\frac{1}{2}}\right) + 3i, (1+i)\left(2 + \varkappa^{\frac{1}{2}}\right) + 3i\right]$. Since left and right endpoint mappings $Y_*(\varkappa, i) = i\left(2 - \varkappa^{\frac{1}{2}}\right)$, $Y^*(\varkappa,\ i) = (2-i)\left(2 + \varkappa^{\frac{1}{2}}\right)$, are λ-convex and λ-concave mappings with $\lambda(m) = m$, for each $m \in [0,\ 1]$ respectively, we have $Y(\varkappa)$ is U·D λ-convex F-N·V·M. We clearly see that $Y \in L([u,\ z], \mathbb{E}_C)$ and

$$\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)} Y_*\left(\frac{u+z}{2},\ i\right) = Y_*\left(\frac{5}{2},\ i\right) = (1-i)\left(8 - 2\sqrt{10}\right) + 12i$$

$$\frac{1}{\alpha\lambda\left(\frac{1}{2}\right)} Y^*\left(\frac{u+z}{2},\ i\right) = Y^*\left(\frac{5}{2},\ i\right) = (1+i)\left(8 + 2\sqrt{10}\right) + 12i$$

$$(Y_*(u,i) + Y_*(z,\ i)) \int_0^1 m^{\alpha-1}[\lambda(m) - \lambda(1-m)]dm = 2(1-i)\left(4 - \sqrt{2} - \sqrt{3}\right) + 12i$$

$$(Y^*(u,i) + Y^*(z,\ i)) \int_0^1 m^{\alpha-1}[\lambda(m) - \lambda(1-m)]dm = 2(1+i)\left(4 + \sqrt{2} + \sqrt{3}\right) + 12i.$$

Note that

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}^{\alpha}_{u^+}\ Y_*(z,\ i) + \mathcal{I}^{\alpha}_{z^-}\ Y_*(u,i)\right] \\ = \Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{\pi}}\int_2^3 (3-\varkappa)^{\frac{-1}{2}} \,.\, \left((1-i)\left(2 - \varkappa^{\frac{1}{2}}\right) + 3i\right)d\varkappa \\ +\Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{\pi}}\int_2^3 (\varkappa-2)^{\frac{-1}{2}} \,.\, \left((1-i)\left(2 - \varkappa^{\frac{1}{2}}\right) + 3i\right)d\varkappa \\ = (1-i)\frac{16{,}903}{10{,}000} + 12i. \end{array}$$

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}^{\alpha}_{u^+}\ Y^*(z,\ i) + \mathcal{I}^{\alpha}_{z^-}\ Y^*(u,i)\right] \\ = \Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{\pi}}\int_2^3 (3-\varkappa)^{\frac{-1}{2}} \,.\, \left((1+i)\left(2 + \varkappa^{\frac{1}{2}}\right) + 3i\right)d\varkappa \\ +\Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{\pi}}\int_2^3 (\varkappa-2)^{\frac{-1}{2}} \,.\, \left((1+i)\left(2 + \varkappa^{\frac{1}{2}}\right) + 3i\right)d\varkappa \\ = (1+i)\frac{143{,}097}{10{,}000} + 12i. \end{array}$$

Therefore

$$\begin{array}{c} \left[(1-i)\left(8 - 2\sqrt{10}\right) + 12i, (1+i)\left(8 + 2\sqrt{10}\right) + 12i\right] \\ \supseteq_I \left[(1-i)\frac{16{,}903}{10{,}000} + 12i, (1+i)\frac{143{,}097}{10{,}000} + 12i\right] \\ \supseteq_I \left[2(1-i)\left(\left(4 - \sqrt{2} - \sqrt{3}\right)\right) + 12i, 2(1+i)\left(\left(4 + \sqrt{2} + \sqrt{3}\right)\right) + 12i\right], \end{array}$$

and Theorem 5 is verified.

We propose the following Pachpatte-type fractional integral inclusions taking use of the fuzzy-number valued $U{\cdot}D$ λ-convexity:

Theorem 6. *Let $\widetilde{Y}, \widetilde{T} : [u, z] \to \mathbb{E}_C$ be $U{\cdot}D$ λ_1-convex and λ_2-convex F-N·V·Ms on $[u, z]$, respectively, whose i-levels $Y_i, T_i : [u, z] \subset \mathbb{R} \to \mathcal{K}_C^+$ are defined by $Y_i(\varkappa) = [Y_*(\varkappa, i), Y^*(\varkappa, i)]$ and $T_i(\varkappa) = [T_*(\varkappa, i), T^*(\varkappa, i)]$ for all $\varkappa \in [u, z]$ and for all $i \in [0, 1]$. If $\widetilde{Y} \otimes \widetilde{T} \in L([u, z], \mathbb{E})$, then*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(z-u)^\alpha} \odot \left[\mathcal{I}_{u^+}^\alpha \widetilde{Y}(z) \otimes \widetilde{T}(z) \oplus \mathcal{I}_{z^-}^\alpha \widetilde{Y}(u) \otimes \widetilde{T}(u)\right] \\
&\supseteq_{\mathbb{F}} \widetilde{\Delta}(u,z) \odot \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)]dm \\
&\oplus \widetilde{\nabla}(u,z) \odot \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)]dm,
\end{aligned} \tag{47}$$

where $\widetilde{\Delta}(u,z) = \widetilde{Y}(u) \otimes \widetilde{T}(u) \oplus \widetilde{Y}(z) \otimes \widetilde{T}(z)$,, $\widetilde{\nabla}(u,z) = \widetilde{Y}(u) \otimes \widetilde{T}(z) \oplus \widetilde{Y}(z) \otimes \widetilde{T}(u)$, and $\Delta_i(u,z) = [\Delta_*((u,z), i), \Delta^*((u,z), i)]$ and $\nabla_i(u,z) = [\nabla_*((u,z), i), \nabla^*((u,z), i)]$.

Proof. Since $\widetilde{Y}, \widetilde{T}$ both are *U·D* λ_1-convex and λ_2-convex *F-N·V·M*s then, for each $i \in [0, 1]$ we have

$$\begin{aligned}
Y_*(mu + (1-m)z, i) \leq \lambda_1(m)Y_*(u,i) + \lambda_1(1-m)Y_*(z, i) \\
Y^*(mu + (1-m)z, i) \geq \lambda_1(m)Y^*(u,i) + \lambda_1(1-m)Y^*(z, i).
\end{aligned}$$

and

$$\begin{aligned}
T_*(mu + (1-m)z, i) \leq \lambda_2(m)T_*(u,i) + \lambda_2(1-m)T_*(z, i) \\
T^*(mu + (1-m)z, i) \geq \lambda_2(m)T^*(u,i) + \lambda_2(1-m)T^*(z, i).
\end{aligned}$$

From the Definition of *U·D* λ-convex *F-N·V·M*s it follows that $\widetilde{0} \leq_{\mathbb{F}} \widetilde{Y}(\varkappa)$ and $\widetilde{0} \leq_{\mathbb{F}} \widetilde{T}(\varkappa)$, so

$$\begin{aligned}
&Y_*(mu + (1-m)z, i) \times T_*(mu + (1-m)z, i) \\
&\leq \lambda_1(m)\lambda_2(m)Y_*(u,i) \times T_*(u,i) + \lambda_1(1-m)\lambda_2(1-m)Y_*(z, i) \times T_*(z, i) \\
&+\lambda_1(m)\lambda_2(1-m)Y_*(u,i) \times T_*(z, i) + \lambda_1(1-m)\lambda_2(m)Y_*(z, i) \times T_*(u,i) \\
&Y^*(mu + (1-m)z, i) \times T^*(mu + (1-m)z, i) \\
&\geq \lambda_1(m)\lambda_2(m)Y^*(u,i) \times T^*(u,i) + \lambda_1(1-m)\lambda_2(1-m)Y^*(z, i) \times T^*(z, i) \\
&+\lambda_1(m)\lambda_2(1-m)Y^*(u,i) \times T^*(z, i) + \lambda_1(1-m)\lambda_2(m)Y^*(z, i) \times T^*(u,i).
\end{aligned} \tag{48}$$

Analogously, we have

$$\begin{aligned}
&Y_*((1-m)u + mz, i)T_*((1-m)u + mz, i) \\
&\leq \lambda_1(1-m)\lambda_2(1-m)Y_*(u,i) \times T_*(u,i) + \lambda_1(m)\lambda_2(m)Y_*(z, i) \times T_*(z, i) \\
&+\lambda_1(1-m)\lambda_2(m)Y_*(u,i) \times T_*(z, i) + \lambda_1(m)\lambda_2(1-m)Y_*(z, i) \times T_*(u,i) \\
&Y^*((1-m)u + mz, i) \times T^*((1-m)u + mz, i) \\
&\geq \lambda_1(1-m)\lambda_2(1-m)Y^*(u,i) \times T^*(u,i) + \lambda_1(m)\lambda_2(m)Y^*(z, i) \times T^*(z, i) \\
&+\lambda_1(1-m)\lambda_2(m)Y^*(u,i) \times T^*(z, i) + \lambda_1(m)\lambda_2(1-m)Y^*(z, i) \times T^*(u,i).
\end{aligned} \tag{49}$$

Adding (48) and (49), we have

$$\begin{aligned}
&Y_*(mu + (1-m)z, i) \times T_*(mu + (1-m)z, i) \\
&+Y_*((1-m)u + mz, i) \times T_*((1-m)u + mz, i) \\
&\leq [\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)][Y_*(u,i) \times T_*(u,i) + Y_*(z, i) \times T_*(z, i)] \\
&+[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)][Y_*(z, i) \times T_*(u,i) + Y_*(u,i) \times T_*(z, i)] \\
&Y^*(mu + (1-m)z, i) \times T^*(mu + (1-m)z, i) \\
&+Y^*((1-m)u + mz, i) \times T^*((1-m)u + mz, i) \\
&\geq [\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)][Y^*(u,i) \times T^*(u,i) + Y^*(z, i) \times T^*(z, i)] \\
&+[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)][Y^*(z, i) \times T^*(u,i) + Y^*(u,i) \times T^*(z, i)].
\end{aligned} \tag{50}$$

Taking multiplication of (50) with $m^{\alpha-1}$ and integrating the obtained result with respect to m over (0,1), we have

$$\begin{array}{c}\int_0^1 m^{\alpha-1}Y_*(mu+(1-m)z,\ i)\times T_*(mu+(1-m)z,\ i)\\ +m^{\alpha-1}Y_*((1-m)u+mz,\ i)\times T_*((1-m)u+mz,\ i)dm\\ \leq \Delta_*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ +\nabla_*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm\\ \int_0^1 m^{\alpha-1}Y^*(mu+(1-m)z,\ i)\times T^*(mu+(1-m)z,\ i)\\ +m^{\alpha-1}Y^*((1-m)u+mz,\ i)\times T^*((1-m)u+mz,\ i)dm\\ \geq \Delta^*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ +\nabla^*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm.\end{array}$$

It follows that,

$$\begin{array}{c}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y_*(z,\ i)\times T_*(z,\ i)+\mathcal{I}_{z^-}^{\alpha}\ Y_*(u,i)\times T_*(u,i)\right]\\ \leq \Delta_*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ +\nabla_*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm.\\ \frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y^*(z,\ i)\times T^*(z,\ i)+\mathcal{I}_{z^-}^{\alpha}\ Y^*(u,i)\times T^*(u,i)\right]\\ \geq\ \Delta^*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ +\nabla^*((u,z),\ i)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm.\end{array}$$

It follows that

$$\begin{array}{c}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}Y_*(z,\ i)\times T_*(z,\ i)+\mathcal{I}_{z^-}^{\alpha}\ Y_*(u,i)\times T_*(u,i),\ \mathcal{I}_{u^+}^{\alpha}\ Y^*(z,\ i)\times T^*(z,\ i)+\mathcal{I}_{z^-}^{\alpha}\ Y^*(u,i)\times T^*(u,\ i)\right]\\ \supseteq_I [\Delta_*((u,z),\ i),\ \Delta^*((u,z),\ i)]\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ +[\nabla_*((u,z),\ i),\ \nabla^*((u,z),\ i)]\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm,\end{array}$$

that is

$$\begin{array}{c}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y_i(z)\times T_i(z)+\mathcal{I}_{z^-}^{\alpha}Y_i(u)\times T_i(u)\right]\\ \supseteq_I \Delta_i(u,z)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ +\nabla_i(u,z)\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm.\end{array}$$

Thus,

$$\begin{array}{c}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\odot\left[\mathcal{I}_{u^+}^{\alpha}\ \widetilde{Y}(z)\otimes\widetilde{T}(z)\oplus\mathcal{I}_{z^-}^{\alpha}\widetilde{Y}(u)\otimes\widetilde{T}(u)\right]\\ \supseteq_{\mathbb{F}}\widetilde{\Delta}(u,z)\odot\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)]dm\\ \oplus\widetilde{\nabla}(u,z)\odot\int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)]dm.\end{array}$$

and the theorem has been established. □

Example 3. *Let* $[u,z]=[0,2]$, $\alpha=\frac{1}{2}$, *and the F-N·V·Ms* $\widetilde{Y},\widetilde{T}:[u,\ z]=[0,2]\to\mathbb{E}_C$, *defined by*

$$\widetilde{Y}(\varkappa)(\theta)=\begin{cases}\frac{\theta}{\varkappa} & \theta\in[0,\ \varkappa]\\ \frac{2\varkappa-\theta}{\varkappa} & \theta\in(\varkappa,\ 2\varkappa]\\ 0 & otherwise,\end{cases} \quad (51)$$

$$\widetilde{T}(\varkappa)(\theta)=\begin{cases}\frac{\theta-\varkappa}{2-\varkappa} & \theta\in[\varkappa,\ 2]\\ \frac{8-e^{\varkappa}-\theta}{8-e^{\varkappa}-2} & \theta\in\left(2,\ 8-e^{\varkappa}\right]\\ 0 & otherwise.\end{cases} \quad (52)$$

Then, for each $i\in[0,\ 1]$, we have $Y_i(\varkappa)=[i\varkappa,(2-i)\varkappa]$ and $T_i(\varkappa)=\left[(1-i)\varkappa+2i,(1-i)\left(8-e^{\varkappa}\right)+2i\right]$. Since left and right endpoint mappings $Y_*(\varkappa,i)=i\varkappa$, $Y^*(\varkappa,\ i)=(2-i)\varkappa$, $T_*(\varkappa,i)=(1-i)\varkappa+2i$ and $T^*(\varkappa,\ i)=(1-i)\left(8-e^{\varkappa}\right)+2i$ are λ-

convex and λ-concave mappings with $\lambda(m) = m$, for each $i \in [0,\ 1]$, we have $Y(\varkappa)$ and $T(\varkappa)$ both are U·D λ-convex F-N·V·Ms with $\lambda(m) = m$. We clearly see that $\widetilde{Y}(\varkappa) \otimes \widetilde{T}(\varkappa) \in L([u,\ z], \mathbb{F})$ and

$$\begin{gathered}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y_*(z) \times T_*(z) + \mathcal{I}_{z^-}^{\alpha}\ Y_*(u) \times T_*(u)\right] \\ = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (2-\varkappa)^{\frac{-1}{2}} \left(i(1-i)\varkappa^2 + 2i^2\varkappa\right) d\varkappa + \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (\varkappa)^{\frac{-1}{2}} \left(i(1-i)\varkappa^2 + 2i^2\varkappa\right) d\varkappa \\ = \frac{8}{15} i(4i+11),\end{gathered}$$

$$\begin{gathered}\frac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y^*(z) \times T^*(z) + \mathcal{I}_{z^-}^{\alpha}\ Y^*(u) \times T^*(u)\right] \\ = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (2-\varkappa)^{\frac{-1}{2}} .\left[(1-i)(2-i)\varkappa\left(8-e^{\varkappa}\right) + 2i(2-i)\varkappa\right] d\varkappa \\ + \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (\varkappa)^{\frac{-1}{2}} .\left[(1-i)(2-i)\varkappa\left(8-e^{\varkappa}\right) + 2i(2-i)\varkappa\right] d\varkappa \\ \approx \frac{8}{5}(2-i)(8-3i).\end{gathered}$$

Note that

$$\begin{gathered}\Delta_*((u,z),\ i) \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)]dm \\ = [Y_*(u) \times T_*(u) + Y_*(z) \times T_*(z)] \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)]dm = \frac{88}{15} i,\end{gathered}$$

$$\begin{gathered}\Delta^*((u,z),\ i) \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)]dm \\ = [Y^*(u) \times T^*(u) + Y^*(z) \times T^*(z)] \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)]dm \\ = \frac{44}{15}.(2-i)\left[(1-i)\left(8-e^2\right) + 2i\right],\end{gathered}$$

$$\begin{gathered}\nabla_*((u,z),\ i) \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)]dm \\ = [Y_*(u) \times T_*(z) + Y_*(z) \times T_*(u)] \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)]dm \\ = \frac{32}{15} i^2,\end{gathered}$$

$$\begin{gathered}\nabla^*((u,z),\ i) \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)]dm \\ = [Y^*(u) \times T^*(z) + Y^*(z) \times T^*(u)] \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)]dm \\ = \frac{16}{15}(2-i)(7-5i).\end{gathered}$$

Therefore, we have

$$\begin{gathered}\Delta_i(u,z) \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(m) + \lambda_1(1-m)\lambda_2(1-m)]dm \\ + \nabla_i(u,z) \int_0^1 m^{\alpha-1}[\lambda_1(m)\lambda_2(1-m) + \lambda_1(1-m)\lambda_2(m)]dm \\ = \frac{44}{15}\left[2i, (2-i)\left[(1-i)\left(8-e^2\right) + 2i\right]\right] + \frac{16}{15}\left[2i^2, (2-i)(7-5i)\right] \\ = \frac{4}{15}\left[2i(11+4i), (2-i)\left[11(1-i)\left(8-e^2\right) + 2i + 28\right]\right].\end{gathered}$$

It follows that

$$\begin{gathered}\frac{4}{5}\left[\frac{2}{3} i(11+4i), 2(2-i)(8-3i)\right] \\ \supseteq_I \frac{4}{15}\left[2i(11+4i), (2-i)\left[11(1-i)\left(8-e^2\right) + 2i + 28\right]\right]\end{gathered}$$

and Theorem 6 has been demonstrated.

Theorem 7. *Let $\widetilde{Y}, \widetilde{T} : [u,\ z] \to \mathbb{E}_C$ be two U·D λ_1-convex and λ_2-convex F-N·V·Ms, respectively, whose i-levels define the family of I·V·Ms $Y_i,\ T_i : [u,\ z] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)]$ and $T_i(\varkappa) = [T_*(\varkappa, i),\ T^*(\varkappa, i)]$ for all $\varkappa \in [u,\ z]$ and for all $i \in [0,\ 1]$. If $\widetilde{Y} \otimes \widetilde{T} \in L([u,\ z], \mathbb{E})$, then*

$$\begin{gathered}\frac{1}{\alpha\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)} \odot \widetilde{Y}\left(\frac{u+z}{2}\right) \otimes \widetilde{T}\left(\frac{u+z}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\alpha)}{(z-u)^{\alpha}} \odot \left[\mathcal{I}_{u^+}^{\alpha}\ \widetilde{Y}(z) \otimes \widetilde{T}(z) \oplus \mathcal{I}_{z^-}^{\alpha}\ \widetilde{Y}(u) \otimes \widetilde{T}(u)\right] \\ \oplus \widetilde{\nabla}(u,z) \odot \int_0^1 \left[m^{\alpha-1} + (1-m)^{\alpha-1}\right] \lambda_1(m)\lambda_2(1-m)dm \\ \oplus \widetilde{\Delta}(u,z) \odot \int_0^1 \left[m^{\alpha-1} + (1-m)^{\alpha-1}\right] \lambda_1(1-m)\lambda_2(1-m)dm.\end{gathered} \tag{53}$$

where $\widetilde{\Delta}(u,z) = \widetilde{Y}(u) \otimes \widetilde{T}(u) \oplus \widetilde{Y}(z) \otimes \widetilde{T}(z)$, $\widetilde{\nabla}(u,z) = \widetilde{Y}(u) \otimes \widetilde{T}(z) \oplus \widetilde{Y}(z) \otimes \widetilde{T}(u)$, and $\Delta_i(u,z) = [\Delta_*((u,z),\ i),\ \Delta^*((u,z),\ i)]$ and $\nabla_i(u,z) = [\nabla_*((u,z),\ i),\ \nabla^*((u,z),\ i)]$.

Proof. Consider $\widetilde{Y}, \widetilde{T} : [u,\ z] \to \mathbb{E}_C$ are $U{\cdot}D$ λ_1-convex and λ_2-convex $F{\cdot}N{\cdot}V{\cdot}M$s. Then, by hypothesis, for each $i \in [0,\ 1]$, we have

$$\begin{gathered}
Y_*\left(\tfrac{u+z}{2},\ i\right) \times T_*\left(\tfrac{u+z}{2}, i\right) \\
Y^*\left(\tfrac{u+z}{2}, i\right) \times T^*\left(\tfrac{u+z}{2}, i\right)
\end{gathered}$$

$$\begin{aligned}
&\leq \lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y_*(mu+(1-m)z,\ i) \times T_*(mu+(1-m)z,\ i) \\ +Y_*(mu+(1-m)z,\ i) \times T_*((1-m)u+mz,\ i) \end{bmatrix} \\
&+\lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y_*((1-m)u+mz,\ i) \times T_*(mu+(1-m)z,\ i) \\ +Y_*((1-m)u+mz,\ i) \times T_*((1-m)u+mz,\ i) \end{bmatrix} \\
&\geq \lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y^*(mu+(1-m)z,\ i) \times T^*(mu+(1-m)z,\ i) \\ +Y^*(mu+(1-m)z,\ i) \times T^*((1-m)u+mz,\ i) \end{bmatrix} \\
&+\lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y^*((1-m)u+mz,\ i) \times T^*(mu+(1-m)z,\ i) \\ +Y^*((1-m)u+mz,\ i) \times T^*((1-m)u+mz,\ i) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y_*(mu+(1-m)z,\ i) \times T_*(mu+(1-m)z,\ i) \\ +Y_*((1-m)u+mz,\ i) \times T_*((1-m)u+mz,\ i) \end{bmatrix} \\
&+\lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} (mY_*(u,\ i)+(1-m)Y_*(z,\ i)) \\ \times((1-m)T_*(u,\ i)+mT_*(z,\ i)) \\ +((1-m)Y_*(u,\ i)+mY_*(z,\ i)) \\ \times(mT_*(u,\ i)+(1-m)T_*(z,\ i)) \end{bmatrix} \\
&\geq \lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y^*(mu+(1-m)z,\ i) \times T^*(mu+(1-m)z,\ i) \\ +Y^*((1-m)u+mz,\ i) \times T^*((1-m)u+mz,\ i) \end{bmatrix} \\
&+\lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} (mY^*(u,\ i)+(1-m)Y^*(z,\ i)) \\ \times((1-m)T^*(u,\ i)+mT^*(z,\ i)) \\ +((1-m)Y^*(u,\ i)+mY^*(z,\ i)) \\ \times(mT^*(u,\ i)+(1-m)T^*(z,\ i)) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&= \lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y_*(mu+(1-m)z,\ i) \times T_*(mu+(1-m)z,\ i) \\ +Y_*((1-m)u+mz,\ i) \times T_*((1-m)u+mz,\ i) \end{bmatrix} \\
&+\lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)\}\nabla_*((u,z),\ i) \\ +\{\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)\}\Delta_*((u,z),\ i) \end{bmatrix} \\
&= \lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} Y^*(mu+(1-m)z,\ i) \times T^*(mu+(1-m)z,\ i) \\ +Y^*((1-m)u+mz,\ i) \times T^*((1-m)u+mz,\ i) \end{bmatrix} \\
&+\lambda_1\left(\tfrac{1}{2}\right)\lambda_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{\lambda_1(m)\lambda_2(1-m)+\lambda_1(1-m)\lambda_2(m)\}\nabla^*((u,z),\ i) \\ +\{\lambda_1(m)\lambda_2(m)+\lambda_1(1-m)\lambda_2(1-m)\}\Delta^*((u,z),\ i) \end{bmatrix}.
\end{aligned} \tag{54}$$

Taking multiplication of (54) with $m^{\alpha-1}$ and integrating over $(0,\ 1)$, we get

$$\begin{gathered}
\frac{1}{\alpha\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)} Y_*\left(\tfrac{u+z}{2}, i\right) \times T_*\left(\tfrac{u+z}{2}, i\right) \\
\leq \frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ Y_*(z) \times T_*(z) + \mathcal{I}_{z^-}^\alpha\ Y_*(u) \times T_*(u)\right] \\
+\nabla_*((u,z),\ i)\int_0^1 \left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(m)\lambda_2(1-m)dm. \\
+\Delta_*((u,z),\ i)\int_0^1 \left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(1-m)\lambda_2(1-m)dm \\
\frac{1}{\alpha\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)} Y^*\left(\tfrac{u+z}{2}, i\right) \times T^*\left(\tfrac{u+z}{2}, i\right) \\
\geq \frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ Y^*(z) \times T^*(z) + \mathcal{I}_{z^-}^\alpha\ Y^*(u) \times T^*(u)\right] \\
+\nabla^*((u,z),\ i)\int_0^1 \left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(m)\lambda_2(1-m)dm \\
+\Delta^*((u,z),\ i)\int_0^1 \left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(1-m)\lambda_2(1-m)dm.
\end{gathered}$$

It follows that

$$\begin{aligned}&\frac{1}{\alpha\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)}Y_i\left(\frac{u+z}{2}\right)\times T_i\left(\frac{u+z}{2}\right)\\ &\supseteq_I \frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\ Y_i(z)\times T_i(z)+\mathcal{I}^\alpha_{z^-}\ Y_i(u)\times T_i(u)\right]\\ &+\nabla_i(u,z)\int_0^1\left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(1-m)\lambda_2(1-m)dm\\ &+\Delta_i(u,z)\int_0^1\left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(1-m)\lambda_2(1-m)dm,\end{aligned}$$

that is

$$\begin{aligned}&\frac{1}{\alpha\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)}\widetilde{Y}\left(\frac{u+z}{2}\right)\widetilde{\times}\widetilde{T}\left(\frac{u+z}{2}\right)\\ &\supseteq_{\mathbb{F}} \frac{\Gamma(\alpha)}{(z-u)^\alpha}\odot\left[\mathcal{I}^\alpha_{u^+}\ \widetilde{Y}(z)\otimes\widetilde{T}(z)\oplus\mathcal{I}^\alpha_{z^-}\ \widetilde{Y}(u)\otimes\widetilde{T}(u)\right]\\ &\oplus\widetilde{\nabla}(u,z)\odot\int_0^1\left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(1-m)\lambda_2(1-m)dm\\ &\oplus\widetilde{\Delta}(u,z)\odot\int_0^1\left[m^{\alpha-1}+(1-m)^{\alpha-1}\right]\lambda_1(1-m)\lambda_2(1-m)dm.\end{aligned}$$

Hence, the required result. □

Let us introduce a new version of fuzzy fractional H·H-type-Fejér inequality with the help of $U{\cdot}D$ λ-convex $F\text{-}N{\cdot}V{\cdot}M$. Firstly, we obtain the right part of classical H·H Fejér inequality through fuzzy Riemann–Liouville fractional integral is known as the second fuzzy fractional H·H Fejér inequality.

Theorem 8. *(Second fuzzy fractional H·H Fejér type inequality) Let* $\widetilde{Y}:[u,\ z]\to\mathbb{E}_C$ *be a* $U{\cdot}D$ λ*-convex* $F\text{-}N{\cdot}V{\cdot}M$ *with* $u<z$*, whose i-levels define the family of* $I{\cdot}V{\cdot}Ms$ $Y_i:[u,\ z]\subset\mathbb{R}\to\mathcal{K}_C^+$ *are given by* $Y_i(\varkappa)=[Y_*(\varkappa,i),\ Y^*(\varkappa,i)]$ *for all* $\varkappa\in[u,\ z]$ *and for all* $i\in[0,\ 1]$*. If* $\widetilde{Y}\in L([u,\ z],\mathbb{E}_C)$ *and* $\mathfrak{F}:[u,\ z]\to\mathbb{R}$, $\mathfrak{F}(\varkappa)\geq 0$, *symmetric with respect to* $\frac{u+z}{2}$*, then*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(z-u)^\alpha}\odot\left[\mathcal{I}^\alpha_{u^+}\ \widetilde{Y}\mathfrak{F}(z)\oplus\mathcal{I}^\alpha_{z^-}\ \widetilde{Y}\mathfrak{F}(u)\right]\\ &\supseteq_{\mathbb{F}}\frac{\widetilde{Y}(u)\oplus\widetilde{Y}(z)}{2}\odot\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]F((1-m)u+mz)dm.\end{aligned}\tag{55}$$

If $\widetilde{Y}$ is $U{\cdot}D$ λ-concave $F\text{-}N{\cdot}V{\cdot}M$, then inequality (55) is reversed.

Proof. Let $\widetilde{Y}$ be a $U{\cdot}D$ λ-convex $F\text{-}N{\cdot}V{\cdot}M$ and $m^{\alpha-1}\mathfrak{F}(mu+(1-m)z)\geq 0$. Then, for each $i\in[0,\ 1]$, we have

$$\begin{aligned}&m^{\alpha-1}Y_*(mu+(1-m)z,\ i)F(mu+(1-m)z)\\ &\leq m^{\alpha-1}(\lambda(m)Y_*(u,\ i)+\lambda(1-m)Y_*(z,\ i))\mathfrak{F}(mu+(1-m)z)\\ &m^{\alpha-1}Y^*(mu+(1-m)z,\ i)\mathfrak{F}(mu+(1-m)z)\\ &\geq m^{\alpha-1}(\lambda(m)Y^*(u,\ i)+\lambda(1-m)Y^*(z,\ i))\mathfrak{F}(mu+(1-m)z),\end{aligned}\tag{56}$$

and

$$\begin{aligned}&m^{\alpha-1}Y_*((1-m)u+mz,\ i)\mathfrak{F}((1-m)u+mz)\\ &\leq m^{\alpha-1}(\lambda(1-m)Y_*(u,\ i)+\lambda(m)Y_*(z,\ i))\mathfrak{F}((1-m)u+mz)\\ &m^{\alpha-1}Y^*((1-m)u+mz,\ i)\mathfrak{F}((1-m)u+mz)\\ &\geq m^{\alpha-1}(\lambda(1-m)Y^*(u,\ i)+\lambda(m)Y^*(z,\ i))\mathfrak{F}((1-m)u+mz).\end{aligned}\tag{57}$$

After adding (56) and (57), and integrating over $[0,\ 1]$, we get

$$\begin{gathered}
\int_0^1 m^{\alpha-1} Y_*(mu+(1-m)z,\ i)\mathfrak{F}(mu+(1-m)z)dm \\
+\int_0^1 m^{\alpha-1} Y_*((1-m)u+mz,\ i)\mathfrak{F}((1-m)u+mz)dm \\
\leq \int_0^1 \left[\begin{array}{c} m^{\alpha-1} Y_*(u,\ i)\{\lambda(m)\mathfrak{F}(mu+(1-m)z)+\lambda(1-m)\mathfrak{F}((1-m)u+mz)\} \\ +m^{\alpha-1} Y_*(z,\ i)\{\lambda(1-m)\mathfrak{F}(mu+(1-m)z)+\lambda(m)\mathfrak{F}((1-m)u+mz)\}\end{array}\right] dm \\
= Y_*(u,\ i)\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]\mathfrak{F}(mu+(1-m)z)dm \\
+Y_*(z,\ i)\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]\mathfrak{F}((1-m)u+mz)dm, \\
\int_0^1 m^{\alpha-1} Y^*((1-m)u+mz,\ i)\mathfrak{F}((1-m)u+mz)dm \\
+\int_0^1 m^{\alpha-1} Y^*(mu+(1-m)z,\ i)\mathfrak{F}(mu+(1-m)z)dm \\
\geq \int_0^1 \left[\begin{array}{c} m^{\alpha-1} Y^*(u,\ i)\{\lambda(m)\mathfrak{F}(mu+(1-m)z)+\lambda(1-m)\mathfrak{F}((1-m)u+mz)\} \\ +m^{\alpha-1} Y^*(z,\ i)\{\lambda(1-m)\mathfrak{F}(mu+(1-m)z)+\lambda(m)\mathfrak{F}((1-m)u+mz)\}\end{array}\right] dm \\
= Y^*(u,\ i)\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]\mathfrak{F}(mu+(1-m)z)dm \\
+Y^*(z,\ i)\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]\mathfrak{F}((1-m)u+mz)dm.
\end{gathered} \tag{58}$$

Taking right hand side of inequality (58), we have

$$\begin{gathered}
\int_0^1 m^{\alpha-1} Y_*(mu+(1-m)z,\ i)\mathfrak{F}((1-m)u+mz)dm \\
+\int_0^1 m^{\alpha-1} Y_*((1-m)u+mz,\ i)\mathfrak{F}((1-m)u+mz)dm \\
= \frac{1}{(z-u)^\alpha}\int_u^z (\varkappa-u)^{\alpha-1} Y_*(u+z-\varkappa, i)\mathfrak{F}(\varkappa)d\varkappa \\
+\frac{1}{(z-u)^\alpha}\int_u^z (\varkappa-u)^{\alpha-1} Y_*(\varkappa, i)\mathfrak{F}(\varkappa)d\varkappa \\
= \frac{1}{(z-u)^\alpha}\int_u^z (z-u)^{\alpha-1} Y_*(\varkappa, i)\mathfrak{F}(u+z-\varkappa)d\varkappa \\
+\frac{1}{(z-u)^\alpha}\int_u^z (\varkappa-u)^{\alpha-1} Y_*(\varkappa, i)\mathfrak{F}(\varkappa)d\varkappa \\
= \frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ Y_*\mathfrak{F}(z)+\mathcal{I}_{z^-}^\alpha\ Y_*\mathfrak{F}(u)\right], \\
\int_0^1 m^{\alpha-1} Y^*(mu+(1-m)z,\ i)\mathfrak{F}((1-m)u+mz)dm \\
+\int_0^1 m^{\alpha-1} Y^*((1-m)u+mz,\ i)\mathfrak{F}((1-m)u+mz)dm \\
= \frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ Y^*\mathfrak{F}(z)+\mathcal{I}_{z^-}^\alpha\ Y^*\mathfrak{F}(u)\right].
\end{gathered} \tag{59}$$

From (59), we have

$$\begin{gathered}
\frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha Y_*\mathfrak{F}(z)+\mathcal{I}_{z^-}^\alpha Y_*\mathfrak{F}(u)\right] \\
\leq [Y_*(u,\ i)+Y_*(z,\ i)]\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]F((1-m)u+mz) \\
\frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha Y^*\mathfrak{F}(z)+\mathcal{I}_{z^-}^\alpha Y^*\mathfrak{F}(u)\right] \\
\geq [Y^*(u,\ i)+Y^*(z,\ i)]\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]F((1-m)u+mz),
\end{gathered}$$

that is

$$\begin{gathered}
\frac{\Gamma(\alpha)}{(z-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ Y_*\mathfrak{F}(z)+\mathcal{I}_{z^-}^\alpha\ Y_*\mathfrak{F}(u),\ \mathcal{I}_{u^+}^\alpha\ Y^*\mathfrak{F}(z)+\mathcal{I}_{z^-}^\alpha\ Y^*\mathfrak{F}(u)\right] \\
\supseteq_I [Y_*(u,\ i)+Y_*(z,\ i),\ Y^*(u,\ i)+Y^*(z,\ i)]\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]\mathfrak{F}((1-m)u+mz)dm,
\end{gathered}$$

hence

$$\begin{gathered}
\frac{\Gamma(\alpha)}{(z-u)^\alpha}\odot\left[\mathcal{I}_{u^+}^\alpha\ \widetilde{Y}\mathfrak{F}(z)\oplus\mathcal{I}_{z^-}^\alpha\ \widetilde{Y}\mathfrak{F}(u)\right] \\
\supseteq_{\mathbb{F}}\left[\widetilde{Y}(u)\oplus\widetilde{Y}(z)\right]\odot\int_0^1 m^{\alpha-1}[\lambda(m)+\lambda(1-m)]\mathfrak{F}((1-m)u+mz)dm.
\end{gathered}$$

Now, we obtain the following result connected with left part of classical H·H Fejér inequality for $U{\cdot}D$ λ-convex F-$N{\cdot}V{\cdot}M$ through fuzzy order inclusion relation which is known as first fuzzy fractional H·H Fejér inequality. □

Theorem 9. *(First fuzzy fractional H·H Fejér inequality) Let $\widetilde{Y} : [u, z] \to \mathbb{E}_C$ be a U·D λ-convex F-N·V·M with $u < z$, whose i-levels define the family of I·V·Ms $Y_i : [u, z] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $Y_i(\varkappa) = [Y_*(\varkappa, i),\ Y^*(\varkappa, i)]$ for all $\varkappa \in [u, z]$ and for all $i \in [0, 1]$. If $\widetilde{Y} \in L([u, z], \mathbb{E}_C)$ and $\mathfrak{F} : [u, z] \to \mathbb{R}$, $\mathfrak{F}(\varkappa) \geq 0$, symmetric with respect to $\frac{u+z}{2}$, then*

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{Y}\left(\frac{u+z}{2}\right) \odot \left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right] \supseteq_{\mathbb{F}} \left[\mathcal{I}_{u^+}^{\alpha}\ \widetilde{Y}\mathfrak{F}(z) \oplus \mathcal{I}_{z^-}^{\alpha}\ \widetilde{Y}\mathfrak{F}(u)\right]. \tag{60}$$

If $\widetilde{Y}$ is *U·D* λ-concave *F-N·V·M*, then inequality (60) is reversed.

Proof. Since $\widetilde{Y}$ is a *U·D* λ-convex *F-N·V·M*, then for $i \in [0, 1]$, we have

$$\begin{aligned} Y_*\left(\tfrac{u+z}{2},\ i\right) &\leq \lambda\left(\tfrac{1}{2}\right)(Y_*(mu + (1-m)z,\ i) + Y_*((1-m)u + mz,\ i)) \\ Y^*\left(\tfrac{u+z}{2},\ i\right) &\geq \lambda\left(\tfrac{1}{2}\right)(Y^*(mu + (1-m)z,\ i) + Y^*((1-m)u + mz,\ i)). \end{aligned} \tag{61}$$

Since $\mathfrak{F}(mu + (1-m)z) = \mathfrak{F}((1-m)u + mz)$, then by multiplying (61) by $m^{\alpha-1}\mathfrak{F}((1-m)u + mz)$ and integrate it with respect to m over $[0, 1]$, we obtain

$$\begin{aligned} &Y_*\left(\tfrac{u+z}{2},\ i\right) \int_0^1 m^{\alpha-1}\mathfrak{F}((1-m)u + mz)dm \\ &\leq \lambda\left(\tfrac{1}{2}\right)\begin{pmatrix} \int_0^1 m^{\alpha-1}Y_*(mu + (1-m)z,\ i)\mathfrak{F}((1-m)u + mz)dm \\ + \int_0^1 m^{\alpha-1}Y_*((1-m)u + mz,\ i)\mathfrak{F}((1-m)u + mz)dm \end{pmatrix}, \\ &Y^*\left(\tfrac{u+z}{2},\ i\right) \int_0^1 m^{\alpha-1}\mathfrak{F}((1-m)u + mz)dm \\ &\geq \lambda\left(\tfrac{1}{2}\right)\begin{pmatrix} \int_0^1 m^{\alpha-1}Y^*(mu + (1-m)z,\ i)\mathfrak{F}((1-m)u + mz)dm \\ + \int_0^1 m^{\alpha-1}Y^*((1-m)u + mz,\ i)\mathfrak{F}((1-m)u + mz)dm \end{pmatrix}. \end{aligned} \tag{62}$$

Let $\varkappa = (1-m)u + mz$. Then, right hand side of inequality (62), we have

$$\begin{aligned} &\int_0^1 m^{\alpha-1}Y_*(mu + (1-m)z,\ i)\mathfrak{F}((1-m)u + mz)dm \\ &\quad + \int_0^1 m^{\alpha-1}Y_*((1-m)u + mz,\ i)\mathfrak{F}((1-m)u + mz)dm \\ &\quad = \tfrac{1}{(z-u)^{\alpha}} \int_u^z (\varkappa - u)^{\alpha-1}Y_*(u - z - \varkappa, i)\mathfrak{F}(\varkappa)d\varkappa \\ &\quad\quad + \tfrac{1}{(z-u)^{\alpha}} \int_u^z (\varkappa - u)^{\alpha-1}Y_*(\varkappa, i)\mathfrak{F}(\varkappa)d\varkappa \\ &\quad = \tfrac{1}{(z-u)^{\alpha}} \int_u^z (z - u)^{\alpha-1}Y_*(\varkappa, i)\mathfrak{F}(u - z - \varkappa)d\varkappa \\ &\quad\quad + \tfrac{1}{(z-u)^{\alpha}} \int_u^z (\varkappa - u)^{\alpha-1}Y_*(\varkappa, i)\mathfrak{F}(\varkappa)d\varkappa \\ &\quad = \tfrac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y_*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y_*\mathfrak{F}(u)\right], \\ &\int_0^1 m^{\alpha-1}Y^*(mu + (1-m)z,\ i)\mathfrak{F}((1-m)u + mz)d \\ &\quad + \int_0^1 m^{\alpha-1}Y^*((1-m)u + mz,\ i)\mathfrak{F}((1-m)u + mz)dm \\ &\quad = \tfrac{\Gamma(\alpha)}{(z-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ Y^*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y^*\mathfrak{F}(u)\right]. \end{aligned} \tag{63}$$

Then from (63), we have

$$\begin{aligned} \tfrac{1}{2\lambda\left(\frac{1}{2}\right)} Y_*\left(\tfrac{u+z}{2},\ i\right)\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right] &\leq \left[\mathcal{I}_{u^+}^{\alpha}\ Y_*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y_*\mathfrak{F}(u)\right] \\ \tfrac{1}{2\lambda\left(\frac{1}{2}\right)} Y^*\left(\tfrac{u+z}{2},\ i\right)\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right] &\geq \left[\mathcal{I}_{u^+}^{\alpha}\ Y^*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y^*\mathfrak{F}(u)\right], \end{aligned}$$

from which, we have

$$\begin{aligned} &\tfrac{1}{2\lambda\left(\frac{1}{2}\right)}\left[Y_*\left(\tfrac{u+z}{2},\ i\right),\ \ Y^*\left(\tfrac{u+z}{2},\ i\right)\right]\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right] \\ &\quad \supseteq_I \left[\ \mathcal{I}_{u^+}^{\alpha}\ Y_*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y_*\mathfrak{F}(u),\ \ \mathcal{I}_{u^+}^{\alpha}\ Y^*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y^*\mathfrak{F}(u)\right], \end{aligned}$$

it follows that

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\Upsilon_i\left(\frac{u+z}{2}\right)\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right]\supseteq_I\left[\mathcal{I}_{u^+}^{\alpha}\ \Upsilon_i\mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \Upsilon_i\mathfrak{F}(u)\right],$$

that is

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\odot\widetilde{\Upsilon}\left(\frac{u+z}{2}\right)\odot\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z)\oplus\mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right]\supseteq_{\mathbb{F}}\left[\mathcal{I}_{u^+}^{\alpha}\ \widetilde{\Upsilon}\mathfrak{F}(z)\oplus\mathcal{I}_{z^-}^{\alpha}\ \widetilde{\Upsilon}\mathfrak{F}(u)\right].$$

This completes the proof. □

Remark 6. *If $\mathfrak{F}(\varkappa)=1$, then from Theorem 8 and Theorem 9, we get Theorem 5.*

If $\widetilde{\Upsilon}$ is lower $U{\cdot}D$ λ-convex F-$N{\cdot}V{\cdot}M$ with $\lambda(m)=m$, then from Theorem 8 and Theorem 9, we obtain the following factional H·H Fejér inequality given in [53]:

$$\begin{aligned}\widetilde{\Upsilon}\left(\tfrac{u+z}{2}\right)\odot\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right]&\leq_{\mathbb{F}}\left[\mathcal{I}_{u^+}^{\alpha}\ \widetilde{\Upsilon}\mathfrak{F}(z)\oplus\mathcal{I}_{z^-}^{\alpha}\ \widetilde{\Upsilon}\mathfrak{F}(u)\right]\\&\leq_{\mathbb{F}}\tfrac{\widetilde{\Upsilon}(u)\oplus\widetilde{\Upsilon}(z)}{2}\odot\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right].\end{aligned}\tag{64}$$

Let $\widetilde{\Upsilon}$ is lower $U{\cdot}D$ λ-convex F-$N{\cdot}V{\cdot}M$ with $\lambda(m)=m$ and $\alpha=1$. Then, from Theorem 8 and Theorem 9, we obtain the following H·H Fejér inequality for convex F-$N{\cdot}V{\cdot}M$, see [52]:

$$\widetilde{\Upsilon}\left(\frac{u+z}{2}\right)\leq_{\mathbb{F}}\frac{1}{\int_u^z\mathfrak{F}(\varkappa)d\varkappa}\odot(FA)\int_u^z\widetilde{\Upsilon}(\varkappa)\mathfrak{F}(\varkappa)d\varkappa\leq_{\mathbb{F}}\frac{\widetilde{\Upsilon}(u)\oplus\widetilde{\Upsilon}(z)}{2}.\tag{65}$$

Let $\widetilde{\Upsilon}$ is lower $U{\cdot}D$ λ-convex F-$N{\cdot}V{\cdot}M$ with $\lambda(m)=m$ and $\alpha=1=\mathfrak{F}(\varkappa)$. Then, from Theorem 8 and Theorem 9, we obtain the following H·H inequality for convex F-$N{\cdot}V{\cdot}M$ given in [52]:

$$\widetilde{\Upsilon}\left(\frac{u+z}{2}\right)\leq_{\mathbb{F}}(FA)\int_u^z\widetilde{\Upsilon}(\varkappa)d\varkappa\leq_{\mathbb{F}}\frac{\widetilde{\Upsilon}(u)\oplus\widetilde{\Upsilon}(z)}{2}.\tag{66}$$

If $\Upsilon_*(\varkappa,\ i)=\Upsilon^*(\varkappa,\ i)$ and $1=i$ and $\lambda(m)=m$, then from Theorem 8 and Theorem 9, following H·H Fejér inequality for classical mapping following inequality given in [69]:

$$\begin{aligned}\Upsilon\left(\tfrac{u+z}{2}\right)\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right]&\leq\left[\mathcal{I}_{u^+}^{\alpha}\ \Upsilon\mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \Upsilon\mathfrak{F}(u)\right]\\&\leq\tfrac{\Upsilon(u)+\Upsilon(z)}{2}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z)+\mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right].\end{aligned}\tag{67}$$

If $\Upsilon_*(\varkappa,\ i)=\Upsilon^*(\varkappa,\ i)$ and $\alpha=1=i$ and $\lambda(m)=m$, then from Theorem 8 and Theorem 9, we obtain the classical H·H Fejér inequality.

If $\Upsilon_*(\varkappa,\ i)=\Upsilon^*(\varkappa,\ i)$ and $\mathfrak{F}(\varkappa)=\alpha=1=i$ and $\lambda(m)=m$, then from Theorem 9 and Theorem 9, we obtain the classical H·H inequality.

Example 4. *We consider the F-$N{\cdot}V{\cdot}M$ $\Upsilon:[0,\ 2]\to\mathbb{E}_C$ defined by,*

$$\Upsilon(\varkappa)(\theta)=\begin{cases}\frac{\theta-2+\varkappa^{\frac{1}{2}}}{\frac{3}{2}-2-\varkappa^{\frac{1}{2}}} & \theta\in\left[2-\varkappa^{\frac{1}{2}},\ \frac{3}{2}\right]\\ \frac{2+\varkappa^{\frac{1}{2}}-\theta}{2+\varkappa^{\frac{1}{2}}-\frac{3}{2}} & \theta\in\left(\frac{3}{2},\ 2+\varkappa^{\frac{1}{2}}\right]\\ 0 & otherwise,\end{cases}\tag{68}$$

Then, for each $i\in[0,\ 1]$, we have $\Upsilon_i(\varkappa)=\left[(1-i)\left(2-\varkappa^{\frac{1}{2}}\right)+\frac{3}{2}i,(1+i)\left(2+\varkappa^{\frac{1}{2}}\right)+\frac{3}{2}i\right]$. Since end-point mappings $\Upsilon_*(\varkappa,i)$, $\Upsilon^*(\varkappa,i)$ are λ-convex and λ-concave mappings with

$\lambda(m) = m$, respectively, for each $i \in [0,\ 1]$, we have $Y(\varkappa)$ is U·D λ-convex F-N·V·M with $\lambda(m) = m$. If

$$\mathfrak{F}(\varkappa) = \begin{cases} \sqrt{\varkappa}, & \sigma \in [0,1], \\ \sqrt{2-\varkappa}, & \sigma \in (1,\ 2], \end{cases} \tag{69}$$

then $\mathfrak{F}(2-\varkappa) = \mathfrak{F}(\varkappa) \geq 0$, for all $\varkappa \in [0,\ 2]$. Since $Y_*(\varkappa, i) = (1-i)\left(2 - \varkappa^{\frac{1}{2}}\right) + \frac{3}{2}i$ and $Y^*(\varkappa, i) = (1+i)\left(2 + \varkappa^{\frac{1}{2}}\right) + \frac{3}{2}i$. If $\alpha = \frac{1}{2}$, then we compute the following:

$$\begin{aligned} &[Y_*(u,\ i) + Y_*(z,\ i)] \cdot \int_0^1 m^{\alpha-1}[i(m) + \lambda(1-m)]\mathfrak{F}((1-m)u + mz)dm \\ &= (1-i)\sqrt{\pi}\left(\tfrac{4-\sqrt{2}}{2}\right) + \tfrac{3}{2}\sqrt{\pi}i, \\ &[Y^*(u,\ i) + Y^*(z,\ i)] \cdot \int_0^1 m^{\alpha-1}[\lambda(m) + \lambda(1-m)]\mathfrak{F}((1-m)u + mz)dm \\ &= (1+i)\sqrt{\pi}\left(\tfrac{4+\sqrt{2}}{2}\right) + \tfrac{3}{2}\sqrt{\pi}i, \end{aligned} \tag{70}$$

and

$$\begin{aligned} \tfrac{\Gamma(\alpha)}{(z-u)^\alpha} \cdot \left[\mathcal{I}_{u^+}^{\alpha}\ Y_*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y_*\mathfrak{F}(u)\right] &= \tfrac{1}{\sqrt{2}}(1-i)\left(2\pi + \tfrac{4-8\sqrt{2}}{3}\right) + \tfrac{3}{2\cdot\sqrt{2}}\pi i, \\ \tfrac{\Gamma(\alpha)}{(z-u)^\alpha} \cdot \left[\mathcal{I}_{u^+}^{\alpha}\ Y^*\mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ Y^*\mathfrak{F}(u)\right] &= \tfrac{1}{\sqrt{2}}(1+i)\left(2\pi + \tfrac{8\sqrt{2}-4}{3}\right) + \tfrac{3}{2\cdot\sqrt{2}}\pi i. \end{aligned} \tag{71}$$

From (70) and (71), we have

$$\begin{aligned} &\tfrac{1}{\sqrt{2}}\left[(1-i)\left(2\pi + \tfrac{4-8\sqrt{2}}{3}\right) + \tfrac{3}{2}\pi i,\ (1+i)\left(2\pi + \tfrac{8\sqrt{2}-4}{3}\right) + \tfrac{3}{2}\pi i\right] \\ &\supseteq_I \sqrt{\pi}\left[(1-i)\left(\tfrac{4-\sqrt{2}}{2}\right) + \tfrac{3}{2}i,\ (1+i)\left(\tfrac{4+\sqrt{2}}{2}\right) + \tfrac{3}{2}i\right], \end{aligned}$$

for each $i \in [0,\ 1]$.

Hence, Theorem 8 is verified.

For Theorem 9, we have

$$\begin{aligned} \tfrac{1}{2\lambda\left(\frac{1}{2}\right)} Y_*\left(\tfrac{u+z}{2},\ i\right) \cdot \left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right] &= (1-i)\sqrt{\pi} + \tfrac{3}{2}\sqrt{\pi}i, \\ \tfrac{1}{2\lambda\left(\frac{1}{2}\right)} Y^*\left(\tfrac{u+z}{2},\ i\right) \cdot \left[\mathcal{I}_{u^+}^{\alpha}\ \mathfrak{F}(z) + \mathcal{I}_{z^-}^{\alpha}\ \mathfrak{F}(u)\right] &= 3(1+i)\sqrt{\pi} + \tfrac{3}{2}\sqrt{\pi}i. \end{aligned} \tag{72}$$

From (71) and (72), we have

$$\sqrt{\pi}\left[(1-i) + \frac{3}{2}i,\ 3(1+i) + \frac{3}{2}i\right] \supseteq_I \left[\frac{(1-i)}{\sqrt{\pi}}\left(2\pi + \frac{4-8\sqrt{2}}{3}\right) + \frac{3}{2}\sqrt{\pi}i,\ \frac{(1+i)}{\sqrt{\pi}}\left(2\pi + \frac{8\sqrt{2}-4}{3}\right) + \frac{3}{2}\sqrt{\pi}i\right],$$

for each $i \in [0,\ 1]$.

4. Conclusions

This study is the first to address fuzzy fractional inclusion relations including fuzzy-number valued λ-convexity, as far as we are aware. Here, we derive the fuzzy fractional integral inclusions for the recently proposed family of mappings together with the H·H- and Pachpatte-type inequality. Specifically, for the fuzzy-number valued λ-convex mappings, we provide an enhanced version of the fuzzy H·H-type integral inclusions. This study's fuzzy integral inclusion relations are significant expansions of the findings made by Tunç in [68]. We would like to underline the wide variety of uses for fuzzy interval analysis in practical mathematics, particularly in the area of fuzzy optimality analysis; for more information, check the published publications [54,71,85]. In certain ways, more studies should be conducted on the significant field of fuzzy-number-valued analytic research that is connected to fuzzy fractional integral operators.

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