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Distributed Optimization for Fractional-Order Multi-Agent Systems Based on Adaptive Backstepping Dynamic Surface Control Technology

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Abstract: In this article, the distributed optimization problem is studied for a class of fractional-order nonlinear uncertain multi-agent systems (MASs) with unmeasured states. Each agent is represented through a system with unknown nonlinearities, unmeasurable states and a local objective function described by a quadratic polynomial function. A penalty function is constructed by a sum of local objective functions and integrating consensus conditions of the MASs. Radial basis function Neural-networks (RBFNNs) and Neural networks (NN) state observer are applied to approximate the unknown nonlinear dynamics and estimate unmeasured states, respectively. By combining the NN state observer and the penalty function, and the stability theory of the Lyapunov function, the distributed observer-based adaptive optimized backstepping dynamic surface control protocol is proposed to ensure the outputs of all agents asymptotically reach consensus to the optimal solution of the global objective function. Simulations demonstrate the effectiveness of the proposed control scheme.

Keywords: fractional order multiagent systems (FOMASs); distributed optimization; dynamic surface control (DSC); neural networks; observer

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1. Introduction

In past few years, the distributed cooperative control of MASs has received a great deal of interest and has became one of the research hotspots due to its potential applications in various fields, including formation control [1], smart grids [2], sensor networks [3], distributed energy resources [4], robotic systems [5], fractional-order systems [6–8], multi-satellites system [9], multiple spacecraft [10,11] and so on. As one of the underlying issues with cooperative control, the consensus refers to the construction of the appropriate decentralized algorithms, which can make the states of all agents ultimately come to an agreement. As an extension of the MASs' consensus issue, the distributed optimization consensus problem considers optimization on the basis of consensus, which is reasonable. For instance, in the aviation mission completed by multiple aircraft cooperatively, there are optimization problems in tasks such as path planning, optimal coverage, and minimum fuel. In the distributed optimization problem, the global optimization objective is a sum of all agents' local optimization objectives.

The main objective of the distributed optimization consensus of MASs is to design appropriate controllers, which is driving all agents to converge to the optimal solution of the optimization problem cooperatively. For different problems, various control protocols have been developed in the past several years. In [12], a distributed active antidisturbance control algorithm was developed to solve the distributed optimization problem of second-order MASs with both mismatched and matched disturbances. Based on the state-integral feedback and adaptive control method, a two-layer control framework was developed for the distributed optimization problem of second-order MASs with unmatched constant disturbances in [13]. In [14], an integral sliding mode controller was designed to give robustness to MASs affected by perturbations and uncertain agent dynamics. An adaptive distributed method was designed to handle the distributed optimization problem for a class of the heterogenous nonlinear MASs on the weight-balanced directed graph in [15]. In [16], continuous distributed algorithms were proposed for the finite-time distributed convex optimization problems of MASs with local disturbance signals. Specifically, an adaptive backstepping method was presented to handle the distributed optimization problem of nonlinear MASs, in which each agent is denoted through the high-order nonlinear strict-feedback form limited by mismatched parametric uncertainties in [17].

However, the aforementioned optimization algorithms may not work when there are nonlinear uncertain functions in the MASs. Due to the inaccuracy of modeling or the existence of unknown disturbance, nonlinear uncertainty exists objectively. Fortunately, NNs and fuzzy logic systems (FLSs) are used to approximate arbitrary nonlinear functions, and achieve arbitrary approximation accuracy. Combined with NNs or FLSs, a variety of adaptive intelligent control methods are developed for nonlinear MASs [18–24]. The distributed dynamic surface technique was developed to design the local consensus controller, and the NNs were employed for function approximation in [25]. A distributed adaptive NNs backstepping controllers were constructed for MASs with nonlinear input in [26]. In [27], a finite-time adaptive NNs controller was developed by using the command filter approach for uncertain nonlinear MASs with prescribed performance and input saturation. In order to deal with the unmeasured states, several observer-based fuzzy or NNs adaptive distributed control schemes were developed for uncertain nonlinear MASs [28–32]. Based on high-gain observer theory and fuzzy technique, a decentralized control approach for double-integrator uncertain MASs with disturbances, unmeasured states and unknown nonlinear dynamics was presented in [33]. In [34], a distributed adaptive control protocol based on the command filtered backstepping method was developed for nonlinear MASs with input saturation. After literature investigation, it is worth mentioning that we have not found that the adaptive intelligent control is used for multi-agent optimization problems. Therefore, how to develop an adaptive intelligent output feedback control method that can make the uncertain MASs with nonlinear dynamics ensure the consensus conditions and satisfy the optimization objective function at the same time is a challenging and meaningful scientific problem.

Furthermore, the above research is limited to integer-order MASs. Actually, it is reasonable to regard the fractional-order MASs (FOMASs) as a generalization of integer-order MASs. FOMASs have a wide range of potential applications in reality [35], such as robotic systems operating on muddy roads, swamp flotation devices and supercoils, and so on. The consensus control of FOMASs has drawn much attention and has become another research hotspot [36–40]. The consensus of FOMASs with linear models using the observer technique was proposed in [41]. In [42], an observer-based Linear Matrix Inequality controller for the consensus of FOMASs depicted by the general linear dynamics with positive constraint was designed. However, to the best of our knowledge, the distributed optimization consensus problem of FOMASs with unmeasured states and nonlinear uncertain dynamics has not been investigated in the existing study, which gives us great motivation for the research presented in this article.

Motivated by the above observations, the purpose of this paper lies in the design of an adaptive backstepping DSC algorithm to solve a distributed constrained optimization problem for FOMASs with unmeasured states and unknown nonlinearities. In the MASs, it is assumed that each agent only accesses its local objective function and is constrained by a quadratic polynomial function. By combining a global objective function and consensus conditions, a penalty function is established. The RBFNNs are used to approximate the unknown nonlinear dynamics, and a NN state observer is utilized to estimate the unmeasured states of each agent. A distributed optimized controller for each agent can be derived by applying the adaptive backstepping DSC technology to accomplish consensus tracking of the optimal solution, which is obtained by minimizing the penalty function. Compared with the aforementioned work, the novel contributions of this paper are:

(1) Compared with [19–23], where the NNs-based adaptive backstepping DSC design algorithm was proposed for the consensus problem of MASs, we solve the distributed optimization problem for the FOMASs with unmeasured states. To accomplish this difficult task, based on the penalty function and negative gradient method, we construct a Lyapunov function for the control strategy. The distributed control method proposed in this paper ensures that all the agents' outputs reach consensus to the optimal solution of the global objective function asymptotically instead of tracking the reference trajectory.
(2) Different from [12,16], in which the distributed optimization algorithms are developed for first-order and second-order MASs, this article mainly focuses on the distributed optimization problem for fractional high-order MASs with unmeasured states and the unknown nonlinear functions. To overcome this challenge, we propose the distributed optimized adaptive backstepping controller based on the fractional DSC method, which is widely used in fractional high-order MASs, and the controller we propose has excellent performance in fractional-order MASs, which will be reflected in the Simulation section when comparing with [16].
(3) In contrast to [17], which investigates an adaptive backstepping protocol to the distributed optimization for the integer order MASs, with each agent modeled by the strict-feedback form, this article will address the FOMASs with unmeasured states. The unknown nonlinear functions are considered in our article and will be approximated by the NNs.

The reminder of this study is constructed as follows. Section 2 gives the formulation and preliminaries of this paper. In Section 3, by employing the adaptive DSC technology, a NN state observer-based distributed controller is proposed. Section 4 provides the numerical simulations to elaborate on the feasibility of the proposed control algorithms. Section 5 gives the conclusions.

2. Preliminaries

2.1. Fractional Calculus

Define the R-L fractional derivative as

$$ {}_0^{RL}I_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau \tag{1} $$

where $n \in N$ and $n-1 < \alpha \leq n$, $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is the Gamma function. $f(t)$ is an arbitrary integrable smooth function on $[0, t]$.

In this paper, we mainly adopt the Caputo fractional derivative [43] as defined with

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau. \tag{2} $$

Remark 1. *To simplify the notation, we set* ${}_0^{C}D_t^{\alpha} f(t) = D^{\alpha} f(t)$.

For a two-parameter function of the Mittag-Leffler type

$$ E_{\alpha,\beta}(\varsigma) = \sum_{k=0}^{\infty} \frac{\varsigma^k}{\Gamma(\alpha k + \beta)}, (\alpha > 0), (\beta > 0) $$

we have the next Lemma.

Lemma 1 ([44]). *For real numbers β, α and v satisfying $\alpha \in (0,1)$*

$$\frac{\pi\alpha}{2} < v < \pi\alpha \tag{3}$$

and for integers $n \geq 1$, it is obtained that

$$E_{\alpha,\beta}(\varsigma) = -\sum_{j=1}^{n} \frac{\zeta^{-j}}{\Gamma(\beta - \alpha j)} + o\left(\frac{1}{|\varsigma|^{n+1}}\right) \tag{4}$$

when $|\varsigma| \to \infty, v \leq |\arg(\varsigma)| \leq \pi$.

Lemma 2 ([44]). *Let $\alpha \in (0,2)$ and β be an arbitrary real number. For $(\pi\alpha/2) < v \leq \min\{\pi, \pi\alpha\}$, it is proven that*

$$\left|E_{\alpha,\beta}(\varsigma)\right| \leq \frac{\mu}{1+|\varsigma|} \tag{5}$$

where $\mu > 0$, $v \leq |\arg(\varsigma)| \leq \pi$, and $|\varsigma| \geq 0$.

Lemma 3 ([45]). *Define a vector of continuous and differentiable functions $x(t) = [x_1(t), \ldots, x_n(t)] \in R^n$. Then, the following relationship holds*

$$\begin{array}{l} \frac{1}{2}D^{\alpha}\left(x^T(t)Px(t)\right) \leq x^T(t)PD^{\alpha}x(t) \\ x(0) = x_0, \forall\alpha \in (0,1), \forall t > t_0 \end{array} \tag{6}$$

where $t_0 = 0$, P=$diag(p_1, p_2, \ldots, p_n)$ and $p_i > 0$, $i = 1, 2, \ldots, n$.

Lemma 4 ([46]). *For any $x, y \in R^n$, the following inequality relationship holds*

$$x^T y \leq \frac{c^a}{a}\|x\|^a + \frac{1}{bc^b}\|y\|^b \tag{7}$$

where $a > 1, b > 1, c > 0$, and $(a-1)(b-1) = 1$.

Lemma 5 ([43]). *In fractional-order nonlinear system, if the α-order derivative of Lyapunov function $V(t,x)$ is satisfying*

$$D^{\alpha}V(t,x) \leq -CV(t,x) + \zeta$$

then

$$V(t,x) \leq V(0)E_{\alpha}(-Ct^{\alpha}) + \tfrac{\zeta\mu}{C}, \quad t \geq 0 \tag{8}$$

and $0 < \alpha < 1$, $C > 0$, and $\zeta \geq 0$. Then, $V(t,x)$ is bounded on $[0,t]$ and fractional order systems are stable and μ is defined in Lemma 2.

2.2. Graph Theory

Suppose that there exist N agents. A directed graph $\mathcal{G} = (\mathcal{W}, \mathcal{E}, \bar{A})$ is used for the information exchange between agents, where $\mathcal{W} = \{1, \ldots, N\}$ is a node set and $\mathcal{E} = \{(i,j) \in \mathcal{W} \times \mathcal{W}\}$ is an edge set. $\bar{A} = \{a_{ij}\} \in R^{N \times N}$ is the adjacency matrix and an edge $(i,j) \in \mathcal{E}$, if and only if $a_{ij} = 1$, means that there is information exchange between node i and node j. Assume that there is no self-loop in diagraph, therefore $a_{ii} = 0$ for $i = 1, \ldots, N$. Utilize $N_i = \{j|(i,j) \in \mathcal{E}\}$ as the neighbor set of node i. Define the matrix $D = diag\{d_1, d_2, \ldots, d_N\}$ as the degree matrix of directed graph $\mathcal{G}$, in which $d_i = \sum_{j \in N_i} a_{ij}$. Define the Laplacian matrix as $L = D - \bar{A}$.

Lemma 6 ([47]). *Define a column vector 1_N with N elements and all elements being one. Denote a symmetric matrix $L \in R^{N \times N}$ as the Laplacian matrix of a directed graph $\mathcal{G}$. 1_N is the eigenvector for eigenvalue 0 of Laplacian matrix.*

2.3. Fractional-Order Nonlinear Multi-Agent System

In this paper, the following FOMAS with nonlinear uncertain dynamics for agent i is considered.

$$\begin{cases} D^{\alpha}x_{i,1}(t) = x_{i,2}(t) + g_{i,1}(x_{i,1}(t)) \\ D^{\alpha}x_{i,l}(t) = x_{i,l+1}(t) + g_{i,l}(x_{i,1}(t), x_{i,2}(t), \cdots, x_{i,l}(t)) \\ D^{\alpha}x_{i,n}(t) = u_i(t) + g_{i,n}(x_{i,1}(t), x_{i,2}(t), \cdots, x_{i,n}(t)) \\ y_i = x_{i,1}(t) \end{cases} \tag{9}$$

where $l = 2, \cdots, n-1$, u_i is the control input, y_i is the system output and $g_{i,l}(x_{i,l}(t), x_{i,2}(t), \ldots, x_{i,l}(t))$ is an unknown nonlinear function defined on the system state vector. Define $X_{i,l} = (x_{i,1}(t), x_{i,2}(t), \cdots, x_{i,l}(t))^T \in R^l$ are the system state vectors for agent i.

Rewrite the system of agent i:

$$\begin{aligned} D^{\alpha}X_{i,n} &= A_iX_{i,n} + K_iy_i + \sum_{l=1}^{n} B_{i,l}[g_{i,l}(X_{i,l})] + B_iu_i(t) \\ y_i &= C_iX_{i,n} \end{aligned} \tag{10}$$

where $A_i = \begin{bmatrix} -k_{i,1} & & & \\ \vdots & & I_{n-1} & \\ -k_{i,n} & 0 & \cdots & 0 \end{bmatrix}$, $K_i = \begin{bmatrix} k_{i,1} \\ \vdots \\ k_{i,n} \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$, $B_{i,l} = [0 \cdots 1 \cdots 0]^T$, $C_i = [1 \quad 0 \cdots 0]$.

For a given positive matrix $Q_i^T = Q_i$, there exists a positive matrix $P_i^T = P_i$ satisfying

$$A_i^TP_i + P_iA_i = -2Q_i. \tag{11}$$

Remark 2. *To simplify the notation, we set* $x_{i,j} = x_{i,j}(t)$.

2.4. Convex Analysis

A function $f(\cdot) : R^n \to R$ is convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y),\ \forall x, y \in R^n, 0 \leq \alpha \leq 1. \tag{12}$$

A differentiable function $f(\cdot) : R^n \to R$ is strong convex on R^n if

$$(x-y)^T(\nabla f(x) - \nabla f(y)) \geq \omega||x-y||^2, \forall x, y \in R^n, \omega > 0. \tag{13}$$

A function $f(\cdot) : R^n \to R$ is Λ-Lipschitz($\Lambda > 0$) on R^n if

$$\|f(x) - f(y)\| \leq \Lambda\|x-y\|, \forall x, y \in R^n. \tag{14}$$

2.5. Problem Formulation

In this paper, we solve the distributed optimization problem for N agents with path tracking into the account. The local objective function $f_i : R^n \to R$ for each agent is defined as

$$\begin{aligned} f_i(x_{i,1}) &= a_i(x_{i,1} - x_d)^2 + c \\ &= a_ix_{i,1}^2 + b_ix_{i,1} + c_i \end{aligned} \tag{15}$$

where x_d is the reference signal for agents to track, $a_i > 0$, $b_i = -2a_ix_d$, $c_i = a_ix_d^2 + c$, $1 \leq i \leq N$ and a_i, c are scalars. Define the global objective function $f : R \to R$ as

$$f(x_{i,1}) = \sum_{i=1}^{N} f_i(x_{i,1}). \tag{16}$$

Considering that the local objective function f_i is differentiable and strictly convex, the global objective function f is differentiable and strictly convex as well. Define $x_1 = [x_{1,1}\ x_{2,1}\ \cdots\ x_{N,1}]^T$. According to Lemma 6, for some $\alpha \in R$, if $x_1 = \alpha \cdot 1_N$, we obtain

$$Lx_1 = 0. \tag{17}$$

Then, we design the penalty term as follows

$$x_1^T L x_1 = 0. \tag{18}$$

Penalty function is defined as

$$P(x_1) = \sum_{i=1}^{N} f_i(x_{i,1}) + x_1^T L x_1. \tag{19}$$

Due to the global objective function being strictly convex, we draw the conclusion that the penalty function is convex as well.

In this paper, we aim at designing controllers $(u_1, \ldots, u_N)$ in order that for every $i = 1, \ldots, N$, $lim_{t\to\infty} x_{i,1} \to x_{i,1}^*$. Define $x_1^* = \left(x_{1,1}^*, \cdots, x_{N,1}^*\right)$. The optimal solution $x_{i,1}^*$ is defined as

$$\left(x_{1,1}^*, \cdots, x_{N,1}^*\right) = \underset{(x_{1,1},\ldots,x_{N,1})}{\arg\min}\ P(x_1). \tag{20}$$

Remark 3. *According to (19), we find that the penalty function consists of two parts. The first part is $\sum_{i=1}^{N} f_i(x_{i,1})$, which is used to ensure the system to minimize the global objective function and track the reference signal. The second part is $x_i^T L x_1$, which is used to reach consensus for all agents. By minimizing the penalty function, the distributed optimization consensus problem can be solved.*

Control objectives: This paper focuses on proposing a distributed optimized controller based on observer-based adaptive neural network DSC technology to make sure that all the agents' signals remain bounded, the output errors between all agents converge to near zero and the tracking error is as small as possible in the closed-loop system, while keeping the consensus to the optimal solution of the global objective function.

3. Main Results

3.1. State Observer Design

Due to the nonlinear functions $g_{i,l}(X_{i,l})$ being unknown, the RBFNNs technology is utilized to detect nonlinear functions. The following two assumptions are needed.

Assumption 1. *The unknown functions $g_{i,l}(X_{i,l})$, $i = 1, \cdots, n$ are expressed as*

$$g_{i,l}(X_{i,l}|\theta_{i,l}) = \theta_{i,l}^T \psi_{i,l}(X_{i,l}), 1 \le i \le n \tag{21}$$

where the node number of NN is q, $\theta_{i,l}$ is the unknown constant vector and $\psi_{i,l}(X_{i,l}) = \left[\psi_{i,l}^1(X_{i,l}), \psi_{i,l}^2(X_{i,l}), \ldots, \psi_{i,l}^q(X_{i,l})\right]^T$ represents the radial basis function vector. A typical basis Gaussian function is given by

$$\psi_{i,l}^p(X_{i,l}) = exp\left(\frac{\left\|X_{i,l} - c_{i,l}^p\right\|^2}{b_{i,l}^2}\right), p = 1, \ldots, q \tag{22}$$

where $c_{i,l}^p \in R^l$ is the center of the receptive field and $b_{i,l} \in R$ is the width of the Gaussian function. Define $c_{i,l} = \left[c_{i,l}^1, c_{i,l}^2, \ldots, c_{i,l}^q\right]$. According to [48,49], more nodes mean more accurate approximation.

In this paper, we assume that the state variables of the system (9) are not available. In this case, we need to estimate the system states, and the state observer for agent i is designed as follows

$$\begin{aligned} D^{\alpha}\widehat{X}_{i,n} &= A_i\widehat{X}_{i,n} + K_i y_i + \sum_{l=1}^{n} B_{i,l}\left[\hat{g}_{i,l}\left(\widehat{X}_{i,l}|\theta_{i,l}\right)\right] + B_i u_i(t) \\ \hat{y}_i &= C_i\widehat{X}_{i,n} \end{aligned} \tag{23}$$

where $C_i = [1\ldots 0\ldots 0]$ and $\widehat{X}_{i,l} = (\hat{x}_{i,1}, \hat{x}_{i,2}, \ldots, \hat{x}_{i,l})^T$ are the estimated values of $X_{i,l}$.

Define the state observation errors vector as $e_i = X_{i,n} - \widehat{X}_{i,n}$. According to Equations (10) and (23), we obtain

$$D^{\alpha}e_i = A_i e_i + \sum_{l=1}^{n} B_{i,l}\left[g_{i,l}\left(\widehat{X}_{i,l}\right) - \hat{g}_{i,l}\left(\widehat{X}_{i,l}|\theta_{i,l}\right) + \Delta g_{i,l}\right] \tag{24}$$

where $\Delta g_{i,l} = g_{i,l}(X_{i,l}) - g_{i,l}\left(\widehat{X}_{i,l}\right)$.

By Assumption 1, we obtain

$$\hat{g}_{i,l}\left(\widehat{X}_{i,l}|\theta_{i,l}\right) = \theta_{i,l}^T\psi_{i,l}\left(\widehat{X}_{i,l}\right). \tag{25}$$

Then, the vector of optimal parameters is defined as

$$\theta_{i,l}^* = \arg\min_{\theta_{i,l}\in\Omega_{i,l}}\left[\sup_{\widehat{X}_{i,l}\in U_{i,l}}\left|\hat{g}_{i,l}\left(\widehat{X}_{i,l}|\theta_{i,l}\right) - g_{i,l}\left(\widehat{X}_{i,l}\right)\right|\right] \tag{26}$$

where $1 \le l \le n$, $\Omega_{i,l}$ and $U_{i,l}$ are compact regions for $\theta_{i,l}$, $X_{i,l}$ and $\widehat{X}_{i,l}$.

Define the optimal approximation error $\delta_{i,l}$ and the parameter estimation error $\widetilde{\theta}_{i,l}$ as

$$\begin{aligned} \delta_{i,l} &= g_{i,l}\left(\widehat{X}_{i,l}\right) - \hat{g}_{i,l}\left(\widehat{X}_{i,l}\middle|\theta_{i,l}^*\right) \\ \widetilde{\theta}_{i,l} &= \theta_{i,l}^* - \theta_{i,l}, l = 1, 2, \ldots, n. \end{aligned} \tag{27}$$

Assumption 2. *The optimal approximation errors remain bounded and there exist positive constants δ_{i0}, satisfying $\left|\delta_{i,l}\right| \le \delta_{i0}$.*

Assumption 3. *There exists a set of known constants $\gamma_{i,l}$ and the following relationship holds*

$$\left|g_{i,l}(X_{i,l}) - g_{i,l}\left(\widehat{X}_{i,l}\right)\right| \le \gamma_{i,l}\left\|X_{i,l} - \widehat{X}_{i,l}\right\|. \tag{28}$$

By Equations (24) and (27), we have

$$\begin{aligned} D^{\alpha}e_i &= A_i e_i + \sum_{l=1}^{n} B_{i,l}\left[g_{i,l}\left(\widehat{X}_{i,l}\right) - \hat{g}_{i,l}\left(\widehat{X}_{i,l}|\theta_{i,l}\right) + \Delta g_{i,l}\right] \\ &= A_i e_i + \sum_{l=1}^{n} B_{i,l}\left[\delta_{i,l} + \Delta g_{i,l} + \widetilde{\theta}_{i,l}^T\psi_{i,l}\left(\widehat{X}_{i,l}\right)\right] \\ &= A_i e_i + \Delta g_i + \delta_i + \sum_{l=1}^{n} B_{i,l}\left[\widetilde{\theta}_{i,l}^T\psi_{i,l}\left(\widehat{X}_{i,l}\right)\right] \end{aligned} \tag{29}$$

where $\delta_i = [\delta_{i,1}, \ldots, \delta_{i,n}]^T$, $\Delta g_i = [\Delta g_1, \ldots, \Delta g_n]^T$.

Construct the first Lyapunov function:

$$V_0 = \sum_{i=1}^{N} V_{i,0} = \sum_{i=1}^{N} \frac{1}{2} e_i^T P_i e_i.$$

According to Lemma 3 and (29), we have

$$\begin{aligned} D^{\alpha} V_0 &\le \sum_{i=1}^{N} \left\{ \frac{1}{2} e_i^T \left(P_i A_i^T + A_i P_i \right) e_i + e_i^T P_i (\delta_i + \Delta g_i) + \sum_{l=1}^{n} e_i^T P_i B_{i,l} \left[\widetilde{\theta}_{i,l}^T \psi_{i,l} \left(\widehat{X}_{i,l} \right) \right] \right\} \\ &\le \sum_{i=1}^{N} \left\{ -e_i^T Q_i e_i + e_i^T P_i (\delta_i + \Delta g_i) + e_i^T P_i \sum_{l=1}^{n} B_{i,l} \widetilde{\theta}_{i,l}^T \psi_{i,l} \left(\widehat{X}_{i,l} \right) \right\}. \end{aligned} \tag{30}$$

By Lemma 4 and Assumption 3, we have

$$\begin{aligned} e_i^T P_i (\delta_i + \Delta g_i) &\le \left| e_i^T P_i \delta_i \right| + \left| e_i^T P_i \Delta g_i \right| \\ &\le \frac{1}{2} \|e_i\|^2 + \frac{1}{2} \|P_i \delta_i\|^2 + \frac{1}{2} \|e_i\|^2 + \frac{1}{2} \|P_i\|^2 \|\Delta g_i\|^2 \\ &\le \|e_i\|^2 + \frac{1}{2} \|P_i \delta_i\|^2 + \frac{1}{2} \|P_i\|^2 \sum_{l=1}^{n} \left| \Delta g_{i,l} \right|^2 \\ &\le \|e_i\|^2 + \frac{1}{2} \|e_i\|^2 \|P_i\|^2 \sum_{l=1}^{n} {\gamma_{i,l}}^2 + \frac{1}{2} \|P_i \delta_i\|^2 \\ &\le \|e_i\|^2 \left(1 + \frac{1}{2} \|P_i\|^2 \sum_{l=1}^{n} {\gamma_{i,l}}^2 \right) + \frac{1}{2} \|P_i \delta_i\|^2 \end{aligned} \tag{31}$$

and

$$\begin{aligned} e_i^T P_i \sum_{l=1}^{n} B_{i,l} \widetilde{\theta}_{i,l}^T \psi_{i,l} \left(\widehat{X}_{i,l} \right) &\le \frac{1}{2} e_i^T P_i^T P_i e_i + \frac{1}{2} \sum_{l=1}^{n} \widetilde{\theta}_{i,l}^T \psi_{i,l} \left(\widehat{X}_{i,l} \right) \psi_{i,l}^T \left(\widehat{X}_{i,l} \right) \widetilde{\theta}_{i,l} \\ &\le \frac{1}{2} \lambda_{i,\max}^2 (P_i) \|e_i\|^2 + \frac{1}{2} \sum_{l=1}^{n} \widetilde{\theta}_{i,l}^T \widetilde{\theta}_{i,l}, \end{aligned} \tag{32}$$

where $0 < \psi_{i,l}(\cdot) \psi_{i,l}^T(\cdot) \le 1$ and $\lambda_{i,max}(P_i)$ is the maximum eigenvalue of positive matrix P_i. By Equations (30)–(32), we obtain

$$D^{\alpha} V_0 \le \sum_{i=1}^{N} \left(-q_{i,0} \|e_i\|^2 + \frac{1}{2} \|P_i \delta_i^*\|^2 + \frac{1}{2} \sum_{l=1}^{n} \widetilde{\theta}_{i,l}^T \widetilde{\theta}_{i,l} \right), \tag{33}$$

where $q_{i,0} = \lambda_{i,\min}(Q_i) - \left(1 + \frac{1}{2} \|P_i\|^2 \sum\limits_{l=1}^{n} \gamma_{i,l}^2 + \frac{1}{2} \lambda_{i,\max}^2 (P_i) \right)$.

Then, we can obtain

$$D^{\alpha} V_0 \le -q_0 \|e\|^2 + \frac{1}{2} \|P \varepsilon\|^2 + \sum_{i=1}^{N} \sum_{l=1}^{n} \frac{1}{2} \widetilde{\theta}_{i,l}^T \widetilde{\theta}_{i,l}, \tag{34}$$

where $q_0 = \sum_{i=1}^{N} q_{i,0}$.

3.2. Controller Design

Theorem 1. *Considering the uncertain nonlinear FOMASs (9) under Assumptions 1–3, there exist state observer (23), virtual control laws (52), (67), (83), adaptive laws (53), (68), (84), (104) and an observer-based adaptive optimized NN dynamic surface controller (103), such that: (1) signals $x_{i,1}$ in the closed-loop system remain semi-global uniformly ultimately bounded, (2) signals $x_{i,1}$ converge to the optimal solution $x_{i,1}^*$ of the distributed optimization problem.*

Proof. Step 1. Define the error variable as follows:

$$\begin{aligned} s_{i,1} &= x_{i,1} - x_{i,1}^* \\ s_{i,l} &= \hat{x}_{i,l} - v_{i,l} \\ w_{i,l} &= v_{i,l} - x_{i,l}^* \quad l = 2, \cdots, n \end{aligned} \tag{35}$$

where $s_{i,l}$ is the tracking error, $v_{i,l}$ is the output of a filter, which can be obtained through virtual controller $x_{i,l}^*$, and $w_{i,l}$ is the output error between the filter output $v_{i,l}$ and the virtual controller $x_{i,l}^*$ and $\hat{x}_{i,l}$ is the estimation of $x_{i,l}$.

First, calculate the gradient of the penalty function given with (19)

$$\frac{\partial P(x_1)}{\partial x_1} = vec\left(\frac{\partial f_i(x_{i,1}(t))}{\partial x_{i,1}}\right) + Lx_1 \tag{36}$$

where $vec\left(\frac{\partial f_i(x_{i,1}(t))}{\partial x_{i,1}}\right)$ is a column vector.

Considering that the penalty function $P(x_1)$ is strictly convex, the necessary condition for the optimal solution to the distributed optimization problem is

$$\frac{\partial P(x_1^*)}{\partial x_1^*} = 0.$$

Then, from (19) and (36), for the agent i, we have

$$\frac{\partial f_i(x_{i,1}^*(t))}{\partial x_{i,1}^*} + \sum_{j \in N_i} a_{ij}(x_{i,1}^* - x_{j,1}^*) = 0. \tag{37}$$

According to (15) and (37),we have

$$2a_i(x_{i,1}^* - x_d) + \sum_{j \in N_i} a_{ij}(x_{i,1}^* - x_{j,1}^*) = 0. \tag{38}$$

Then, according to (35) and (38), we have

$$\begin{aligned} \frac{\partial P(x_1)}{\partial x_{i,1}} =& \frac{\partial f_i(x_{i,1}(t))}{\partial x_{i,1}} + \sum_{j \in N_i} a_{ij}(x_{i,1} - x_{j,1}) \\ =& 2a_i(x_{i,1} - x_d) + \sum_{j \in N_i} a_{ij}(x_{i,1} - x_{j,1}) \\ =& 2a_i(x_{i,1} - x_d) + \sum_{j \in N_i} a_{ij}(x_{i,1} - x_{j,1}) - 2a_i(x_{i,1}^* - x_d) + \sum_{j \in N_i} a_{ij}(x_{i,1}^* - x_{j,1}^*) \\ =& 2a_i s_{i,1} + \sum_{j \in N_i} a_{ij}(s_{i,1} - s_{j,1}). \end{aligned} \tag{39}$$

Let $s_1 = [s_{1,1} \ \cdots \ s_{N,1}]^T$, according to (39), we have

$$\frac{\partial P(x_1)}{\partial x_1} = Hs_1$$

where $H = \mathcal{A} + L$ and $\mathcal{A} = diag\{2a_i\}$.

Construct Lyapunov function:

$$
\begin{aligned}
V_1 &= V_0 + \frac{1}{2}\left(\frac{\partial P(x_1)}{\partial x_1}\right)^T H^{-1}\left(\frac{\partial P(x_1)}{\partial x_1}\right) + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\widetilde{\theta}_{i,1} \\
&= V_0 + \frac{1}{2}s_1^T H s_1 + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\widetilde{\theta}_{i,1}
\end{aligned}
\tag{40}
$$

where $s_1 = [s_{1,1} \ \cdots \ s_{N,1}]^T$ and $\sigma_{i,1}$ is a designed parameter. According to (9), (27) and (35), we have

$$
D^\alpha s_{i,1} = \hat{x}_{i,2} + \theta_{i,1}^T\psi_{i,1} + \widetilde{\theta}_{i,1}^T\psi_{i,1} + \Delta g_{i,1} + \delta_{i,1} + e_{i,2}. \tag{41}
$$

Then, according to (40) and (41), we can obtain

$$
\begin{aligned}
D^\alpha V_1 =& D^\alpha V_0 + s_1^T H D^\alpha s_1 + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T D^\alpha\widetilde{\theta}_{i,1} \\
=& D^\alpha V_0 + s_1^T H\left(\hat{x}_2 + vec\left(\theta_{i,1}^T\psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T\psi_{i,1}\right) + \Delta g_1 + \delta_1 + e_2\right) + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T D^\alpha\widetilde{\theta}_{i,1} \\
=& D^\alpha V_0 + s_1^T H\left(s_2 + w_2 + x_2^* + vec\left(\theta_{i,1}^T\psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T\psi_{i,1}\right) + \Delta g_1 + \delta_1 + e_2\right) \\
&+ \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T D^\alpha\widetilde{\theta}_{i,1} \\
=& D^\alpha V_0 + s_1^T H s_2 + s_1^T H w_2 + s_1^T H\left(x_2^* + vec\left(\theta_{i,1}^T\psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T\psi_{i,1}\right)\right) + s_1^T H\Delta g_1 \\
&+ s_1^T H\varepsilon_1 + s_1^T H e_2 - \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T D^\alpha\theta_{i,1}
\end{aligned}
\tag{42}
$$

where $s_2 = [s_{1,2} \ s_{2,2} \ \cdots \ s_{N,2}]^T$, $w_2 = [w_{1,2} \ w_{2,2} \ \cdots \ w_{N,2}]^T$, $x_2^* = [x_{1,2}^* \ x_{2,2}^* \ \cdots \ x_{N,2}^*]^T$, $\Delta g_1 = [\Delta g_{1,1} \ \Delta g_{2,1} \ \cdots \ \Delta g_{N,1}]^T$, $\delta_1 = [\delta_{1,1} \ \delta_{2,1} \ \cdots \ \delta_{N,1}]^T$, $e_2 = [e_{1,2} \ e_{2,2} \ \cdots \ e_{N,2}]^T$, $vec\left(\theta_{i,1}^T\psi_{i,1}\right)$ and $vec\left(\widetilde{\theta}_{i,1}^T\psi_{i,1}\right)$ are column vectors.

According to Lemma 4, we have

$$
s_1^T H s_2 \le \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}s_2^T s_2 \tag{43}
$$

$$
s_1^T H w_2 \le \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}w_2^T w_2 \tag{44}
$$

$$
s_1^T H\Delta g_1 \le s_1^T H\gamma_1 e_1 \le \frac{1}{2}s_1^T H\gamma_1\gamma_1^T H^T s_1 + \frac{1}{2}e_1^T e_1 \tag{45}
$$

$$
s_1^T H\varepsilon_1 \le \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}\delta_1^T\delta_1 \tag{46}
$$

$$
s_1^T H e_2 \le \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}e_2^T e_2 \tag{47}
$$

where $\gamma_1 = diag[\gamma_{i,1}]$, $e_1 = [e_{1,1}\ e_{2,1}\ \cdots\ e_{N,1}]^T$. Substituting (43)–(47) into (42), we have

$$\begin{aligned}D^\alpha V_1 \leq & D^\alpha V_0 + s_1^T H\left(x_2^* + vec\left(\theta_{i,1}^T \psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T \psi_{i,1}\right)\right) \\ & + \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}w_2^T w_2 + \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}s_2^T s_2 \\ & + \frac{1}{2}s_1^T H\gamma_1\gamma_1 H^T s_1 + \frac{1}{2}e_1^T e_1 + \frac{1}{2}s_1^T HH^T s_1 \\ & + \frac{1}{2}\delta_1^T \delta_1 + \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}e_2^T e_2 - \sum_{i=1}^{N} \frac{1}{\sigma_{i,1}} \widetilde{\theta}_{i,1}^T D^\alpha \theta_{i,1}.\end{aligned} \tag{48}$$

According to the definition of H, we have

$$\begin{aligned}s_1^T H = & s_1^T \mathcal{A} + s_1^T L \\ = & [s_{1,1}\ s_{2,1}\ \cdots\ s_{N,1}] \begin{bmatrix} 2a_1 & & & \\ & 2a_2 & & \\ & & \ddots & \\ & & & 2a_N \end{bmatrix} + [s_{1,1}\ s_{2,1}\ \cdots\ s_{N,1}] \begin{bmatrix} \sum_{j\in N_i} a_{1j} & \cdots & -a_{1N} \\ -a_{21} & \cdots & -a_{2N} \\ \vdots & \ddots & \\ -a_{N1} & \cdots & \sum_{j\in N_i} a_{Nj} \end{bmatrix} \\ = & [2a_1 s_{1,1}\ 2a_2 s_{2,1} \cdots\ 2a_N s_{N,1}] + [s_{1,1}\ s_{2,1}\ \cdots\ s_{N,1}] \begin{bmatrix} \sum_{j\in N_i} a_{1j} & \cdots & -a_{N1} \\ -a_{12} & \cdots & -a_{N2} \\ \vdots & \ddots & \\ -a_{1N} & \cdots & \sum_{j\in N_i} a_{Nj} \end{bmatrix} \\ = & [2a_1 s_{1,1}\ 2a_2 s_{2,1} \cdots\ 2a_N s_{N,1}] + \left[\sum_{j\in N_i} a_{1j}(s_{1,1} - s_{j,1}) \cdots \sum_{j\in N_i} a_{Nj}(s_{N,1} - s_{j,1})\right] \\ = & \left[2a_1 s_{1,1} + \sum_{j\in N_i} a_{1j}(s_{1,1} - s_{j,1})\ \cdots 2a_N s_{N,1} + \sum_{j\in N_i} a_{Nj}(s_{N,1} - s_{j,1})\right].\end{aligned} \tag{49}$$

Then, we have

$$\begin{aligned}s_1^T HH^T s_1 = & \left(2a_1 s_{1,1} + \sum_{j\in N_i} a_{1j}(s_{1,1} - s_{j,1})\right)^2 + \cdots + \left(2a_N s_{N,1} + \sum_{j\in N_i} a_{Nj}(s_{N,1} - s_{j,1})\right)^2 \\ = & \sum_{i=1}^{N} \left[2a_i(x_{i,1} - x_d) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})\right]^2,\end{aligned} \tag{50}$$

$$\begin{aligned}s_1^T H\gamma_1\gamma_1^T H^T s_1 = & \gamma_{1,1}^2 \left(2a_1 s_{1,1} + \sum_{j\in N_i} a_{1j}(s_{1,1} - s_{j,1})\right)^2 + \cdots \\ & + \gamma_{N,1}^2 \left(2a_N s_{N,1} + \sum_{j\in N_i} a_{Nj}(s_{N,1} - s_{j,1})\right)^2 \\ = & \sum_{i=1}^{N} \gamma_{i,1}^2 \left[2a_i(x_{i,1} - x_d) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})\right]^2.\end{aligned} \tag{51}$$

According to (48), (50) and (51), design the first virtual controller $x_{i,2}^*$ and update law $\theta_{i,1}$ as

$$x_{i,2}^* = -c_{i,1}\left[2a_i(x_{i,1} - x_d) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})\right] - \theta_{i,1}^T \psi_{i,1} \tag{52}$$

$$D^\alpha \theta_{i,1} = \sigma_{i,1}\psi_{i,1}\left[2a_i(x_{i,1} - x_d) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})\right] - \rho_{i,1}\theta_{i,1} \tag{53}$$

where $c_{i,1} = 2 + \frac{\gamma_{i,1}^2}{2}$ and $\rho_{i,1}$ is the designed parameters. Substituting (52) and (53) into (48), after (34) we obtain

$$\begin{aligned} D^\alpha V_1 \le & - q_0\|e\|^2 + \frac{1}{2}\|P\varepsilon\|^2 + \sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,l}^T\widetilde{\theta}_{i,l} + \frac{1}{2}e_1^T e_1 \\ & + \frac{1}{2}e_2^T e_2 + \frac{1}{2}\delta_1^T\varepsilon_1 + \sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\theta_{i,1} + \sum_{i=1}^{N}\frac{1}{2}s_{i,2}^2 + \sum_{i=1}^{N}\frac{1}{2}w_{i,2}^2 \\ \le & - q_1\|e\|^2 + \eta_1 + \sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,l}^T\widetilde{\theta}_{i,l} + \sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\theta_{i,1} + \sum_{i=1}^{N}\frac{1}{2}s_{i,2}^2 + \sum_{i=1}^{N}\frac{1}{2}w_{i,2}^2 \end{aligned} \tag{54}$$

where $q_1 = q_0 - N$, $\eta_1 = \frac{1}{2}\|P\varepsilon\|^2 + \frac{1}{2}\delta_1^T\varepsilon_1$.

Based on DSC technique, we have the state variable $v_{i,2}$ as a solution of the fractal differential equation

$$\lambda_{i,2}D^\alpha v_{i,2} + v_{i,2} = x_{i,2}^* \quad v_{i,2}(0) = x_{i,2}^*(0). \tag{55}$$

According to Equations (35) and (55), we obtain

$$\begin{aligned} D^\alpha w_{i,2} &= D^\alpha v_{i,2} - D^\alpha x_{i,2}^* \\ &= -\frac{v_{i,2} - x_{i,2}^*}{\lambda_{i,2}} - D^\alpha x_{i,2}^* \\ &= -\frac{w_{i,2}}{\lambda_{i,2}} + B_{i,2} \end{aligned} \tag{56}$$

where $\lambda_{i,2}$ is designed parameter and $B_{i,2}$ is a continuous function depanding on variables $x_{i,1}$, $x_{j,1}$, $s_{i,2}$, $s_{j,2}$, $w_{i,2}$, $w_{j,2}$, $\theta_{i,1}$, $\theta_{j,1}$, x_d, $D^\alpha x_d$. According to [50,51], there exist constants $M_{i,2} > 0$, $i = 1, \dots, N$, such that $|B_{i,2}| \le M_{i,2}$ holds. □

Remark 4. *In Equation (52), the designed virtual controller contains three parts. The first part $-2c_{i,1}a_i(x_{i,1} - x_d)$ is used to ensure that the system can track reference signal. The second part $-c_{i,1}\sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})$ can make sure that all agents achieve consensus. Third part $-\theta_{i,1}^T\psi_{i,1}$ is used to approximate the unknown nonlinear functions $g_{i,1}(x_{i,1})$. Note that there is no conflict between three parts.*

Step 2. Define the error variable $s_{i,2} = \hat{x}_{i,2} - v_{i,2}$. Then, we obtain after (21) and (25)

$$\begin{aligned} D^\alpha s_{i,2} =& D^\alpha \hat{x}_{i,2} - D^\alpha v_{i,2} \\ =& \hat{x}_{i,3} + k_{i,2}e_{i,1} + \theta_{i,2}^T\psi_{i,2} + \widetilde{\theta}_{i,2}^T\psi_{i,2} + \delta_{i,2} + \Delta g_{i,2} - D^\alpha v_{i,2}. \end{aligned} \tag{57}$$

According to (35), we have

$$\begin{aligned} D^\alpha s_{i,2} =& s_{i,3} + x_{i,3}^* + w_{i,3} + k_{i,2}e_{i,1} + \theta_{i,2}^T\psi_{i,2} \\ & + \widetilde{\theta}_{i,2}^T\psi_{i,2} + \delta_{i,2} + \Delta g_{i,2} - D^\alpha v_{i,2}. \end{aligned} \tag{58}$$

Construct the Lyapunov function

$$\begin{aligned} V_2 &= V_1 + \sum_{i=1}^{N} V_{i,2} \\ &= V_1 + \frac{1}{2}\sum_{i=1}^{N}\left\{ s_{i,2}^2 + \frac{1}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^T\widetilde{\theta}_{i,2} + w_{i,2}^2 \right\} \end{aligned} \tag{59}$$

where $\sigma_{i,2}$ is designed parameter. Then, we have

$$D^\alpha V_2 = D^\alpha V_1 + \sum_{i=1}^{N}\left\{ s_{i,2}D^\alpha s_{i,2} + \frac{1}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^T D^\alpha \widetilde{\theta}_{i,2} + w_{i,2}D^\alpha w_{i,2} \right\}. \tag{60}$$

Substituting (58) into (60), we have

$$\begin{aligned} D^\alpha V_2 =& D^\alpha V_1 + \sum_{i=1}^{N}\Big[s_{i,2}\Big(s_{i,3} + x_{i,3}^* + w_{i,3} + k_{i,2}e_{i,1} + \theta_{i,2}^T\psi_{i,2} + \widetilde{\theta}_{i,2}^T\psi_{i,2} \\ &+ \delta_{i,2} + \Delta g_{i,2} - D^\alpha v_{i,2}\Big) + \frac{1}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^T D^\alpha\widetilde{\theta}_{i,2} + w_{i,2}D^\alpha w_{i,2}\Big]. \end{aligned} \tag{61}$$

According to Lemma 4, we have

$$s_{i,2}k_{i,2}e_{i,1} \le \frac{1}{2}s_{i,2}^2 + \frac{1}{2}k_{i,2}^2\|e_{i,1}\|^2 \tag{62}$$

$$s_{i,2}(s_{i,3} + w_{i,3}) \le s_{i,2}^2 + \frac{1}{2}\left(s_{i,3}^2 + w_{i,3}^2\right) \tag{63}$$

$$s_{i,2}\delta_{i,2} \le \frac{1}{2}s_{i,2}^2 + \frac{1}{2}\|\delta_{i,2}\|^2 \tag{64}$$

$$s_{i,2}\Delta g_{i,2} \le \frac{1}{2}s_{i,2}^2 + \frac{1}{2}\gamma_{i,2}^2\|e_{i,2}\|^2. \tag{65}$$

Substituting (62)–(65) into (61), we have

$$\begin{aligned} D^\alpha V_2 \le& D^\alpha V_1 + \sum_{i=1}^{N}\Big[s_{i,2}\Big(x_{i,3}^* + \theta_{i,2}^T\psi_{i,2} + \widetilde{\theta}_{i,32}^T\psi_{i,2} - D^\alpha v_{i,2}\Big) + \frac{5}{2}s_{i,2}^2 \\ &+ \frac{1}{2}\left(s_{i,3}^2 + w_{i,3}^2\right) + \frac{1}{2}k_{i,2}^2\|e_{i,1}\|^2 + \frac{1}{2}\|\delta_{i,2}\|^2 + \frac{1}{2}\gamma_{i,2}^2\|e_{i,2}\|^2 \\ &- \frac{1}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^T D^\alpha\theta_{i,2} + w_{i,2}D^\alpha w_{i,2}\Big]. \end{aligned} \tag{66}$$

According to Theorem 1, the second virtual controller $x_{i,3}^*$ and update laws $\theta_{i,2}$ are designed as follows

$$x_{i,3}^* = -c_{i,2}s_{i,2} - 3s_{i,2} - \theta_{i,2}^T\psi_{i,2} + \frac{x_{i,2}^* - v_{i,2}}{\lambda_{i,2}} \tag{67}$$

$$D^\alpha\theta_{i,2} = \sigma_{i,2}\psi_{i,2}(\hat{X}_{i,2})s_{i,2} - \rho_{i,2}\theta_{i,2} \tag{68}$$

where $\rho_{i,2}$ is designed parameter.

Substituting Equations (67), (68), (54) and (56) into (66), the following inequalities hold

$$\begin{aligned}D^\alpha V_2 \leq & -q_1\|e\|^2+\eta_1+\sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,l}^T\widetilde{\theta}_{i,l}+\sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\theta_{i,1}+\sum_{i=1}^{N}\frac{1}{2}s_{i,2}^2+\sum_{i=1}^{N}\frac{1}{2}w_{i,2}^2\\ &+\sum_{i=1}^{N}\Bigg[s_{i,2}\Big(-c_{i,2}s_{i,2}-3s_{i,2}-\theta_{i,2}^T\psi_{i,2}+\frac{x_{i,2}^*-v_{i,2}}{\lambda_{i,2}}+\theta_{i,2}^T\psi_{i,2}+\widetilde{\theta}_{i,2}^T\psi_{i,2}\\ &-D^\alpha v_{i,2}\Big)+\frac{5}{2}s_{i,2}^2+\frac{1}{2}\Big(s_{i,3}^2+w_{i,3}^2\Big)+\frac{1}{2}k_{i,2}^2\|e_{i,1}\|^2+\frac{1}{2}\|\delta_{i,2}\|^2+\frac{1}{2}\gamma_{i,2}^2\|e_{i,2}\|^2\\ &-\frac{1}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^T(\sigma_{i,2}\psi_{i,2}(\hat{X}_{i,2})s_{i,2}-\rho_{i,2}\theta_{i,2})+w_{i,2}\Big(-\frac{w_{i,2}}{\lambda_{i,2}}+B_{i,2}\Big)\Bigg].\end{aligned} \tag{69}$$

According to Lemma 4, we have $w_{i,2}B_{i,2} \leq \frac{1}{2}w_{i,2}^2+\frac{1}{2}M_{i,2}^2$. Then, we obtain

$$\begin{aligned}D^\alpha V_2 \leq & -q_2\|e\|^2+\eta_2+\sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,l}^T\widetilde{\theta}_{i,l}+\sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\theta_{i,1}+\sum_{i=1}^{N}\frac{\rho_{i,2}}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^T\theta_{i,2}\\ &-\sum_{i=1}^{N}c_{i,2}s_{i,2}^2-\sum_{i=1}^{N}\Big(\frac{1}{\lambda_{i,2}}-1\Big)w_{i,2}^2+\frac{1}{2}\sum_{i=1}^{N}M_{i,2}^2+\frac{1}{2}\sum_{i=1}^{N}\Big(s_{i,3}^2+w_{i,3}^2\Big)\end{aligned} \tag{70}$$

where

$$\begin{aligned}q_2&=q_1-\frac{1}{2}\sum_{i=1}^{N}\Big(k_{i,2}^2+\gamma_{i,2}^2\Big)\\ \eta_2&=\eta_1+\frac{1}{2}\sum_{i=1}^{N}\|\delta_{i,2}\|^2.\end{aligned}$$

By using the DSC technique, we have the next fractal differential equation

$$\lambda_{i,3}D^\alpha v_{i,3}+v_{i,3}=x_{i,3}^*,\quad v_{i,3}(0)=x_{i,3}^*(0). \tag{71}$$

According to (71), we can obtain

$$\begin{aligned}D^\alpha w_{i,3}&=D^\alpha v_{i,3}-D^\alpha x_{i,3}^*\\ &=-\frac{v_{i,3}-x_{i,3}^*}{\lambda_{i,3}}-D^\alpha x_{i,3}^*\\ &=-\frac{w_{i,3}}{\lambda_{i,3}}+B_{i,3}\end{aligned} \tag{72}$$

where $\lambda_{i,3}$ is designed parameter and $B_{i,3}=-D^\alpha x_{i,3}^*$.

Step k. Defining the k-th error variable $s_{i,k}=\hat{x}_{i,k}-v_{i,k}$, we have

$$\begin{aligned}D^\alpha s_{i,k}=&D^\alpha\hat{x}_{i,k}-D^\alpha v_{i,k}\\ =&\hat{x}_{i,k+1}+k_{i,k}e_{i,1}+\theta_{i,k}^T\psi_{i,k}+\widetilde{\theta}_{i,k}^T\psi_{i,k}+\delta_{i,k}+\Delta g_{i,k}-D^\alpha v_{i,k}.\end{aligned} \tag{73}$$

Substituting (35) into (73), we can obtain

$$\begin{aligned}D^\alpha s_{i,k}=&s_{i,k+1}+x_{i,k+1}^*+w_{i,k+1}+k_{i,k}e_{i,1}\\ &+\theta_{i,k}^T\psi_{i,k}+\widetilde{\theta}_{i,k}^T\psi_{i,k}+\delta_{i,k}+\Delta g_{i,k}-D^\alpha v_{i,k}.\end{aligned} \tag{74}$$

Construct the Lyapunov function

$$
\begin{aligned}
V_k &= V_{k-1} + \sum_{i=1}^{N} V_{i,k} \\
&= V_{k-1} + \frac{1}{2}\sum_{i=1}^{N}\left\{ s_{i,k}^2 + \frac{1}{\sigma_{i,k}}\widetilde{\theta}_{i,k}^T\widetilde{\theta}_{i,k} + w_{i,k}^2 \right\}
\end{aligned} \tag{75}
$$

where $\sigma_{i,k}$ is designed parameter. Then, we have

$$
D^\alpha V_k = D^\alpha V_{k-1} + \sum_{i=1}^{N}\left\{ s_{i,k}D^\alpha s_{i,k} + \frac{1}{\sigma_{i,k}}\widetilde{\theta}_{i,k}^T D^\alpha \widetilde{\theta}_{i,k} + w_{i,k}D^\alpha w_{i,k} \right\}. \tag{76}
$$

Substituting (74) into (76), we have

$$
\begin{aligned}
D^\alpha V_k =& D^\alpha V_{k-1} + \sum_{i=1}^{N}\Big[s_{i,k}\Big(s_{i,k+1} + x_{i,k+1}^* + w_{i,k+1} \\
&+ k_{i,k}e_{i,1} + \theta_{i,k}^T\psi_{i,k} + \widetilde{\theta}_{i,k}^T\psi_{i,k} + \delta_{i,k} + \Delta g_{i,k} \\
&- D^\alpha v_{i,k}\Big) + \frac{1}{\sigma_{i,k}}\widetilde{\theta}_{i,k}^T D^\alpha\widetilde{\theta}_{i,k} + w_{i,k}D^\alpha w_{i,k}\Big].
\end{aligned} \tag{77}
$$

According to Lemma 4, we have

$$
s_{i,k}k_{i,k}e_{i,1} \le \frac{1}{2}s_{i,k}^2 + \frac{1}{2}k_{i,k}^2\|e_{i,1}\|^2 \tag{78}
$$

$$
s_{i,k}(s_{i,k+1} + w_{i,k+1}) \le s_{i,k}^2 + \frac{1}{2}\left(s_{i,k+1}^2 + w_{i,k+1}^2 \right) \tag{79}
$$

$$
s_{i,k}\delta_{i,k} \le \frac{1}{2}s_{i,k}^2 + \frac{1}{2}\|\delta_{i,k}\|^2 \tag{80}
$$

$$
s_{i,k}\Delta g_{i,k} \le \frac{1}{2}s_{i,k}^2 + \frac{1}{2}\gamma_{i,k}^2\|e_{i,k}\|^2. \tag{81}
$$

Substituting (78)–(81) into (77), we can obtain

$$
\begin{aligned}
D^\alpha V_k \le& D^\alpha V_{k-1} + \sum_{i=1}^{N}\Big[s_{i,k}\Big(x_{i,k+1}^* + \theta_{i,k}^T\psi_{i,k} + \widetilde{\theta}_{i,k}^T\psi_{i,k} - D^\alpha v_{i,k}\Big) \\
&+ \frac{5}{2}s_{i,k}^2 + \frac{1}{2}(s_{i,k+1}^2 + w_{i,k+1}^2) + \frac{1}{2}k_{i,k}^2\|e_{i,1}\|^2 + \frac{1}{2}\|\delta_{i,k}\|^2 \\
&+ \frac{1}{2}\gamma_{i,k}^2\|e_{i,k}\|^2 - \frac{1}{\sigma_{i,k}}\widetilde{\theta}_{i,k}^T D^\alpha\theta_{i,k} + w_{i,k}D^\alpha w_{i,k}\Big].
\end{aligned} \tag{82}
$$

According to the Theorem 1, design the k-th virtual controller $x_{i,k+1}^*$ and update laws $\theta_{i,k}$, as follows

$$
x_{i,k+1}^* = -c_{i,k}s_{i,k} - 3s_{i,k} - \theta_{i,k}^T\psi_{i,k} + \frac{x_{i,k}^* - v_{i,k}}{\lambda_{i,k}} \tag{83}
$$

$$
D^\alpha\theta_{i,k} = \sigma_{i,k}\psi_{i,k}(\hat{X}_{i,k})s_{i,k} - \rho_{i,k}\theta_{i,k} \tag{84}
$$

where $\rho_{i,k}$ is the designed parameter. Generally, by using the DSC technique, we have the next fractal differential equation

$$
\lambda_{i,k}D^\alpha v_{i,k} + v_{i,k} = x_{i,k}^*, \quad v_{i,k}(0) = x_{i,k}^*(0). \tag{85}
$$

According to Equation (85), we have

$$\begin{aligned} D^\alpha w_{i,k} &= D^\alpha v_{i,k} - D^\alpha x^*_{i,k} \\ &= -\frac{v_{i,k} - x^*_{i,k}}{\lambda_{i,k}} - D^\alpha x^*_{i,k} \\ &= -\frac{w_{i,k}}{\lambda_{i,k}} + B_{i,k} \end{aligned} \tag{86}$$

where $\lambda_{i,k}$ is the designed parameter and $B_{i,k} = -D^\alpha x^*_{i,k}$.

Substituting Equations (83), (84) and (86) into (82), then we have

$$\begin{aligned} D^\alpha V_k \le & D^\alpha V_{k-1} + \sum_{i=1}^{N} \Bigg[s_{i,k}\Big(-c_{i,k}s_{i,k} - 3s_{i,k} - \theta^T_{i,k}\psi_{i,k} + \frac{x^*_{i,k} - v_{i,k}}{\lambda_{i,k}} \\ & + \theta^T_{i,k}\psi_{i,k} + \widetilde{\theta}^T_{i,k}\psi_{i,k} - D^\alpha v_{i,k} \Big) + \frac{5}{2}s^2_{i,k} + \frac{1}{2}(s^2_{i,k+1} + w^2_{i,k+1}) \\ & + \frac{1}{2}k^2_{i,k}\|e_{i,1}\|^2 + \frac{1}{2}\|\delta_{i,k}\|^2 + \frac{1}{2}\gamma^2_{i,k}\|e_{i,k}\|^2 - \frac{1}{\sigma_{i,k}}\widetilde{\theta}^T_{i,k}(\sigma_{i,k}\psi_{i,k}s_{i,k} \\ & - \rho_{i,k}\theta_{i,k}) + w_{i,k}\Big(-\frac{w_{i,k}}{\lambda_{i,k}} + B_{i,k} \Big) \Bigg]. \end{aligned} \tag{87}$$

According to Lemma 4, we have $w_{i,k}B_{i,k} \le \frac{1}{2}w^2_{i,k} + \frac{1}{2}M^2_{i,k}$. The following inequalities hold

$$\begin{aligned} D^\alpha V_k \le & D^\alpha V_{k-1} + \sum_{i=1}^{N} \Bigg[s_{i,k}\Big(-c_{i,k}s_{i,k} - 3s_{i,k} - \theta^T_{i,k}\psi_{i,k} + \frac{x^*_{i,k} - v_{i,k}}{\lambda_{i,k}} \\ & + \theta^T_{i,k}\psi_{i,k} + \widetilde{\theta}^T_{i,k}\psi_{i,k} - D^\alpha v_{i,k} \Big) + \frac{5}{2}s^2_{i,k} + \frac{1}{2}(s^2_{i,k+1} + w^2_{i,k+1}) \\ & + \frac{1}{2}k^2_{i,k}\|e_{i,1}\|^2 + \frac{1}{2}\|\delta_{i,k}\|^2 + \frac{1}{2}\gamma^2_{i,k}\|e_{i,k}\|^2 - \frac{1}{\sigma_{i,k}}\widetilde{\theta}^T_{i,k}(\sigma_{i,k}\psi_{i,k}s_{i,k} \\ & - \rho_{i,k}\theta_{i,k}) - \frac{w^2_{i,k}}{\lambda_{i,k}} + \frac{1}{2}w^2_{i,k} + \frac{1}{2}M^2_{i,k} \Bigg]. \end{aligned} \tag{88}$$

Combining (34), (54) and (70), we can obtain

$$\begin{aligned} D^\alpha V_{k-1} \le & -q_{k-1}\|e\|^2 + \eta_{k-1} + \sum_{i=1}^{N}\sum_{l=1}^{n} \frac{1}{2}\widetilde{\theta}^T_{i,l}\widetilde{\theta}_{i,l} + \sum_{i=1}^{N} \Bigg[\sum_{l=1}^{k-1} \frac{\rho_{i,l}}{\sigma_{i,l}}\widetilde{\theta}^T_{i,l}\theta_{i,l} \\ & - \sum_{l=2}^{k-1} c_{i,l}s^2_{i,l} + \sum_{l=2}^{k-1}\left(\frac{1}{\lambda_{i,l}} - 1 \right) w^2_{i,l} + \frac{1}{2}\sum_{l=2}^{k-1} M^2_{i,k-1} + \frac{1}{2}\left(s^2_{i,k} + w^2_{i,k} \right) \Bigg]. \end{aligned} \tag{89}$$

Substituting (89) into (88), we have

$$\begin{aligned} D^\alpha V_k \le & -q_k\|e\|^2 + \eta_k + \sum_{i=1}^{N}\sum_{l=1}^{n} \frac{1}{2}\widetilde{\theta}^T_{i,l}\widetilde{\theta}_{i,l} + \sum_{i=1}^{N} \Bigg[\sum_{l=1}^{k} \frac{\rho_{i,l}}{\sigma_{i,l}}\widetilde{\theta}^T_{i,l}\theta_{i,l} \\ & - \sum_{l=2}^{k} c_{i,l}s^2_{i,l} - \sum_{l=2}^{k}\left(\frac{1}{\lambda_{i,l}} - 1 \right) w^2_{i,l} + \frac{1}{2}\sum_{l=2}^{k} M^2_{i,k} + \frac{1}{2}\left(s^2_{i,k+1} + w^2_{i,k+1} \right) \Bigg] \end{aligned} \tag{90}$$

where

$$q_k = q_{k-1} - \frac{1}{2}\sum_{i=1}^{N}\left(k_{i,k}^2 + \gamma_{i,k}^2\right)$$
$$\eta_k = \eta_{k-1} + \frac{1}{2}\sum_{i=1}^{N}\|\delta_{i,k}\|^2.$$

Step n. Define the n-th error variable and the output error of the filter, as follows

$$s_{i,n} = \hat{x}_{i,n} - v_{i,n} \tag{91}$$

$$w_{i,n} = v_{i,n} - x_{i,n}^*. \tag{92}$$

Then, we have

$$\begin{aligned} D^\alpha s_{i,n} =& D^\alpha \hat{x}_{i,n} - D^\alpha v_{i,n} \\ =& u_i + k_{i,n}e_{i,1} + \theta_{i,n}^T\psi_{i,n} + \widetilde{\theta}_{i,n}^T\psi_{i,n} + \delta_{i,n} + \Delta g_{i,n} - D^\alpha v_{i,n}. \end{aligned} \tag{93}$$

By using the DSC technique, we have the next fractal differential equation

$$\lambda_{i,n}D^\alpha v_{i,n} + v_{i,n} = x_{i,n}^*, \quad v_{i,n}(0) = x_{i,n}^*(0). \tag{94}$$

By Equation (92), we have

$$D^\alpha w_{i,n} = D^\alpha v_{i,n} - D^\alpha x_{i,n}^* = -\frac{w_{i,n}}{\lambda_{i,n}} + B_{i,n} \tag{95}$$

where $\lambda_{i,n}$ is the designed parameter and $B_{i,n} = -D^\alpha x_{i,n}^*$.

Construct Lyapunov function

$$\begin{aligned} V_n =& V_{n-1} + \sum_{i=1}^{N} V_{i,n} \\ =& V_{n-1} + \frac{1}{2}\sum_{i=1}^{N}\left\{s_{i,n}^2 + \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T\widetilde{\theta}_{i,n} + w_{i,n}^2\right\} \end{aligned} \tag{96}$$

where $\sigma_{i,n}$ is designed parameter. Then, we have

$$D^\alpha V_n = D^\alpha V_{n-1} + \sum_{i=1}^{N}\left\{s_{i,n}D^\alpha s_{i,n} + \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T D^\alpha\widetilde{\theta}_{i,n} + w_{i,n}D^\alpha w_{i,n}\right\}. \tag{97}$$

Substituting (93) into (97), we have

$$\begin{aligned} D^\alpha V_n =& D^\alpha V_{n-1} + \sum_{i=1}^{N}\Big[s_{i,n}\Big(u_i + k_{i,m}e_{i,1} + \theta_{i,n}^T\psi_{i,n} + \widetilde{\theta}_{i,n}^T\psi_{i,n} + \delta_{i,n} + \Delta g_{i,n} - D^\alpha v_{i,n}\Big) \\ &+ \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T D^\alpha\widetilde{\theta}_{i,n} + w_{i,n}D^\alpha w_{i,n}\Big]. \end{aligned} \tag{98}$$

According to Lemma 4, we have

$$s_{i,n}k_{i,n}e_{i,1} \le \frac{1}{2}s_{i,n}^2 + \frac{1}{2}k_{i,n}^2\|e_{i,1}\|^2 \tag{99}$$

$$s_{i,n}\delta_{i,n} \le \frac{1}{2}s_{i,n}^2 + \frac{1}{2}\|\delta_{i,n}\|^2 \tag{100}$$

$$s_{i,n}\Delta g_{i,n} \leq \frac{1}{2}s_{i,n}^2 + \frac{1}{2}\gamma_{i,n}^2\|e_{i,n}\|^2. \tag{101}$$

From (99)–(101), Equation (98) can be written as

$$\begin{aligned} D^\alpha V_n \leq & D^\alpha V_{n-1} + \sum_{i=1}^{N}\Big[s_{i,n}\Big(u_i + \theta_{i,n}^T\psi_{i,n} + \widetilde{\theta}_{i,n}^T\psi_{i,n} - D^\alpha v_{i,n}\Big) + \frac{3}{2}s_{i,n}^2 + \frac{1}{2}k_{i,n}^2\|e_{i,1}\|^2 \\ & + \frac{1}{2}\|\delta_{i,n}\|^2 + \frac{1}{2}\gamma_{i,n}^2\|e_{i,n}\|^2 - \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T D^\alpha\theta_{i,n} + w_{i,n}D^\alpha w_{i,n}\Big]. \end{aligned} \tag{102}$$

Design the controller u_i and update laws $\theta_{i,n}$

$$u_i = -c_{i,n}s_{i,n} - 2s_{i,n} - \theta_{i,n}^T\psi_{i,n} + \frac{x_{i,n}^* - v_{i,n}}{\lambda_{i,n}} \tag{103}$$

$$D^\alpha\theta_{i,n} = \sigma_{i,n}\psi_{i,n}(\hat{X}_{i,n})s_{i,n} - \rho_{i,n}\theta_{i,n} \tag{104}$$

where $\rho_{i,n}$ is designed parameter. According to (90), substituting Equations (95), (103) and (104) into (102), then the following inequalities hold

$$\begin{aligned} D^\alpha V_n \leq & -q_{n-1}\|e\|^2 + \eta_{n-1} + \sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,l}^T\widetilde{\theta}_{i,l} + \sum_{i=1}^{N}\Big[\sum_{l=1}^{n-1}\frac{\rho_{i,l}}{\sigma_{i,l}}\widetilde{\theta}_{i,l}^T\theta_{i,l} \\ & - \sum_{l=2}^{n-1}c_{i,l}s_{i,l}^2 + \sum_{l=2}^{n-1}\Big(\frac{1}{\lambda_{i,l}} - 1\Big)w_{i,l}^2 + \frac{1}{2}\sum_{l=2}^{n-1}M_{i,l}^2 + \frac{1}{2}\Big(s_{i,n}^2 + w_{i,n}^2\Big)\Big] \\ & + \sum_{i=1}^{N}\Big[s_{i,n}\Big(-c_{i,n}s_{i,n} - 2s_{i,n} - \theta_{i,n}^T\psi_{i,n} + \frac{x_{i,n}^* - v_{i,n}}{\lambda_{i,n}} + \theta_{i,n}^T\psi_{i,n} \\ & + \widetilde{\theta}_{i,n}^T\psi_{i,n} - D^\alpha v_{i,n}\Big) + \frac{3}{2}s_{i,n}^2 + \frac{1}{2}k_{i,n}^2\|e_{i,1}\|^2 + \frac{1}{2}\|\delta_{i,n}\|^2 + \frac{1}{2}\gamma_{i,n}^2\|e_{i,n}\|^2 \\ & - \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T\Big(\sigma_{i,n}\psi_{i,n}s_{i,n} - \rho_{i,n}\theta_{i,n}\Big) + w_{i,n}\Big(-\frac{w_{i,n}}{\lambda_{i,n}} + B_{i,n}\Big)\Big]. \end{aligned} \tag{105}$$

According to Lemma 4, we obtain $w_{i,n}B_{i,n} \leq \frac{1}{2}w_{i,n}^2 + \frac{1}{2}M_{i,n}^2$, then we have

$$\begin{aligned} D^\alpha V_n \leq & -q_n\|e\|^2 + \eta_n + \sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,l}^T\widetilde{\theta}_{i,l} + \sum_{i=1}^{N}\Big[\sum_{l=1}^{n}\frac{\rho_{i,l}}{\sigma_{i,l}}\widetilde{\theta}_{i,l}^T\theta_{i,l} \\ & - \sum_{l=2}^{n}c_{i,l}s_{i,l}^2 - \sum_{l=2}^{n}\Big(\frac{1}{\lambda_{i,l}} - 1\Big)w_{i,l}^2 + \frac{1}{2}\sum_{l=2}^{n}M_{i,l}^2\Big] \end{aligned} \tag{106}$$

where

$$\begin{aligned} q_n &= q_{n-1} - \frac{1}{2}\sum_{i=1}^{N}\Big(k_{i,n}^2 + \gamma_{i,n}^2\Big) \\ \eta_n &= \eta_{n-1} + \frac{1}{2}\sum_{i=1}^{N}\|\delta_{i,n}\|^2. \end{aligned}$$

According to Lemma 4, we can obtain

$$\widetilde{\theta}_{*,l}^T\theta_{*,l} \leq -\frac{1}{2}\widetilde{\theta}_{*,l}^T\widetilde{\theta}_{*,l} + \frac{1}{2}\theta_{*,l}^{*T}\theta_{*,l}^*. \tag{107}$$

Then, we have

$$
\begin{aligned}
D^{\alpha}V_n \leq & -q_n\|e\|^2 + \eta_n + \sum_{i=1}^{N}\sum_{l=1}^{n}\frac{1}{2}\tilde{\theta}_{i,l}^T\tilde{\theta}_{i,l} + \sum_{i=1}^{N}\Bigg[-\sum_{l=1}^{n}\frac{\rho_{i,l}}{2\sigma_{i,l}}\tilde{\theta}_{i,l}^T\tilde{\theta}_{i,l} \\
& + \sum_{l=1}^{n}\frac{\rho_{i,l}}{2\sigma_{i,l}}\theta_{i,l}^{*T}\theta_{i,l}^{*} - \sum_{l=2}^{n}c_{i,l}s_{i,l}^2 - \sum_{l=2}^{n}\left(\frac{1}{\lambda_{i,l}}-1\right)w_{i,l}^2 + \frac{1}{2}\sum_{l=2}^{n}M_{i,l}^2\Bigg].
\end{aligned} \tag{108}
$$

Define

$$
\zeta = \eta_n + \sum_{i=1}^{N}\left(\sum_{l=1}^{n}\frac{\rho_{i,l}}{2\sigma_{i,l}}\theta_{i,l}^{*T}\theta_{i,l}^{*} + \frac{1}{2}\sum_{l=2}^{n}M_{i,l}^2\right). \tag{109}
$$

Then, according to Equation (108), the following inequalities hold

$$
D^{\alpha}V_n \leq -q_n\|e\|^2 + \sum_{i=1}^{N}\left[-\sum_{l=2}^{n}c_{i,l}s_{i,l}^2 - \sum_{l=1}^{n}\left(\frac{\rho_{i,l}}{2\sigma_{i,l}}-\frac{1}{2}\right)\tilde{\theta}_{i,l}^T\tilde{\theta}_{i,l} - \sum_{l=2}^{n}\left(\frac{1}{\lambda_{i,l}}-1\right)w_{i,l}^2\right] + \zeta \tag{110}
$$

where $c_{i,l} > 0$, $(l = 2, \cdots, n)$, $\left(\frac{\rho_{i,l}}{2\sigma_{i,l}} - \frac{1}{2}\right) > 0$, $(l = 1, \cdots, n)$, $\left(\frac{1}{\lambda_{i,l}} - 1\right) > 0$, $(l = 2, \cdots, n)$.
Define

$$
C = min\left\{2\frac{q_n}{\lambda_{min}(P)}, 2c_{i,l}, 2\left(\frac{\rho_{i,l}}{2\sigma_{i,l}}-\frac{1}{2}\right), 2\left(\frac{1}{\lambda_{i,l}}-1\right)\right\}. \tag{111}
$$

Then, Equation (110) becomes

$$
D^{\alpha}V_n(t,x) \leq -CV_n(t,x) + \zeta. \tag{112}
$$

According to (112) and Lemma 5, we have

$$
V_n \leq V(0)E_{\alpha}(-Ct^{\alpha}) + \frac{\zeta\mu}{C} \quad t \geq 0. \tag{113}
$$

Then, we have

$$
\lim_{t\to\infty}|V_n(t)| \leq \frac{\zeta\mu}{C}. \tag{114}
$$

Since $\frac{1}{2}|s_{i,1}|^2 \leq V_n(T)$, we have

$$
\lim_{t\to\infty}||s_{i,1}|| \leq \sqrt{\frac{2\zeta\mu}{C}}. \tag{115}
$$

Then, we can conclude that all the signals of system (9) remain bounded in the closed-loop system and converge to the optimal solution x^*. The output errors and the consensus tracking errors converge to close to zero.

Remark 5. *The algorithm proposed in this paper represents two kinds of protocols in the existing literature. For example, if the reference signal is time-invariant, it becomes the algorithm solving the distributed time-invariant convex optimization [12–17]. If we set the reference signal as a time-varying signal, it becomes the algorithm solving the distributed time-varying convex optimization [52–55].*

4. Simulation

In this section, two examples are given to verify the validity of the proposed method.

Example 1. *Consider the following fractional Duffing-Holmes chaotic system [56].*

$$\begin{cases} D^{\alpha}x_{i,1} = x_{i,2} + g_{i,1}(x_{i,1}) \\ D^{\alpha}x_{i,2} = u_i + g_{i,2}(x_{i,1}, x_{i,2}) \\ \quad y_i = x_{i,1} \end{cases} \tag{116}$$

with $i = 1,2,3,4,5$ *and the initial states are selected as* $x_1(0) = [0.1, 0.1]$, $x_2(0) = [0.2, 0.2]$, $x_3(0) = [0.3, 0.3]$, $x_4(0) = [0.4, 0.4]$, $x_5(0) = [0.5, 0.5]$. *In this paper, a communication graph for five agents is given by Figure 1. The phase portrait depicting such behavior is shown in Figures 2 and 3.* $x_d = \sin t$ *is defined as the reference signal. Define the unknown functions in system (116) as*

$$\begin{aligned} g_{1,1}(X_{1,1}) =& g_{2,1}(X_{2,1}) = g_{3,1}(X_{3,1}) = g_{4,1}(X_{4,1}) = g_{5,1}(X_{5,1}) = 0 \\ g_{1,2}(X_{1,2}) =& x_{1,1} - 0.25x_{1,2} - x_{1,1}^3 + 0.3\cos(t) \\ g_{2,2}(X_{2,2}) =& x_{2,1} - 0.25x_{2,2} - x_{2,1}^3 + 0.1\left(x_{2,1}^2 + x_{2,2}^2\right)^{1/2} + 0.3\cos(t) \\ g_{3,2}(X_{3,2}) =& x_{3,1} - 0.25x_{3,2} - x_{3,1}^3 + 0.2\sin(t)\left(x_{3,1}^2 + 2x_{3,2}^2\right)^{1/2} + 0.3\cos(t) \\ g_{4,2}(X_{4,2}) =& x_{4,1} - 0.25x_{4,2} - x_{4,1}^3 + 0.2\sin(t)\left(2x_{4,1}^2 + 2x_{4,2}^2\right)^{1/2} + 0.3\cos(t) \\ g_{5,2}(X_{5,2}) =& x_{5,1} - 0.1x_{5,2} - x_{5,1}^3 + 0.2\sin(t)\left(x_{5,1}^2 + x_{5,2}^2\right)^{1/2} + 0.3\cos(t). \end{aligned}$$

The local objective functions of (15) of each of the five agents are given as follows

$$\begin{aligned} f_1(x_{1,1}) &= 3x_{1,1}^2 - 6x_d x_{1,1} + 3x_d^2 + 0.1 \\ f_2(x_{2,1}) &= 4.6x_{2,1}^2 - 9.2x_d x_{2,1} + 4.6x_d^2 + 0.2 \\ f_3(x_{3,1}) &= 3.5x_{3,1}^2 - 7x_d x_{3,1} + 3.5x_d^2 + 1 \\ f_4(x_{4,1}) &= 2.5x_{4,1}^2 - 5x_d x_{4,1} + 2.5x_d^2 + 0.3 \\ f_5(x_{5,1}) &= 2.3x_{5,1}^2 - 4.6x_d x_{5,1} + 2.3x_d^2 + 0.4. \end{aligned}$$

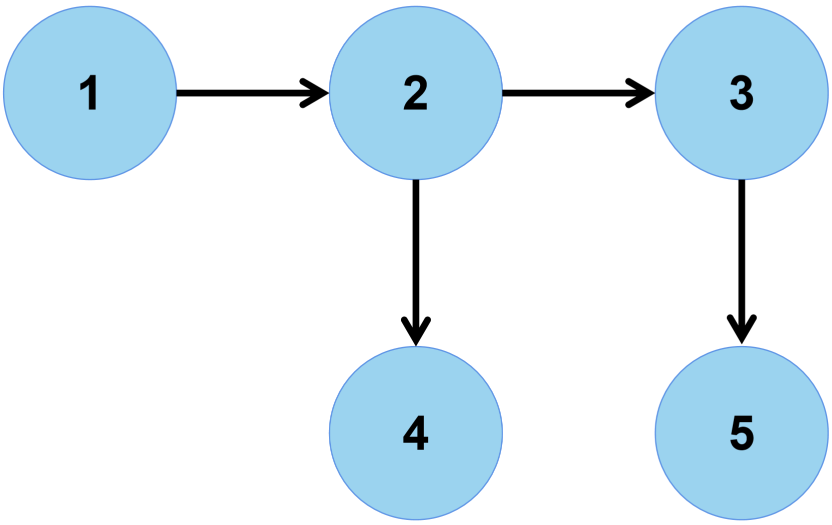

Figure 1. Communication graph.

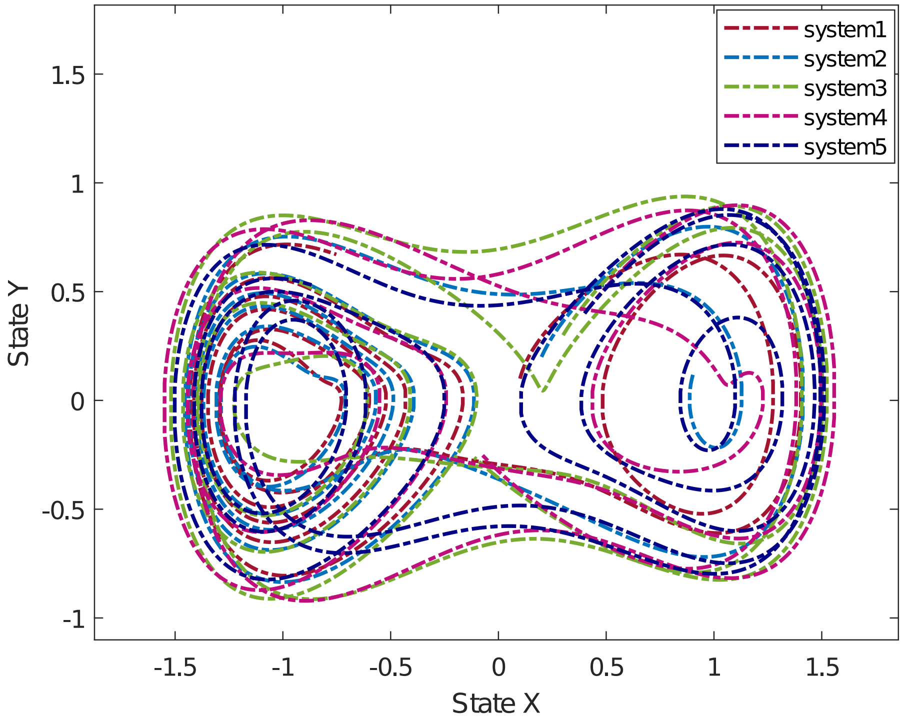

Figure 2. Fractional order Duffing-Holmes chaotic systems trajectories.

Then, define the penalty function

$$P(x_1) = \sum_{i=1}^{5} f_i(x_{i,1}) + x_1^T L x_1. \tag{117}$$

Base on the penalty function (117), the necessary condition for the optimal solution x_1^* to the distributed optimization problem is as follows

$$\frac{\partial P(x_1^*)}{\partial x_1^*} = 0 \tag{118}$$

where $x_1^* = \left[x_{1,1}^*, x_{2,1}^*, \cdots, x_{5,1}^*\right]^T$.

The virtual controller, the parameters update laws and the control input are developed by the observer. For the observer, the design parameters are selected as $k_{1,1} = k_{2,1} = k_{3,1} = k_{4,1} = k_{5,1} = 50$, $k_{1,2} = k_{2,2} = k_{3,2} = k_{4,2} = k_{5,2} = 1000$ and the initial states for 5 agents are selected as $\hat{x}_1 = [0.2, 0.2]$, $\hat{x}_2 = [0.3, 0.3]$, $\hat{x}_3 = [0.4, 0.4]$, $\hat{x}_4 = [0.5, 0.5]$, $\hat{x}_5 = [0.6, 0.6]$. For the RBFNNs, eleven nodes are selected as $q = 11$. We design two different centers of the receptive field. $c_{1,1}$ and $c_{2,1}$ are evenly space in $[-1, 1]$. $c_{1,2}$ and $c_{2,2}$ are selected as $\frac{1}{5}\begin{pmatrix} -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ -6 & -4.8 & -3.6 & -2.4 & -1.2 & 0 & 1.2 & 2.4 & 3.6 & 4.8 & 6 \end{pmatrix}$. The width of the Gaussian function is selected as $b_{1,1} = b_{1,2} = b_{2,1} = b_{2,2} = 5$.

According to Theorem 1 and Equations (52), (53), (103) and (104), the virtual control law, the parameters update laws and the control input are designed, as follows

$$x_{i,2}^* = -c_{i,1}\left[2a_i(x_{i,1} - x_d) + \sum_{j \in N_i} a_{ij}(x_{i,1} - x_{j,1})\right] - \theta_{i,1}^T \psi_{i,1} \tag{119}$$

$$D^{\alpha}\theta_{i,1} = \sigma_{i,1}\left[2a_i(x_{i,1} - x_d) + \sum_{j \in N_i} a_{ij}(x_{i,1} - x_{j,1})\right] - \rho_{i,1}\theta_{i,1} \tag{120}$$

$$u_i = -c_{i,2}s_{i,2} - 2s_{i,2} - \theta_{i,2}^{T}\psi_{i,2} + \frac{x_{i,2}^{*} - v_{i,2}}{\lambda_{i,2}} \tag{121}$$

$$D^{\alpha}\theta_{i,2} = \sigma_{i,2}\psi_{i,2}s_{i,2} - \rho_{i,2}\theta_{i,2}. \tag{122}$$

Choose design parameters as $c_{1,1} = c_{3,1} = c_{4,1} = c_{5,1} = 5$, $c_{2,1} = 4$, $c_{i,2} = 2$, $\sigma_{i,1} = \sigma_{i,2} = 1$, $\rho_{i,1} = 40$, $\rho_{i,2} = 80$, $\lambda_{i,2} = 0.05$.

Figures 2–12 show the simulation results. The trajectory of the system can be observed in Figure 2, where it clearly shows that the system is unstable. In Figure 3 it can be observed as well. Figure 4 shows the trajectories of x_d and $x_{i,1}$. Figure 5 displays the trajectories of the tracking error $s_{i,1}$, which demonstrates that the tracking error $s_{i,1}$ can quickly converge to near zero. Figures 6 and 7 give the trajectories of $\hat{x}_{i,1}$ and $\hat{x}_{i,2}$. Figure 8 shows the trajectories of $x_{i,2}$. In Figures 9 and 10, we use $x_{1,1}$ and $x_{1,2}$ for examples to compare the true value with the estimated value. Figure 11 gives the trajectories of u_i, from which it can be clearly observed that the control input can converge quickly. The value of penalty function is shown in Figure 12, from which we can draw the conclusion that the algorithm successfully minimizes the value of the penalty function.

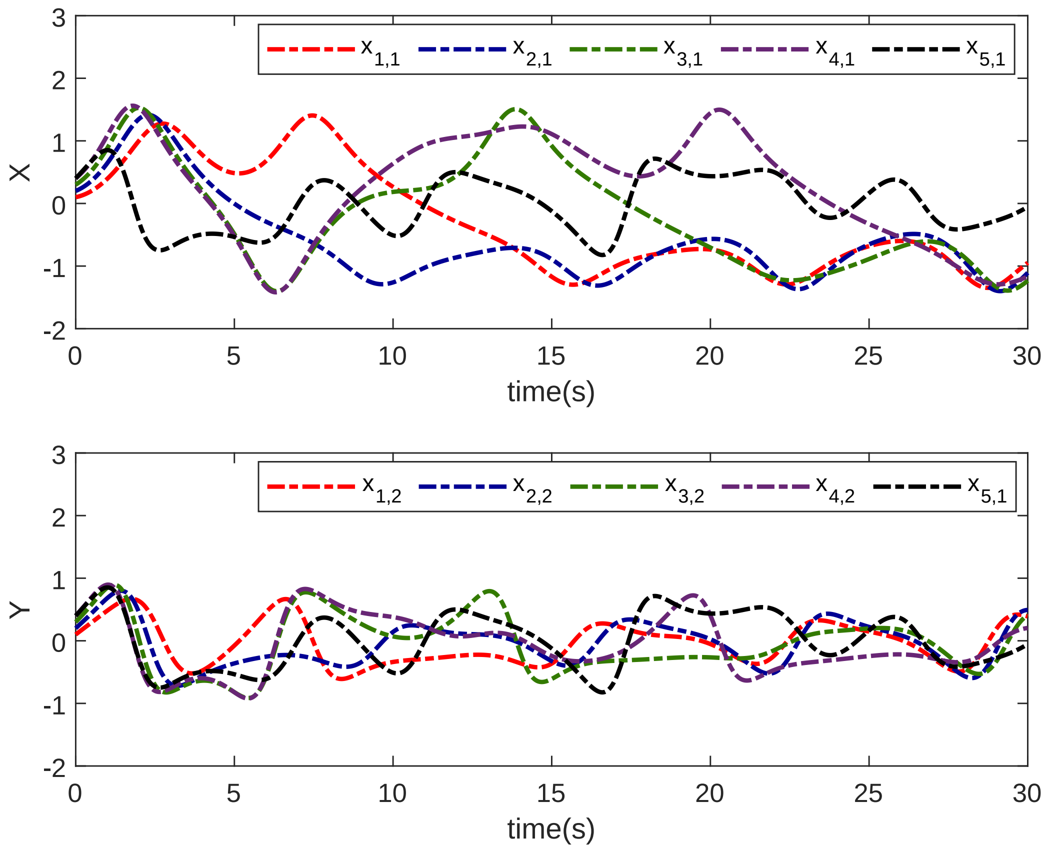

Figure 3. Trajectories of system states without control input.

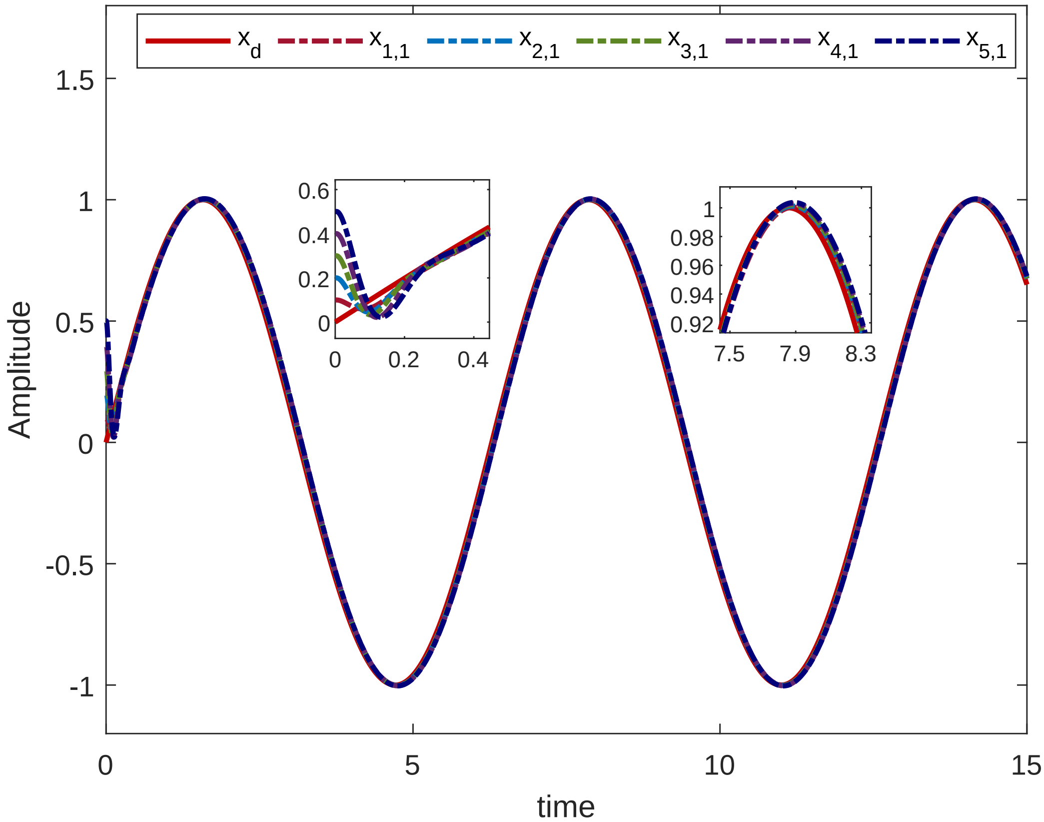

Figure 4. The trajectories of x_d and $x_{i,1}(i = 1, \cdots, 5)$.

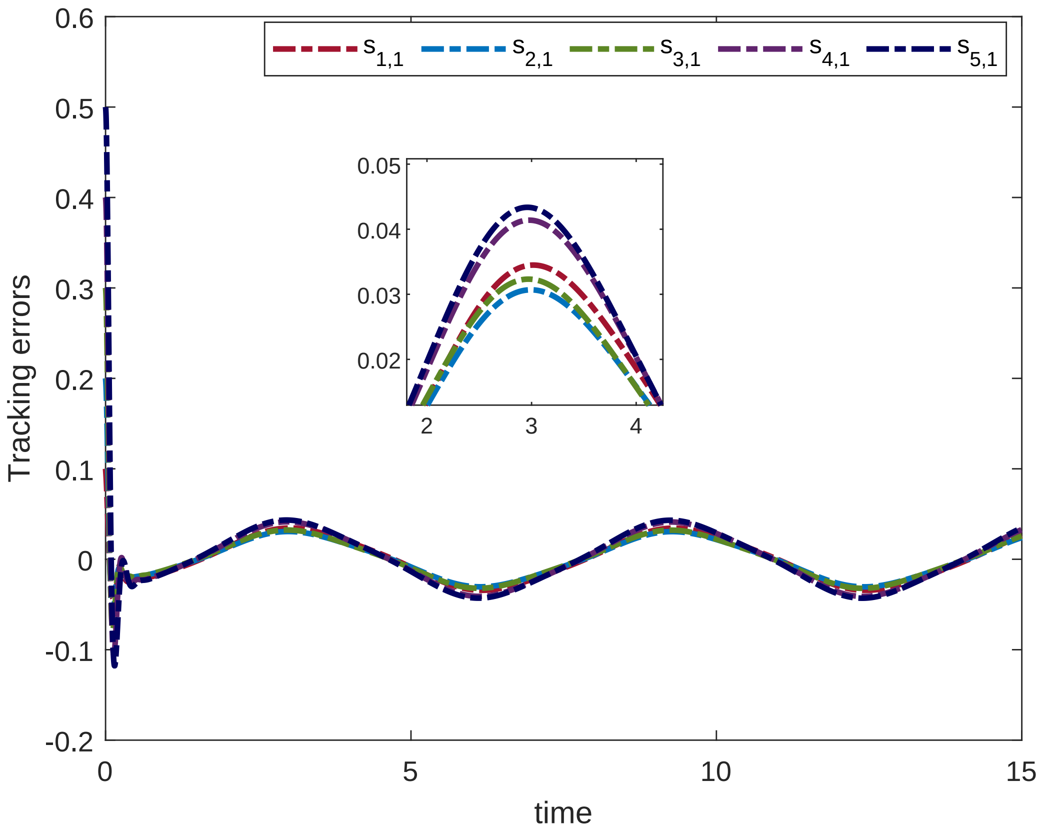

Figure 5. The trajectories of error $s_{i,1}(i = 1, \cdots, 5)$.

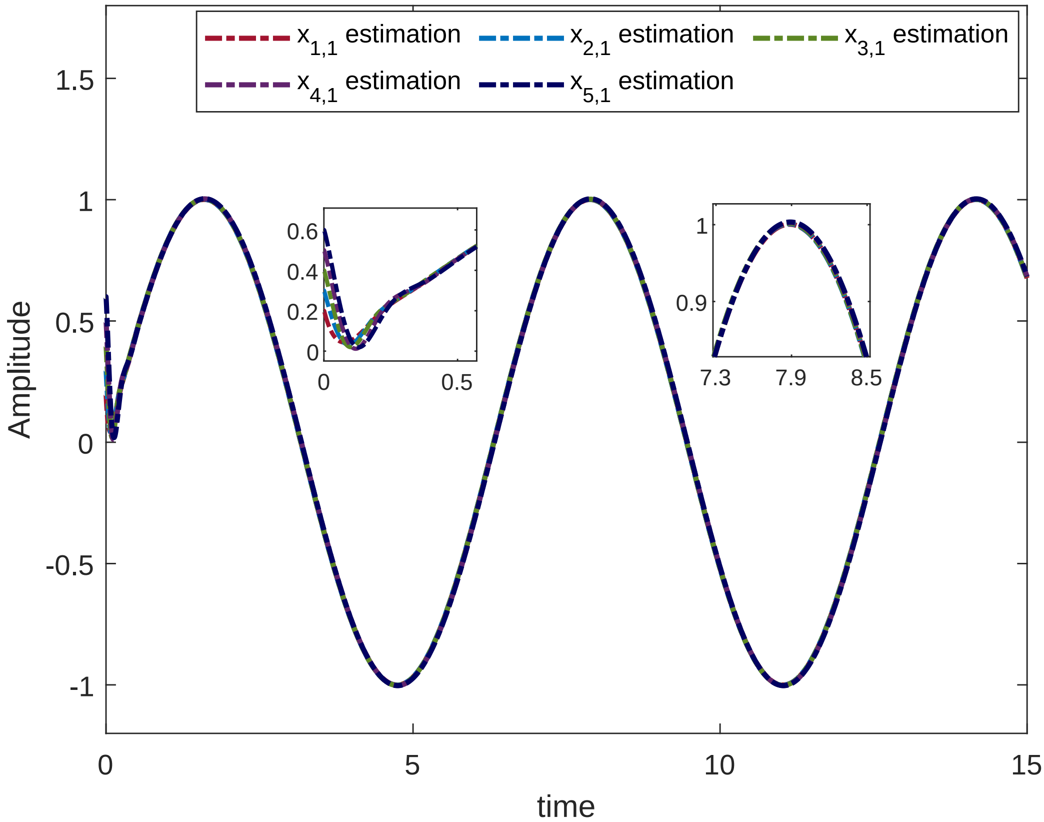

Figure 6. The trajectories of $x_{i,1}(i = 1, \cdots, 5)$ estimation values.

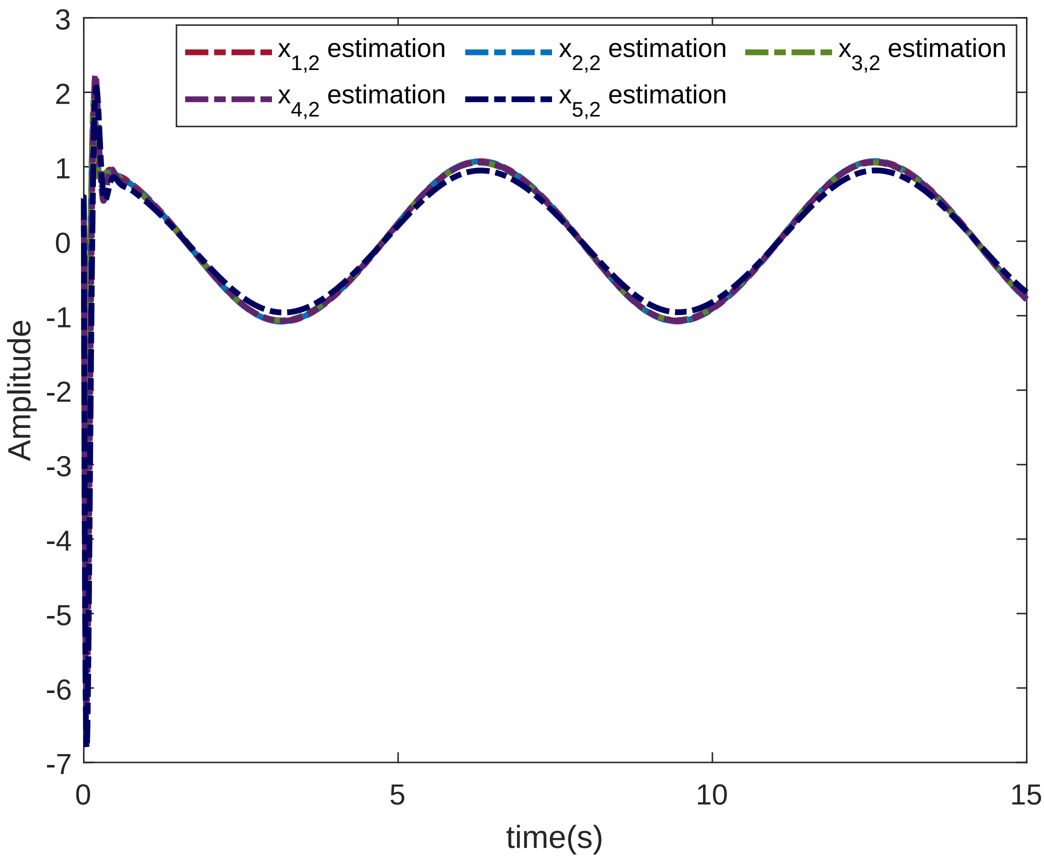

Figure 7. The trajectories of $x_{i,2}(i = 1, \cdots, 5)$ estimation values.

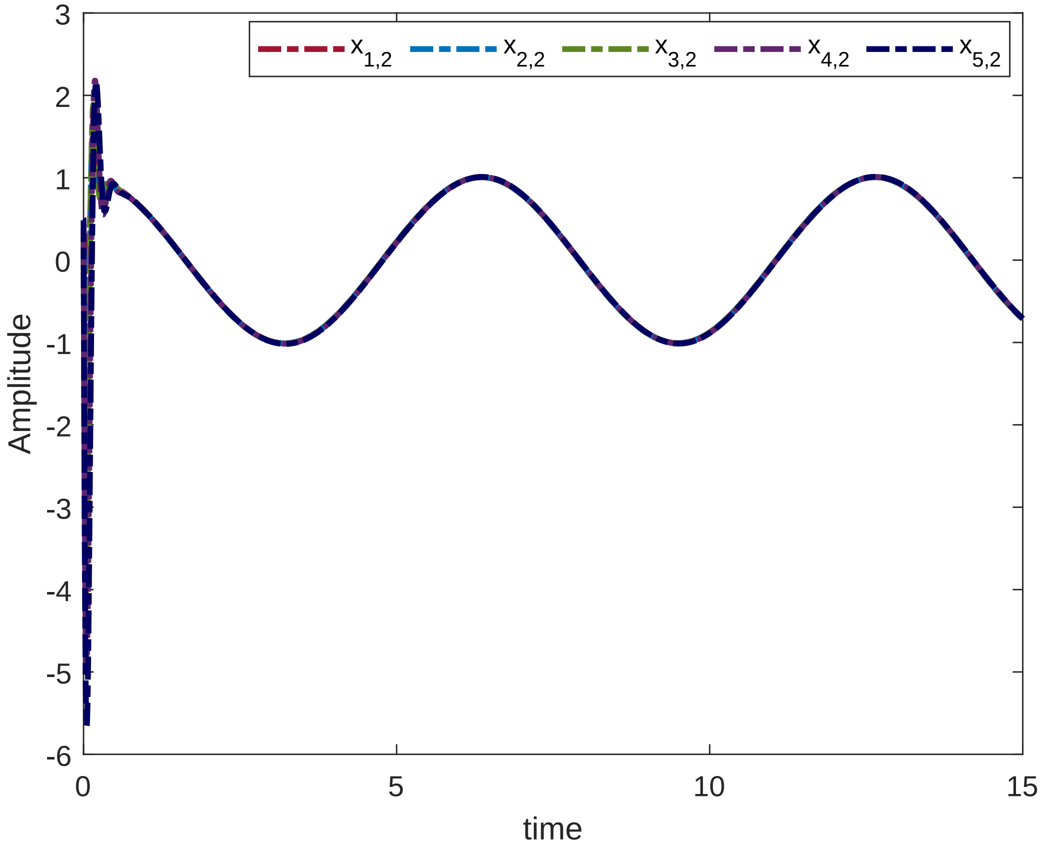

Figure 8. The trajectories of $x_{i,2}(i = 1, \cdots, 5)$.

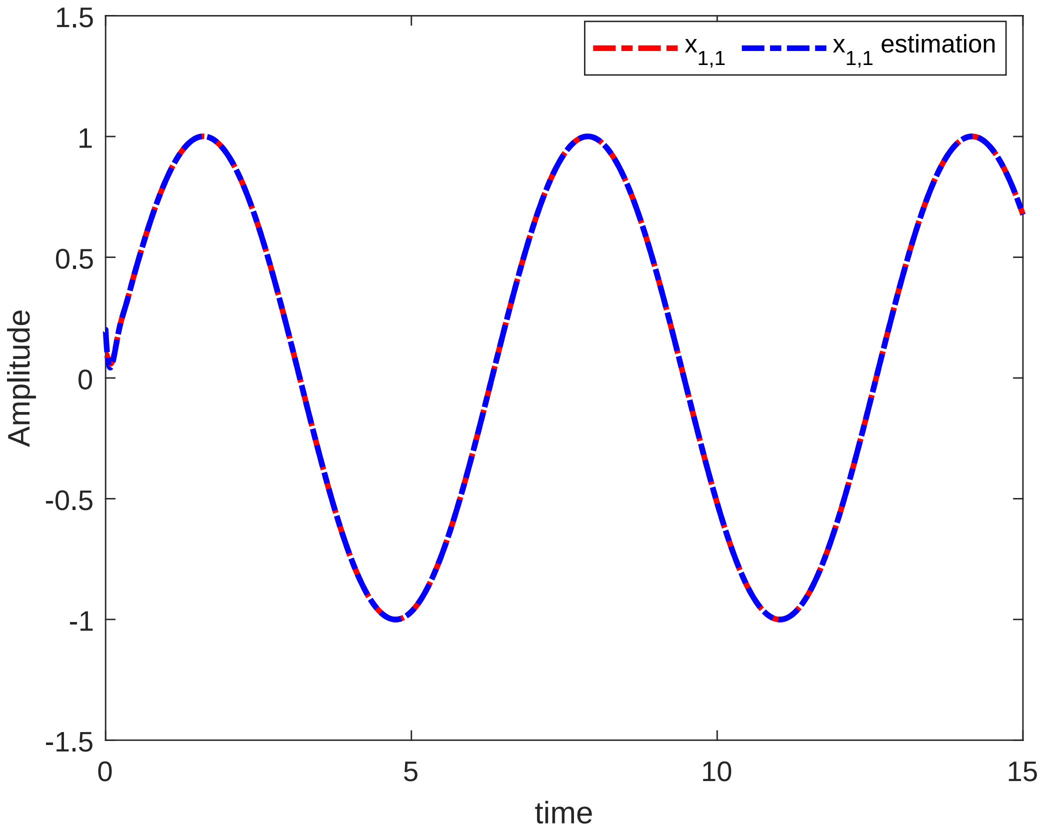

Figure 9. The trajectories of $x_{1,1}$ and its estimation.

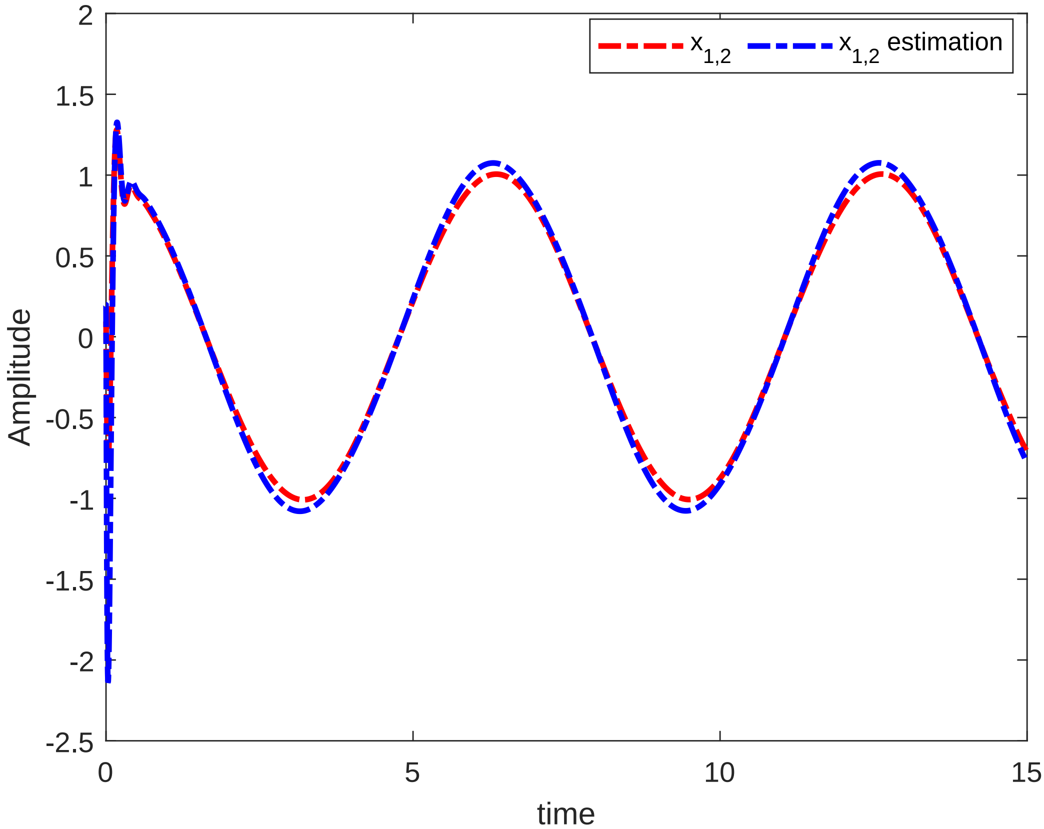

Figure 10. The trajectories of $x_{1,2}$ and its estimation.

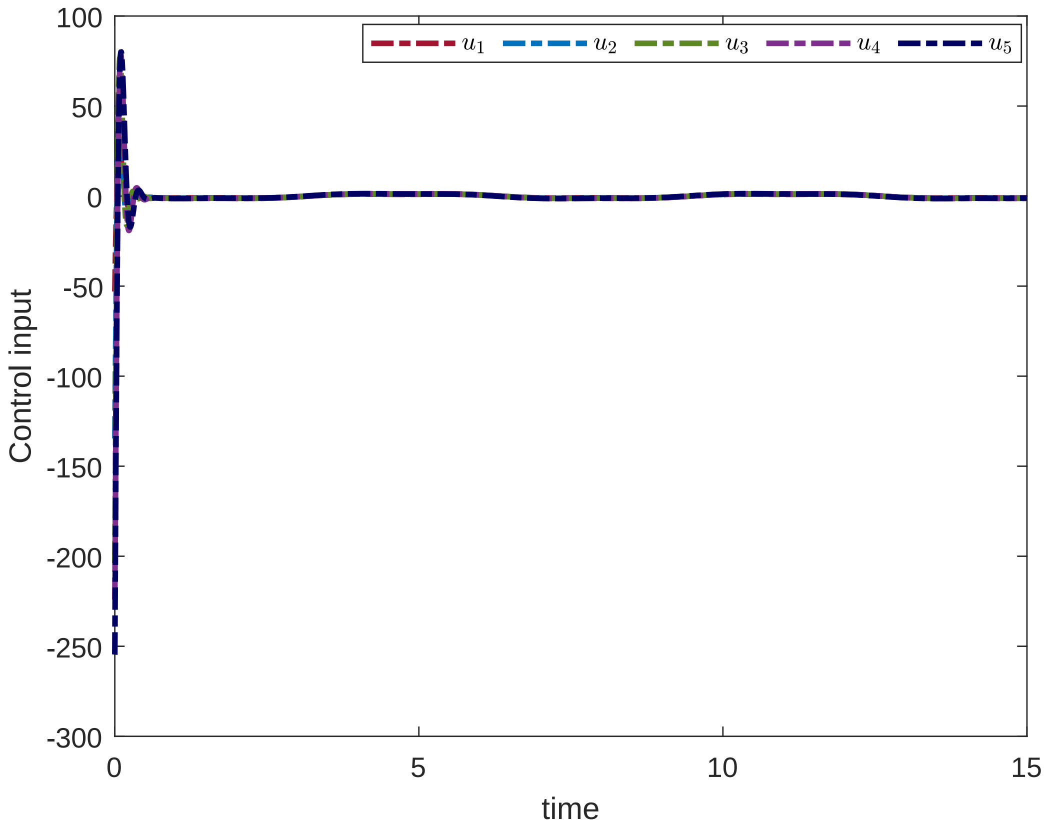

Figure 11. The trajectories of control input u_i.

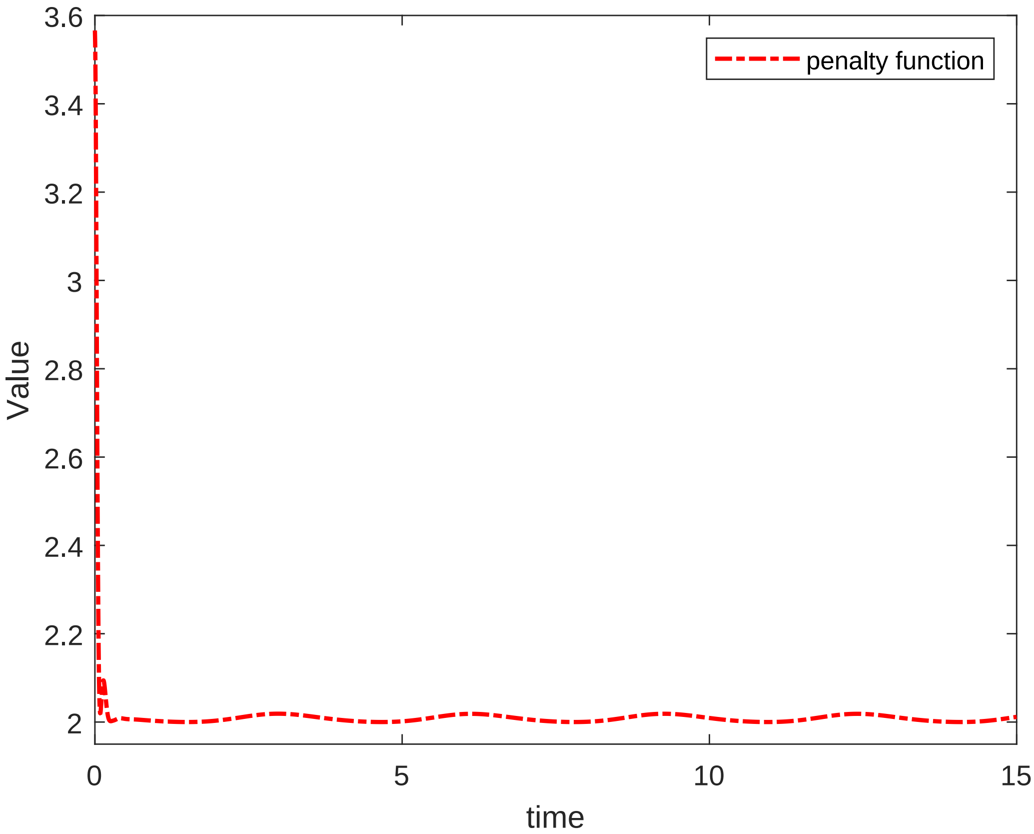

Figure 12. The value of penalty function.

Based on the above simulation results, it is demonstrated that the proposed control method can ensure that the consensus tracking error converges to a small area of the origin quickly, and all agents can synchronize to the reference trajectory with good control performance. Meanwhile, the value of the penalty function converges to the minimum successfully. The controller designed by this method can not only guarantee good tracking performance of all agents, but consider the distributed optimization problem as well.

Example 2. *Consider the following agent dynamics [16]*

$$D^{\alpha}x_i(t) = u_i(t) + g_{i,1}(t), i = 1, 2, \dots, N \tag{123}$$

with $\alpha = 0.98$, $i = 1, 2, 3, 4, 5$ *and the initial states are selected as* $x_1(0) = 0.1$, $x_2(0) = 0.2$, $x_3(0) = 0.3$, $x_4(0) = 0.4$ *and* $x_5(0) = 0.5$. *Define* $x_d = \sin t$ *as the reference signal. Define the unknown functions* $g_{1,1}(t) = -0.1\sin t$, $g_{2,1}(t) = 0.2\sin t$, $g_{3,1}(t) = 0.1\sin t$, $g_{4,1}(t) = 0.2\cos t$ *and* $g_{5,1}(t) = -0.1\cos t$ *in system (123) as the local disturbance signals . A communication graph for five agents is given by Figure 1.*

Define the local objective functions of each of the five agents, as follows

$$\begin{aligned}
f_1(x_{1,1}) &= 4x_{1,1}^2 - 8x_d x_{1,1} + 4x_d^2 + 1.8 \\
f_2(x_{2,1}) &= 4.6x_{1,1}^2 - 9.2x_d x_{1,1} + 4.6x_d^2 + 2.2 \\
f_3(x_{3,1}) &= 3.5x_{1,1}^2 - 7x_d x_{1,1} + 3.5x_d^2 + 1.4 \\
f_4(x_{4,1}) &= 2.5x_{1,1}^2 - 5x_d x_{1,1} + 2.5x_d^2 + 6.6 \\
f_5(x_{5,1}) &= 2.2x_{1,1}^2 - 4.4x_d x_{1,1} + 2.2x_d^2 + 9.
\end{aligned} \tag{124}$$

Define the penalty function

$$P(x_1) = \sum_{i=1}^{5} f_i(x_{i,1}) + x_1^T L x_1. \tag{125}$$

Design the necessary condition for optimal solution to the distributed optimization problem, as follows

$$\frac{\partial P(x_1^*)}{\partial x_1^*} = 0 \tag{126}$$

where $x_1^* = \left[x_{1,1}^*, x_{2,1}^*, \cdots, x_{5,1}^*\right]^T$.

According to Theorem 1 and Equations (52) and (53), the control input and the parameters update laws are developed, as follows

$$u_i = -c_{i,1}\left[2a_i(x_{i,1} - x_d) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})\right] - \theta_{i,1}^T \psi_{i,1} \tag{127}$$

$$D^\alpha \theta_{i,1} = \sigma_{i,1}\left[2a_i(x_{i,1} - x_d) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})\right] - \rho_{i,1}\theta_{i,1}. \tag{128}$$

Choose design parameters as $c_{i,1} = 3$, $\sigma_{1,1} = 9.5$, $\sigma_{2,1} = 40$, $\sigma_{3,1} = 8$, $\sigma_{4,1} = 28$, $\sigma_{5,1} = 5$, $\rho_{1,1} = \rho_{4,1} = 20$ and $\rho_{2,1} = \rho_{3,1} = \rho_{5,1} = 40$.

A communication graph for five agents is given by Figure 1. Figure 13 shows the simulation results of the protocol designed in this paper, from which we can find that the output signals $x_{i,1}$ can track the reference signal x_d. Figure 14 displays the trajectories of tracking error $s_{i,1}$. Figure 15 shows the trajectories of control input u_i, where it can be demonstrated that all the control input are smooth in the whole time. The value of the penalty function is shown in Figure 16.

In [16], a finite-time distributed optimized algorithm is proposed, as follows

$$\begin{aligned}
&u_i = u_i^o + u_i^r, i = 1, 2, \ldots, N \\
&u_i^o = -sig^\alpha\left(\triangledown f_i(x_i) + \gamma\sum_{j=1}^{N} a_{ij}(x_i - x_j)\right) \\
&u_i^r = -k_{1i} sig^{\frac{1}{2}}(s_i) + \phi_i \\
&\dot{\phi}_i = -k_{2i} sign(s_i) \\
&s_i = x_i - x_i(0) - \int_0^t u_i^o(\tau) d(\tau).
\end{aligned} \tag{129}$$

Choose design parameters as $\gamma = 15$, $\alpha = 0.13$, $k_{1i} = 3$ and $k_{2i} = -5$.

A communication graph for five agents is given by Figure 1. The simulation results are displayed in Figure 17. The trajectories of the tracking error $s_{i,1}$ are shown in Figure 18. In Figure 19, all the trajectories of the control input u_i are displayed. The value of the penalty function is shown in Figure 20, from which we can conclude that the algorithm successfully solves the distributed optimization problem.

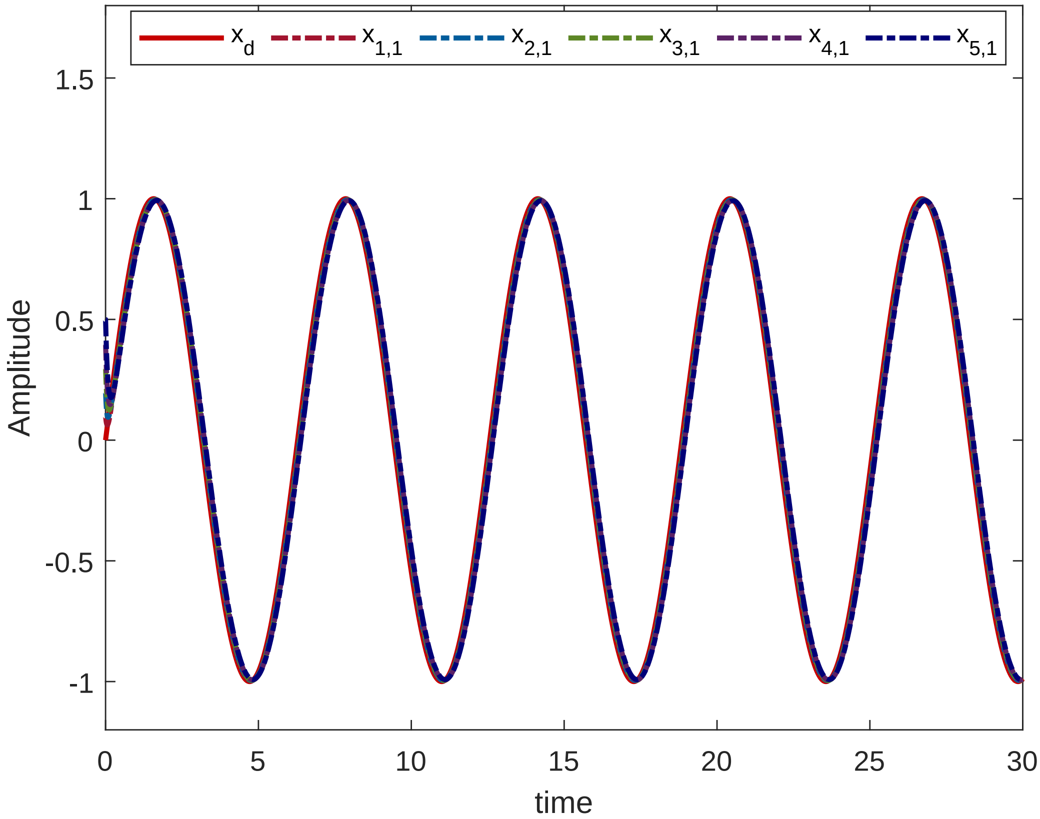

Figure 13. The trajectories of x_d and $x_{i,1}(i = 1, \cdots, 5)$ based on the algorithm proposed in this paper.

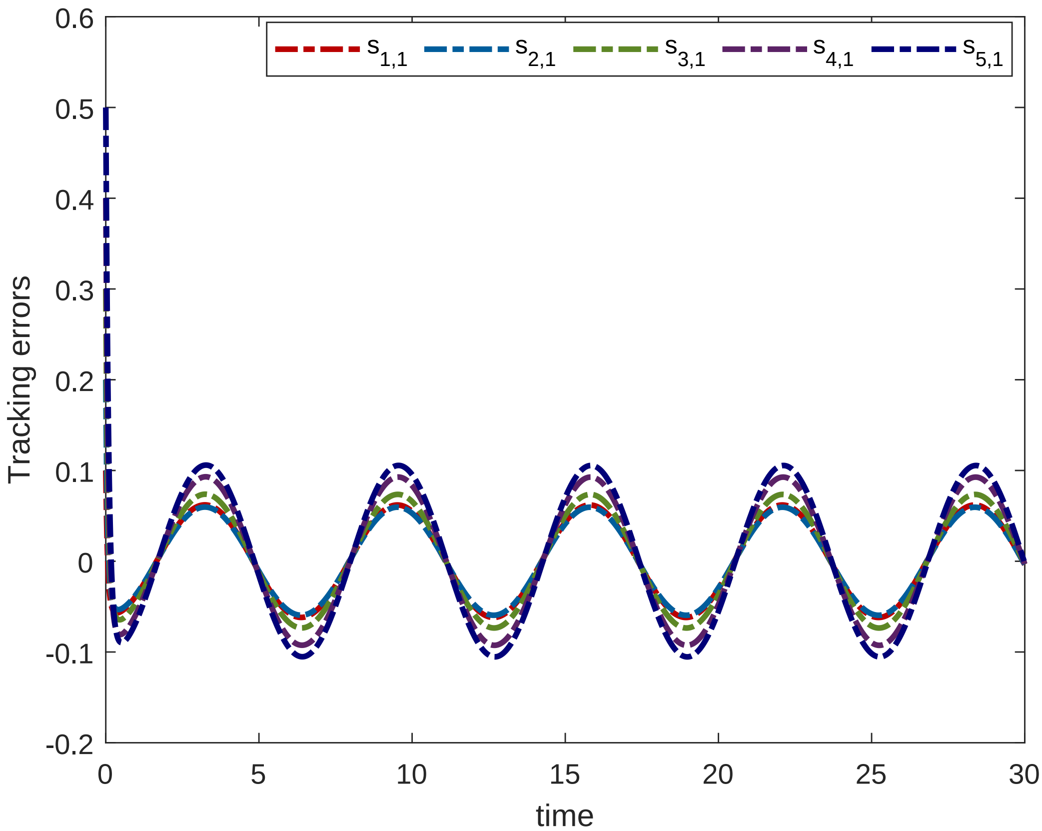

Figure 14. The trajectories of tracking error $s_{i,1}(i = 1, \cdots, 5)$ based on the algorithm proposed in this paper.

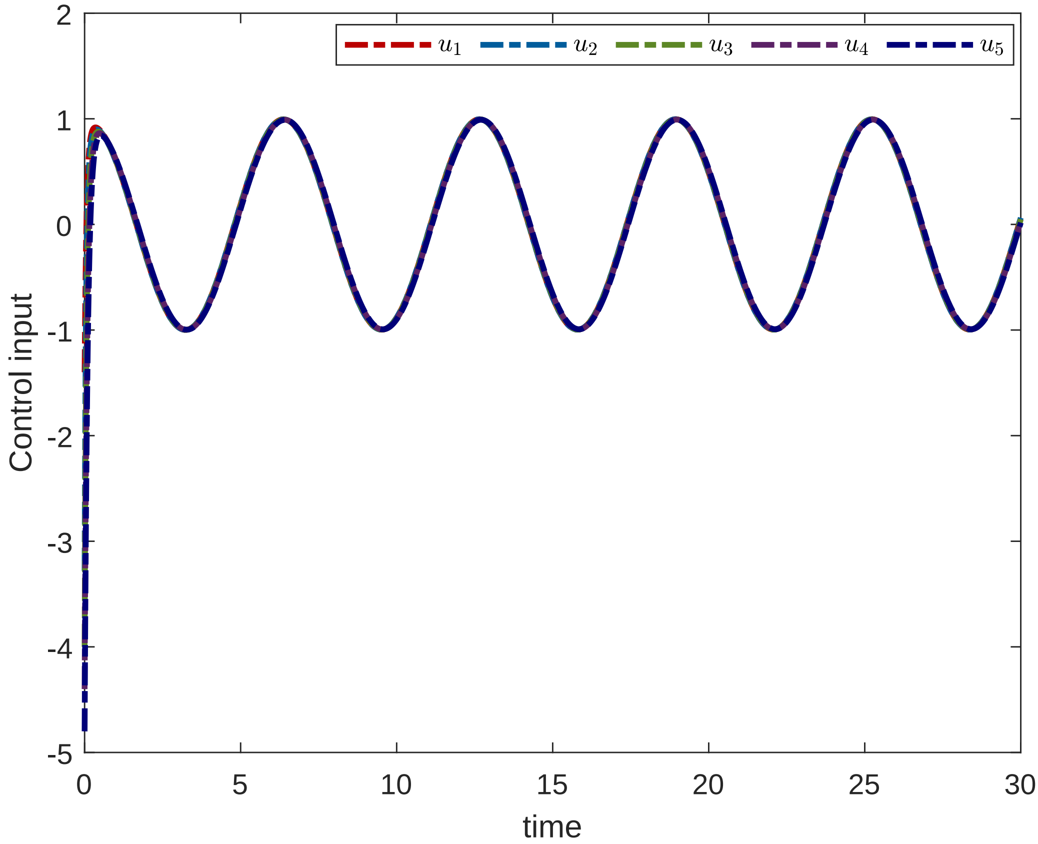

Figure 15. The trajectories of control input u_i based on the algorithm proposed in this paper.

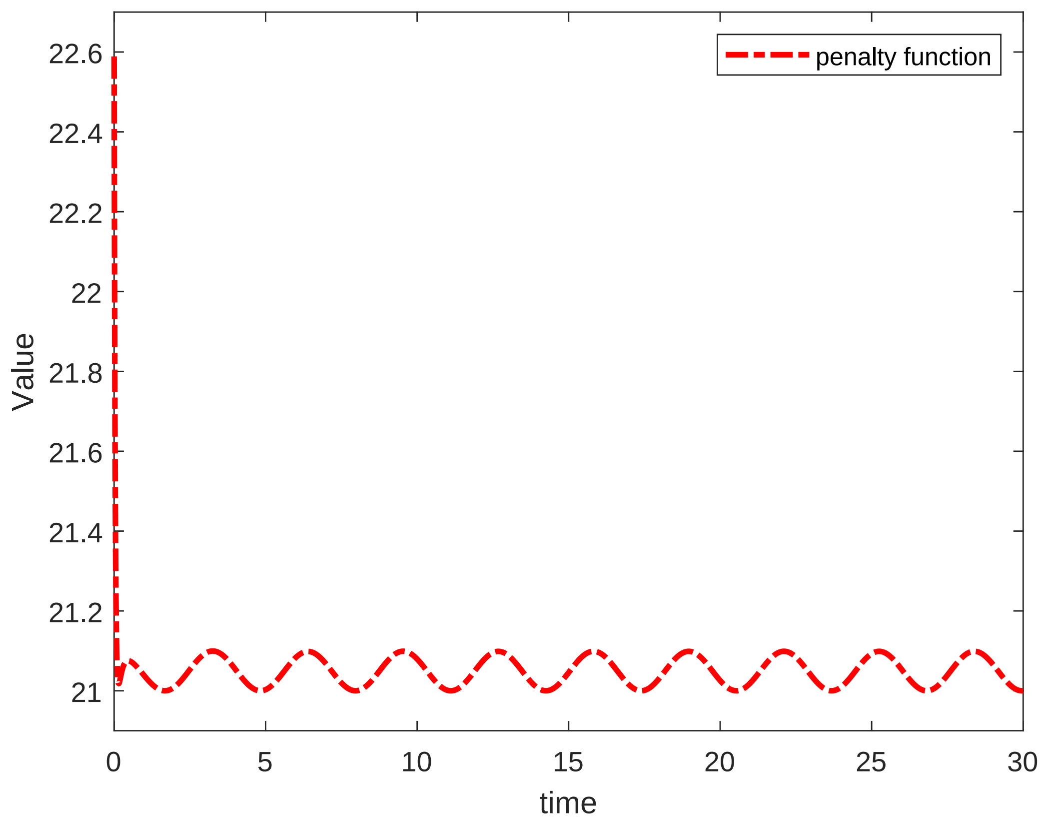

Figure 16. The value of penalty function based on the algorithm proposed in this paper.

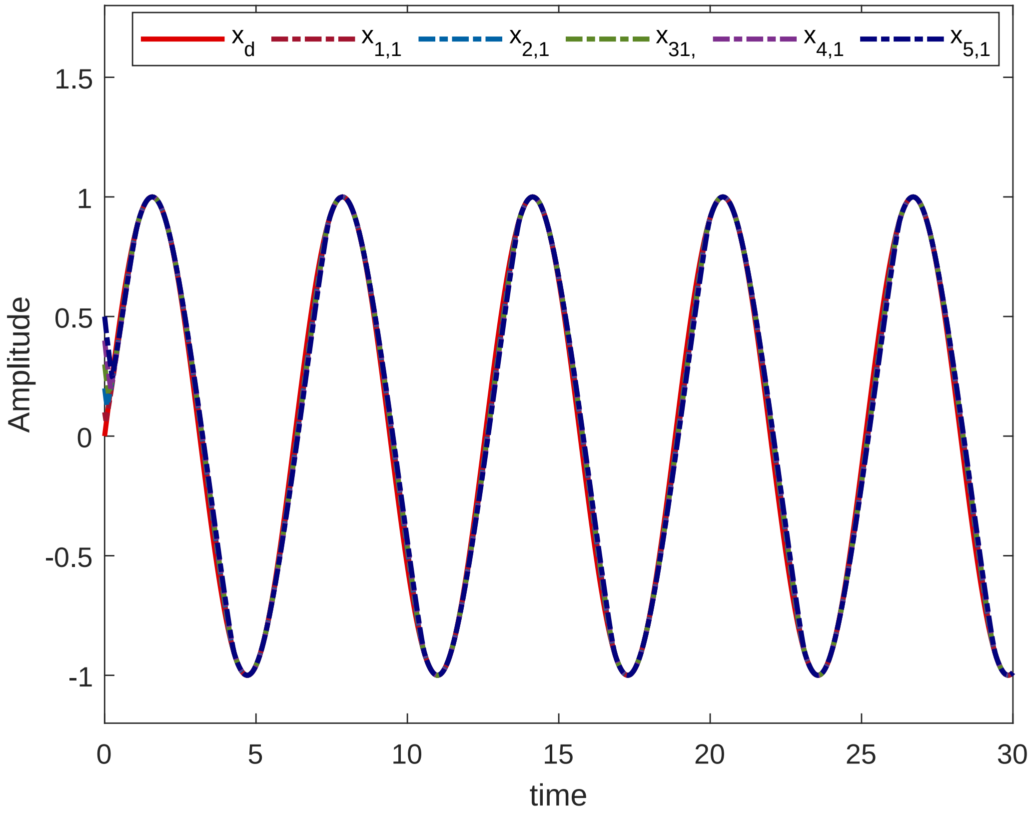

Figure 17. The trajectories of x_d and $x_{i,1}(i = 1, \cdots, 5)$ based on the finite-time distributed algorithm.

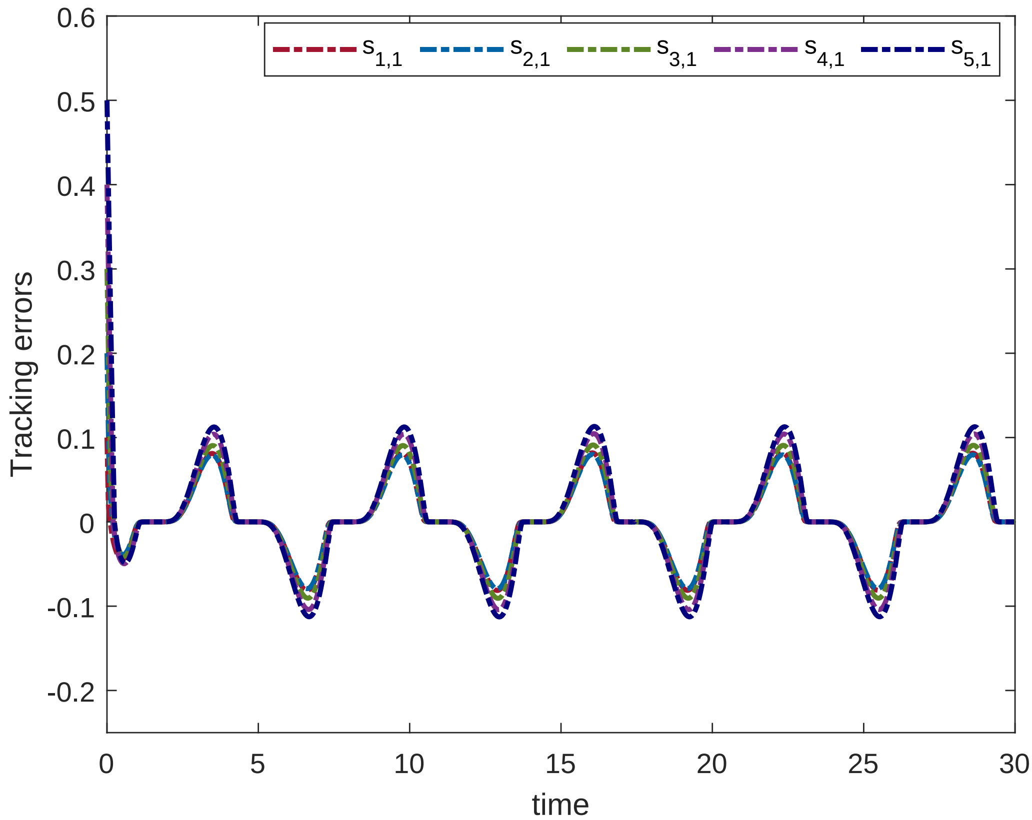

Figure 18. The trajectories of error $s_{i,1}(i = 1, \cdots, 5)$ based on the finite-time distributed algorithm.

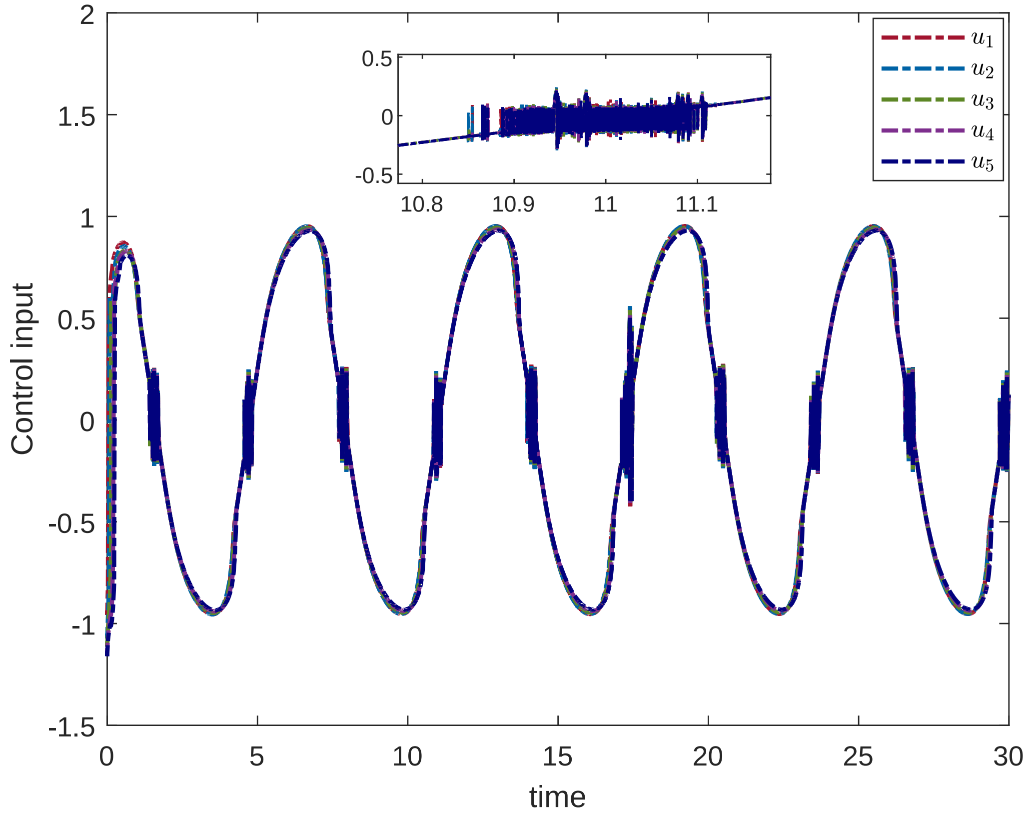

Figure 19. The trajectories of control input u_i based on the finite-time distributed algorithm.

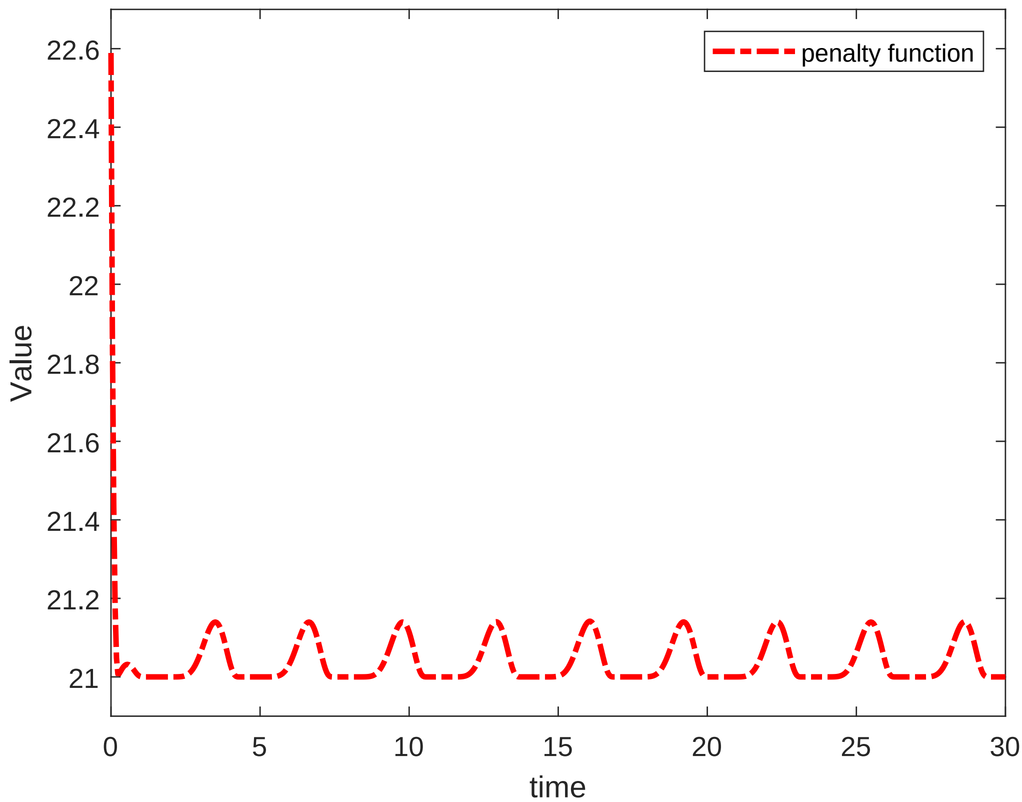

Figure 20. The value of penalty function based on the finite-time distributed algorithm.

From Figures 13 and 17, we found that both the methods proposed in this paper and in [16] achieve the goal of tracking the reference signal x_d. In Figures 14 and 18, the tracking error between two algorithms is very close, in the range of -0.1 to 0.1. In Figure 15, it is clearly observed that based on the algorithm proposed in this paper, the control input is smooth in the whole process of simulation. However, the control input shown in Figure 19 is found to be not smooth at some times, from which we conclude that the method proposed in [16] is not extraordinarily suitable for the fractional-order distributed optimization consensus problem. Based on the simulation results, it is demonstrated that the algorithm designed in this paper makes sure that all agents can be synchronized to the optimal solution x^*, while rejecting local disturbance signals $g_{i,1}(t)$.

5. Conclusions

This article has investigated the distributed optimization problem of FOMASs with nonlinear uncertain dynamics. Each agent is described by the fractional-order nonlinear system containing unmeasured states, unknown nonlinear functions, and is constrained by a local objective function denoted by a quadratic polynomial function. To estimate the unmeasured states, we construct the NN state observer, and exploit RBFNNs to approximate unknown nonlinear functions, respectively. The distributed optimization problem of FOMASs is transformed into an optimization problem with equality constraints, and a corresponding penalty function is constructed. Example and simulation results demonstrate that all the agents' outputs are steered to the optimal solution of the global objective function based on the observer-based adaptive NNs backstepping DSC algorithm proposed in this paper. Compared with the traditional distributed algorithm developed for the integer-order MASs, our protocol is feasibility and effectiveness due to its smoother inputs. Future research will focus on the adaptive distributed optimized NNs control algorithm design for the distributed optimization containment problem of FOMASs on the basis of this paper's results.

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