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An Investigation on the Optimal Control for Hilfer Fractional Neutral Stochastic Integrodifferential Systems with Infinite Delay

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Abstract: The main concern of this manuscript is to study the optimal control problem for Hilfer fractional neutral stochastic integrodifferential systems with infinite delay. Initially, we establish the existence of mild solutions for the Hilfer fractional stochastic integrodifferential system with infinite delay via applying fractional calculus, semigroups, stochastic analysis techniques, and the Banach fixed point theorem. In addition, we establish the existence of mild solutions of the Hilfer fractional neutral stochastic delay integrodifferential system. Further, we investigate the existence of optimal pairs for the Hilfer fractional neutral stochastic delay integrodifferential systems. We provide an illustration to clarify our results.

Keywords: Hilfer fractional derivative; stochastic differential equation; optimal control; mild solutions; neutral integrodifferential equations

MSC: 34A08; 58C30; 60H10; 93E20

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1. Introduction

In mathematics, the study of fractional calculus and fractional differential equations has received more attention in the last two decades. The memory and heredity properties of several essential materials and processes can be described using differential equations of arbitrary order. Several physical problems that cannot be solved by differential equations of integer order are solved by differential equations of fractional order. Such problems have been successfully modeled in various areas of science and engineering, such as biomechanics, electrochemistry, electromagnetic processes, electrical circuits, fluid mechanics, and viscoelasticity. For more details, we refer interested readers to the monographs [1–3] and the references therein. Many articles have been devoted to the existence of solutions for fractional differential equations, for instance [4–6]. Recently, the authors of [7] investigated the existence of Atangana–Baleanu semilinear fractional integrodifferential equations with noninstantaneous impulses.

On the other hand, along with Riemann–Liouville and Caputo fractional derivatives, Hilfer [8] pioneered the Hilfer fractional derivative, which is a generalization of the Riemann–Liouville derivative. The Hilfer fractional derivative is used, for example, in the theoretical simulation of dielectric relaxation in glass-forming materials. Many researchers have focused on these Hilfer fractional differential equations. For more information on Hilfer fractional differential equations and their application, readers may refer to [9–13]. As deterministic models often fluctuate due to noise, naturally such models must be extended to take into account stochastic models, where the corresponding parameters are accounted for as suitable Brownian motion and stochastic processes. Instead of deterministic equations, stochastic differential equations explain the modeling of the majority of issues in real-world contexts. Moreover, using different fixed point theorems with stochastic analysis theory, fractional calculus, operator semigroup theory, and cosine families, authors have studied the approximate controllability of stochastic systems [14–19]. For more details, we refer readers to the monograph [20] and the references therein. Stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both the physical and social sciences [21,22].

The optimal control problem plays a significant role in the design and analysis of control systems. It has several applications in diverse fields, for example, robotics, the control of chemical processes, power plants, and space technology. In [23], the authors studied optimal control and time-optimal control problems for a class of semilinear evolution systems with infinite delay. The existence and optimal controls for fractional stochastic evolution equations of the Sobolev type have been studied in [24] using fractional resolvent operators. The problems of optimal control have been investigated in [25] through sectorial operators, fractional calculus, the fixed point technique, and the Wiener process for the stochastic fractional in infinite dimensions with non-instantaneous impulse. For more details about optimal controls, we refer readers to [26–29]. In addition, integrodifferential equations are used in a variety of scientific fields where an effect or delay must be considered, including biology, control theory, and medicine. By means of noncompact measures and Mönch's fixed point approaches, in [30] the authors verified the existence of the mild solution of a Hilfer fractional integrodifferential system. The authors of [31] evaluated the existence of mild solutions for the fractional integrodifferential systems of mixed type through a family of solution operators and the contraction mapping principle. Due to wide application of integrodifferential systems, fractional integrodifferential systems have been studied by many scholars and have been reported in the literature [32–35].

Recently, the author of [36] investigated fractional optimal control of a semilinear system with fixed delay in a reflexive Banach space. Furthermore, the authors of [37] evaluated the solvability and optimal controls of a class of fractional integrodifferential evolution systems with infinite delay. By utilizing analytic resolvent operators, the solvability and optimal controls for impulsive fractional stochastic integrodifferential equations have been investigated in [38]. Moreover, in [39,40], the authors examined the outcomes for approximate controllability with infinite delay of order $r \in (1,2)$ and verified the existence of an optimal control for the Lagrange problem. However, there have only been a few studies on the existence, stability, and optimal control of Hilfer fractional stochastic differential systems. It is therefore essential to extend the concept of optimal control to such systems.

In particular, in [41,42], the authors investigated the existence of Hilfer fractional stochastic differential systems both of the Sobolev type and not by referring to fractional calculus, Hölder inequality, stochastic analysis, and fixed point theorems. In addition, they discussed the existence of optimal pairs for the corresponding Lagrange control systems. To the best of our knowledge, there are no results in the literature on the optimal control for Hilfer fractional stochastic integrodifferential systems with infinite delay and neutral systems using Banach fixed point theorem.

Motivated by this consideration, in this paper we study the optimal control of the following Hilfer fractional stochastic integrodifferential system with infinite delay:

$$\begin{cases} D_{0+}^{\delta,r} y(\mathfrak{z}) = Ay(\mathfrak{z}) + \mathbb{B}(\mathfrak{z})z(\mathfrak{z}) + f\left(\mathfrak{z}, y_{\mathfrak{z}}, \int_0^{\mathfrak{z}} g(\mathfrak{z},\varkappa,y_{\varkappa})d\varkappa\right) \\ \qquad\qquad +h\left(\mathfrak{z}, y_{\mathfrak{z}}, \int_0^{\mathfrak{z}} \widetilde{g}(\mathfrak{z},\varkappa,y_{\varkappa})d\varkappa\right)\frac{dW(\mathfrak{z})}{d\mathfrak{z}}, \mathfrak{z} \in \mathcal{E}' = (0,\vartheta], \\ I_{0+}^{(1-\delta)(1-r)} y(\mathfrak{z}) = \zeta(\mathfrak{z}) \in L^2(\Omega, \mathscr{G}_J),\ \mathfrak{z} \in (-\infty, 0], \end{cases} \tag{1}$$

where $D_{0+}^{\delta,r}$ is the Hilfer fractional derivative of type $\delta \in [0,1]$ and order $r \in (0,1)$ and $A : \mathscr{D}(A) \subset \mathcal{Y} \to \mathcal{Y}$ stands for an infinitesimal generator of a strongly continuous semigroup $\{\mathcal{G}(\mathfrak{z})\}_{\mathfrak{z}\geq 0}$ on a separable Hilbert space $\mathcal{Y}$ with $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ and norm $\|\cdot\|_{\mathcal{Y}}$. Let $\mathcal{E} = [0,\vartheta]$; then, control function z receives values from another separable reflexive Hilbert space $\mathcal{K}$. Here, $\mathbb{B} : \mathcal{K} \to \mathcal{Y}$ is the bounded linear operator, $f : \mathcal{E} \times \mathscr{G}_J \times \mathcal{Y} \to \mathcal{Y}$, $h : \mathcal{E} \times \mathscr{G}_J \times \mathcal{Y} \to L_2^0$, and $g, \widetilde{g} : \mathcal{E} \times \mathcal{E} \times \mathscr{G}_J \to \mathcal{Y}$ are appropriate functions. Let $\mathcal{Z}$ be another real separable

Hilbert space with $\langle\cdot,\cdot\rangle_{\mathcal{Z}}$ and the norm $\|\cdot\|_{\mathcal{Z}}$. Assume that $\{W(\mathfrak{z}),\ \mathfrak{z}\geq 0\}$ is a $\mathcal{Z}$-valued Brownian motion or Wiener process with a finite-trace nuclear covariance operator $\mathcal{Q}\geq 0$. The element $y_{\mathfrak{z}}:(-\infty,0]\to\mathcal{Y}$ is described by $y_{\mathfrak{z}}(\varkappa)=y(\mathfrak{z}+\varkappa)$ and belongs to the abstract phase space $\mathscr{G}_{\jmath}$. The initial condition $\zeta=\{\zeta(\mathfrak{z}):\mathfrak{z}\in(-\infty,0]\}$ is an $\mathfrak{J}_0$-measurable and $\mathscr{G}_{\jmath}$-valued random variable independent of the Wiener process $\{W(\mathfrak{z})\}$ with a finite second moment.

The contents of the rest of this manuscript are as follows. In Section 2, we present a few necessary preliminaries related to our study. In Section 3, we discuss the existence and uniqueness results of mild solutions for system (1). In Section 4, we prove the main results concerned with the existence results of mild solutions for the neutral system (7) using Banach fixed point theorem. The existence of the optimal control for the corresponding Lagrange problem is examined in Section 5. The last section shows the applicability of our obtained theory.

2. Preliminaries

Let $(\Omega,\mathfrak{J},\widehat{\mathbb{P}})$ be a complete probability space and let the normal filtration be $\{\mathfrak{J}_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathcal{E}}$, which is right continuous, and let $\{\mathfrak{J}_0\}$ contain all $\widehat{\mathbb{P}}$-null sets. Assume that $\mathcal{Y},\mathcal{Z}$ are separable Hilbert spaces and W is a $\mathcal{Q}$-Wiener process on $(\Omega,\mathfrak{J}_{\vartheta},\widehat{\mathbb{P}})$ with the covariance operator $\mathcal{Q}$ such that $Tr\mathcal{Q}<\infty$. Consider the case where there exists a basis $\{\breve{\xi}_{\Bbbk}\}_{\Bbbk\geq 1}$ in $\mathcal{Z}$, with complete orthonormal and a bounded sequence of non-negative real number $\{\hbar_{\Bbbk}\}\geq 0$ such that $\mathcal{Q}\breve{\xi}_{\Bbbk}=\hbar_{\Bbbk}\breve{\xi}_{\Bbbk}$, $\quad \Bbbk=1,2,\cdots$, and a sequence $\{w_{\Bbbk}\}_{\Bbbk\geq 1}$ of independent Brownian motions such that $\langle W(\mathfrak{z}),\breve{\xi}\rangle_{\mathcal{Z}}=\sum_{\Bbbk=1}^{\infty}\sqrt{\hbar_{\Bbbk}}\langle\breve{\xi}_{\Bbbk},\breve{\xi}\rangle w_{\Bbbk}(\mathfrak{z})$, $\breve{\xi}\in\mathcal{Z}$, $\mathfrak{z}\in\mathcal{E}$.

Consider $L_2^0=L_2(\mathcal{Q}^{1/2}\mathcal{Z};\mathcal{Y})$ to be the space of all Hilbert–Schmidt operators from $\mathcal{Q}^{1/2}\mathcal{Z}$ to $\mathcal{Y}$ with $\|\varphi\|^2_{\mathcal{Q}}=Tr(\varphi\mathcal{Q}\varphi^*)$, where the adjoint of the operator φ is φ^*. The set of all strongly measurable square integrable $\mathcal{Y}$-valued random variables is represented by $L_2(\Omega,\mathfrak{J},\widehat{\mathbb{P}};\mathcal{Y})\equiv L_2(\Omega;\mathcal{Y})$, which is a Banach space equipped with the norm $\|y(\cdot)\|_{L_2}=(E\|y(\cdot;x_0)\|^2_{\mathcal{Y}})^{1/2}$, where $E\|\check{h}_0\|=\int_{\Omega}\check{h}_0(x_0)d\widehat{\mathbb{P}}$ defines the expectation E. Let $C\big(\mathcal{E},L_2(\Omega;\mathcal{Y})\big)$ demonstrate the Banach space of all the continuous functions from $\mathcal{E}$ into $L_2(\Omega;\mathcal{Y})$ that fulfill $\sup_{\mathfrak{z}\in\mathcal{E}}E\|y(\mathfrak{z})\|^2<\infty$, and let $L_2^0(\Omega,\mathcal{Y})$ denote the family of all $\mathfrak{J}_0$-measurable $\mathcal{Y}$-valued random variables.

Definition 1 ([43])**.** *The fractional integral of order r for $h:[0,\infty)\to\mathbb{R}$ with the lower limit zero is represented by*

$$I^r_{0^+}h(\mathfrak{z})=\frac{1}{\Gamma(r)}\int_0^{\mathfrak{z}}\frac{h(\varkappa)}{(\mathfrak{z}-\varkappa)^{1-r}}d\varkappa,\quad \mathfrak{z}>0,\quad r\in\mathbb{R}^+,$$

provided that the RHS is point-wise determined on $[0,\infty)$.

Definition 2. *Riemann-Liouville's derivative of order r for $h:[0,\infty)\to\mathbb{R}$ with the lower limit zero is represented by*

$${}^LD^r_{0^+}h(\mathfrak{z})=\frac{1}{\Gamma(m-r)}\frac{d^m}{d\mathfrak{z}^m}\int_0^{\mathfrak{z}}\frac{h(\varkappa)}{(\mathfrak{z}-\varkappa)^{r+1-m}}d\varkappa,\quad \mathfrak{z}>0,\ m-1<r<m,\ r\in\mathbb{R}^+.$$

Definition 3 ([43])**.** *Caputo's derivative of order r for $h:[0,\infty)\to\mathbb{R}$ with the lower limit zero is represented by*

$${}^CD^r_{0^+}h(\mathfrak{z})=\frac{1}{\Gamma(m-r)}\int_0^{\mathfrak{z}}\frac{h^{(m)}(\varkappa)}{(\mathfrak{z}-\varkappa)^{r+1-m}}d\varkappa,\quad \mathfrak{z}>0,\ m-1<r<m,\ r\in\mathbb{R}^+.$$

Definition 4 ([43]). *The Hilfer Fractional Derivative of type $0 \leq \delta \leq 1$ and order $0 < r < 1$ with the lower limit zero is represented by*

$$D_{0^+}^{\delta,r} h(\mathfrak{z}) = I_{0^+}^{\delta(1-r)} \frac{d}{d\mathfrak{z}} I_{0^+}^{(1-\delta)(1-r)} h(\mathfrak{z}),\ \mathfrak{z} > 0.$$

Remark 1 ([8]). *The Hilfer fractional derivative is related to the classical Riemann-Liouville fractional derivative and the classical Caputo fractional derivative as follows:*

$$D_{0^+}^{\delta,r} h(\mathfrak{z}) = \begin{cases} \frac{d}{d\mathfrak{z}} I_{0^+}^{1-r} h(\mathfrak{z}) = {}^{L}D^{r}{}_{0^+} h(\mathfrak{z}),\ \delta = 0, 0 < r < 1; \\ I_{0^+}^{1-r} \frac{d}{d\mathfrak{z}} h(\mathfrak{z}) = {}^{C}D^{r}{}_{0^+} h(\mathfrak{z}),\ \delta = 1, 0 < r < 1. \end{cases}$$

$\mathcal{W}_r(\xi)$ is a Wright function that is described as follows:

$$\mathcal{W}_r(\mathfrak{z}) = \sum_{m=1}^{\infty} \frac{(-\xi)^{m-1}}{(m-1)!\Gamma(1-rm)},\ 0 < r < 1,\ \xi \in \mathbb{C},$$

and fulfills

$$\int_0^\infty \xi^v \mathcal{W}_r(\xi) d\xi = \frac{\Gamma(1+v)}{\Gamma(1+rv)}; \quad \int_0^\infty \mathcal{W}_r(\xi) d\xi = 1, \quad \xi \geq 0.$$

For $y \in \mathcal{Y}$, we define $\{\mathcal{S}_r(\mathfrak{z}) : \mathfrak{z} > 0\}$ and $\{\mathcal{G}_{\delta,r}(\mathfrak{z}) : \mathfrak{z} > 0\}$ by

$$\mathcal{S}_r(\mathfrak{z}) = \mathfrak{z}^{r-1} \mathcal{N}_r(\mathfrak{z}); \quad \mathcal{G}_{\delta,r}(\mathfrak{z}) = I_{0^+}^{\delta(1-r)} \mathcal{S}_r(\mathfrak{z}); \quad \mathcal{N}_r(\mathfrak{z}) = \int_0^\infty r\xi \mathcal{W}_r(\xi) \mathcal{G}(\mathfrak{z}^r \xi) d\xi.$$

Now, we present the abstract phase space $\mathscr{G}_\jmath$, which has previously been used in [44,45]. Let $\jmath : (-\infty, 0] \to (0, +\infty)$ be continuous with $\ell = \int_{-\infty}^0 \jmath(\mathfrak{z}) d\mathfrak{z} < +\infty$. The abstract phase space $\mathscr{G}_\jmath$ is defined as follows:

$$\mathscr{G}_\jmath = \begin{cases} \zeta : (-\infty, 0] \to \mathcal{Y},\ \forall\, c > 0,\ \left(E\|\zeta(\eta)\|^2\right)^{\frac{1}{2}} \text{is a bounded and measurable} \\ \text{function on } [-c, 0] \text{ with } \int_{-\infty}^0 \jmath(\varkappa) \sup_{\varkappa \leq \eta \leq 0} \left(E\|\zeta(\eta)\|^2\right)^{\frac{1}{2}} d\varkappa < +\infty, \end{cases}$$

and

$$\|\zeta\|_{\mathscr{G}_\jmath} = \int_{-\infty}^0 \jmath(\varkappa) \sup_{\varkappa \leq \eta \leq 0} \left(E\|\zeta(\eta)\|^2\right)^{\frac{1}{2}} d\varkappa, \text{ for all } \zeta \in \mathscr{G}_\jmath,$$

therefore, $(\mathscr{G}_\jmath, \|\cdot\|_{\mathscr{G}_\jmath})$ is a Banach space. The space of all continuous $\mathcal{Y}$-valued stochastic processes $\{\xi(\mathfrak{z}) : \mathfrak{z} \in (-\infty, \vartheta]\}$ and is considered as $C((-\infty, \vartheta], \mathcal{Y})$, and

$$\mathscr{G}'_\jmath = \{y : y \in C((-\infty, \vartheta], \mathcal{Y}),\ y_0 = \zeta \in \mathscr{G}_\jmath\}.$$

Let us take $\|\cdot\|_\vartheta$ be a seminorm in $\mathscr{G}'_\jmath$ defined as

$$\|y\|_\vartheta = \|\zeta\|_{\mathscr{G}_\jmath} + \sup_{\varkappa \in [0,\vartheta]} \left(E\|y(\varkappa)\|^2\right)^{\frac{1}{2}},\ y \in \mathscr{G}'_\jmath.$$

Lemma 1 ([44]). *If $y \in \mathscr{G}'_\jmath$, then for $\mathfrak{z} \in \mathcal{E}, y_\mathfrak{z} \in \mathscr{G}_\jmath$. Furthermore,*

$$\ell\left(E\|y(\mathfrak{z})\|^2\right)^{\frac{1}{2}} \leq \|y_\mathfrak{z}\|_{\mathscr{G}_\jmath} \leq \|y_0\|_{\mathscr{G}_\jmath} + \ell \sup_{\varkappa \in [0,\mathfrak{z}]} \left(E\|y(\varkappa)\|^2\right)^{\frac{1}{2}},$$

where $\ell = \int_{-\infty}^0 \jmath(\varkappa) d\varkappa < +\infty$.

Lemma 2. *Provided that $\|\mathcal{P}\|$ is Lebesgue integral, then a measurable function $\mathcal{P} : \mathcal{E} \to \mathcal{Y}$ is a Bochner integrable.*

Definition 5 ([10])**.** *An $\mathfrak{J}_\mathfrak{z}$-adapted stochastic process $y : (-\infty, \vartheta] \to \mathcal{Y}$ is known as a mild solution of (1) provided that $y_0 = \zeta \in L^2(\Omega, \mathscr{G}_J)$ on $(-\infty, 0]$ satisfying $y_0 \in L_2^0(\Omega, \mathcal{Y})$ and the following integral equation*

$$\begin{aligned} y(\mathfrak{z}) = \mathcal{G}_{\delta,r}(\mathfrak{z})\zeta(0) + \int_0^\mathfrak{z} \mathcal{S}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ + \int_0^\mathfrak{z} \mathcal{S}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, y_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, y_\varrho)d\varrho\Big)d\varkappa \\ + \int_0^\mathfrak{z} \mathcal{S}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, y_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, y_\varrho)d\varrho\Big)dW(\varkappa),\ \mathfrak{z} \in \mathcal{E}', \end{aligned}$$

is fulfilled. Because $\mathcal{S}_r(\mathfrak{z}) = \mathfrak{z}^{r-1}\mathcal{N}_r(\mathfrak{z})$ is identical with

$$\begin{aligned} y(\mathfrak{z}) =& \mathcal{G}_{\delta,r}(\mathfrak{z})\zeta(0) + \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ &+ \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, y_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, y_\varrho)d\varrho\Big)d\varkappa \\ &+ \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, y_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, y_\varrho)d\varrho\Big)dW(\varkappa),\ \mathfrak{z} \in \mathcal{E}', \end{aligned} \tag{2}$$

we introduce the following assumption.

(**H0**)*: In the uniform operator topology, $\mathcal{G}(\mathfrak{z})$ is continuous for $\mathfrak{z} > 0$ and $\{\mathcal{G}(\mathfrak{z})\}_{\mathfrak{z}\geq 0}$ is uniformly bounded, i.e., there exists $\mathbb{K} > 1$ such that $\sup_{\mathfrak{z}\in[0,\infty)} \|\mathcal{G}(\mathfrak{z})\| < \mathbb{K}$.*

Lemma 3 ([8,46])**.** *Suppose that* (**H0**) *is fulfilled; then, the characteristics are as follows:*

- *$\{\mathcal{G}_{\delta,r}(\mathfrak{z})\}$, $\{\mathcal{S}_r(\mathfrak{z})\}$ and $\{\mathcal{N}_r(\mathfrak{z})\}$ are strongly continuous for $\mathfrak{z} > 0$.*
- *for any fixed $\mathfrak{z} > 0$, the linear bounded operators $\{\mathcal{G}_{\delta,r}(\mathfrak{z})\}$, $\{\mathcal{S}_r(\mathfrak{z})\}$, and $\mathcal{N}_r(\mathfrak{z})$ are defined as*

$$\begin{aligned} &\|\mathcal{S}_r(\mathfrak{z})y\| \leq \frac{\mathbb{K}\mathfrak{z}^{r-1}}{\Gamma(r)}\|y\|, \quad \|\mathcal{N}_r(\mathfrak{z})y\| \leq \frac{\mathbb{K}}{\Gamma(r)}\|y\| \text{ and} \\ &\|\mathcal{G}_{\delta,r}(\mathfrak{z})y\| \leq \frac{\mathbb{K}\mathfrak{z}^{\beta-1}}{\Gamma(\beta)}\|y\|, \quad \beta = \delta + r - \delta r. \end{aligned}$$

3. Existence of Mild Solution

Useful assumptions are made to investigate the existence of mild solutions to Equation (1) as follows.

(**H1**)$:$ $f : \mathcal{E} \times \mathscr{G}_J \times \mathcal{Y} \to \mathcal{Y}$ is a continuous function and there exist positive constants $\mathbb{K}_f$, $\overline{\mathbb{K}}_f$ such that for $\mathfrak{z} \in \mathcal{E}$, $\mathfrak{u}_1, \bar{\mathfrak{u}}_1 \in \mathscr{G}_J$, $\mathfrak{u}_2, \bar{\mathfrak{u}}_2 \in \mathcal{Y}$

$$\begin{aligned} &E\|f(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) - f(\mathfrak{z}, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2)\|^2 \leq \mathbb{K}_f\big(\|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1\|^2_{\mathscr{G}_J} + E\|\mathfrak{u}_2 - \bar{\mathfrak{u}}_2\|^2\big), \\ &E\|f(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\|^2 \leq \overline{\mathbb{K}}_f\big(1 + \|\mathfrak{u}_1\|^2_{\mathscr{G}_J} + E\|\mathfrak{u}_2\|^2\big). \end{aligned}$$

(**H2**)$:$ $h : \mathcal{E} \times \mathscr{G}_J \times \mathcal{Y} \to L_2^0$ is a continuous function and there exist positive constants $\mathbb{K}_h$, $\overline{\mathbb{K}}_h$ such that for $\mathfrak{z} \in \mathcal{E}$, $\mathfrak{u}_1, \bar{\mathfrak{u}}_1 \in \mathscr{G}_J$, $\mathfrak{u}_2, \bar{\mathfrak{u}}_2 \in \mathcal{Y}$

$$\begin{aligned} &E\|h(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) - h(\mathfrak{z}, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2)\|^2 \leq \mathbb{K}_h\big(\|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1\|^2_{\mathscr{G}_J} + E\|\mathfrak{u}_2 - \bar{\mathfrak{u}}_2\|^2\big), \\ &E\|h(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\|^2 \leq \overline{\mathbb{K}}_h\big(1 + \|\mathfrak{u}_1\|^2_{\mathscr{G}_J} + E\|\mathfrak{u}_2\|^2\big). \end{aligned}$$

(**H3**)$:$ For each $(\mathfrak{z}, \varkappa) \in \mathcal{E} \times \mathcal{E}$, the functions $g, \widetilde{g} : \mathcal{E} \times \mathcal{E} \times \mathscr{G}_J \to \mathcal{Y}$ are continuous and there exist positive constants $m_1, m_2, \overline{m}_1, \overline{m}_2$ such that for all $\mathfrak{u}, \bar{\mathfrak{u}} \in \mathscr{G}_J$

$$
\begin{aligned}
&E\|g(\mathfrak{z},\varkappa,\mathfrak{u})-g(\mathfrak{z},\varkappa,\bar{\mathfrak{u}})\|^2\leq m_1\|\mathfrak{u}-\bar{\mathfrak{u}}\|^2_{\mathscr{G}_J},\\
&E\|\widetilde{g}(\mathfrak{z},\varkappa,\mathfrak{u})-\widetilde{g}(\mathfrak{z},\varkappa,\bar{\mathfrak{u}})\|^2\leq m_2\|\mathfrak{u}-\bar{\mathfrak{u}}\|^2_{\mathscr{G}_J},\\
&E\|g(\mathfrak{z},\varkappa,\mathfrak{u})\|^2\leq \overline{m}_1\big(1+\|\mathfrak{u}\|^2_{\mathscr{G}_J}\big),\\
&E\|\widetilde{g}(\mathfrak{z},\varkappa,\mathfrak{u})\|^2\leq \overline{m}_2\big(1+\|\mathfrak{u}\|^2_{\mathscr{G}_J}\big).
\end{aligned}
$$

(**H4**): Let $z\in\mathcal{K}$ be the control function and the operator $\mathbb{B}(\cdot)\in L_\infty(\mathcal{E},L(\mathcal{K},\mathcal{Y}))$, $\|\mathbb{B}\|_\infty$ denote the norm operator $\mathbb{B}$.

(**H5**): Multivalued maps $\mathbb{A}:\mathcal{E}\to\mathcal{M}(\mathcal{K})$ (where $\mathcal{M}(\mathcal{K})$ is a class of nonempty closed, convex subsets of $\mathcal{K}$) are measurable and $\mathbb{A}(\cdot)\subseteq\Theta$, where Θ is a bounded set of $\mathcal{K}$.

Admissible set $\mathbb{A}_{ad}$ is defined as, the set of all $v(\cdot):\mathcal{E}\times\Omega\to\mathcal{Y}$ such that v is a $\mathfrak{I}_{\mathfrak{z}}$-adapted stochastic process and $E\int_0^\vartheta\|v(\mathfrak{z})\|^2d\mathfrak{z}<\infty$. Clearly, $\mathbb{A}_{ad}\neq\emptyset$ and $\mathbb{A}_{ad}\subset L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})(1<\mathfrak{q}<+\infty)$ is bounded, closed, and convex. It is evident that $\mathbb{B}z\in L^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})$ for all $z\in\mathbb{A}_{ad}$.

Theorem 1. *Under the assumptions* (**H0**)–(**H5**), *Equation* (1) *has a unique mild solution provided that*

$$
12\ell^2\left[\frac{\vartheta^{2-2\beta+2r}\mathbb{K}^2\overline{\mathbb{K}}_f}{\Gamma^2(r+1)}\Big(1+\overline{m}_1\vartheta^2\Big)+\frac{\vartheta^{1-2\beta+2r}\mathbb{K}^2Tr(\mathcal{Q})\overline{\mathbb{K}}_h}{(2r-1)\Gamma^2(r)}\Big(1+\overline{m}_2\vartheta^2\Big)\right]<1.
$$

Proof. Define an operator $\Psi:\mathscr{G}'_J\to\mathscr{G}'_J$ as

$$
\Psi y(\mathfrak{z})=\begin{cases}\zeta(\mathfrak{z}),\quad \mathfrak{z}\in(-\infty,0],\\
+\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa\\
+\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)f\Big(\varkappa,y_\varkappa,\int_0^\varkappa g(\varkappa,\varrho,y_\varrho)d\varrho\Big)d\varkappa\\
+\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)h\Big(\varkappa,y_\varkappa,\int_0^\varkappa \widetilde{g}(\varkappa,\varrho,y_\varrho)d\varrho\Big)dW(\varkappa),\ \text{for } \mathfrak{z}>0.\end{cases}\tag{3}
$$

For $\zeta\in\mathscr{G}_J$, we define $\bar{\zeta}$ as follows:

$$
\bar{\zeta}(\mathfrak{z})=\begin{cases}\zeta(\mathfrak{z}),\quad \mathfrak{z}\in(-\infty,0],\\ \mathcal{G}_{\delta,r}(\mathfrak{z})\zeta(0),\quad \mathfrak{z}\in\mathcal{E},\end{cases}
$$

then, $\bar{\zeta}\in\mathscr{G}'_J$. Let $y(\mathfrak{z})=w(\mathfrak{z})+\bar{\zeta}(\mathfrak{z}),-\infty<\mathfrak{z}\leq\vartheta$. Clearly, y fulfills (3) if and only if w fulfills $w_0=0$ and

$$
\begin{aligned}
w(\mathfrak{z})=&\int_0^{\mathfrak{z}}(\mathfrak{z}-\varrho)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa\\
&+\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)f\Big(\varkappa,w_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa g(\varkappa,\varrho,w_\varrho+\bar{\zeta}_\varrho)d\varrho\Big)d\varkappa\\
&+\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)h\Big(\varkappa,w_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa \widetilde{g}(\varkappa,\varrho,w_\varrho+\bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa).
\end{aligned}
$$

Let us consider $\mathscr{G}''_J=\{w\in\mathscr{G}'_J:w_0=0\in\mathscr{G}_J\}$. For any $w\in\mathscr{G}''_J$,

$$
\begin{aligned}
\|w\|_\vartheta&=\|w_0\|_{\mathscr{G}_J}+\sup_{0\leq\varkappa\leq\vartheta}\big(E\|w(\varkappa)\|^2\big)^{\frac{1}{2}}\\
&=\sup_{0\leq\varkappa\leq\vartheta}\big(E\|w(\varkappa)\|^2\big)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, $(\mathscr{G}_J'', \|\cdot\|_\vartheta)$ is a Banach space. For some $\mathfrak{p} > 0$, we consider $\mathcal{B}_\mathfrak{p} = \{w \in \mathscr{G}_J'' : \|w\|_\vartheta^2 \leq \mathfrak{p}\}$; then, for each $\mathfrak{p}$, $\mathcal{B}_\mathfrak{p} \subseteq \mathscr{G}_J''$ is uniformly bounded, $w \in \mathcal{B}_\mathfrak{p}$, and referring to Lemma 1,

$$
\begin{aligned}
\|w_\mathfrak{z} + \bar{\zeta}_\mathfrak{z}\|_{\mathscr{G}_J}^2 &\leq 2\big(\|w_\mathfrak{z}\|_{\mathscr{G}_J}^2 + \|\bar{\zeta}_\mathfrak{z}\|_{\mathscr{G}_J}^2\big) \\
&\leq 4\Big(\|w_0\|_{\mathscr{G}_J}^2 + \ell^2 \sup_{\varkappa\in[0,\mathfrak{z}]} \big(E\|w(\varkappa)\|^2\big) + \|\bar{\zeta}_0\|_{\mathscr{G}_J}^2 + \ell^2 \sup_{\varkappa\in[0,\mathfrak{z}]} \big(E\|\bar{\zeta}(\varkappa)\|^2\big)\Big) \\
&\leq 4\left(0 + \ell^2\|w\|_\vartheta^2 + \|\zeta\|_{\mathscr{G}_J}^2 + \ell^2 E\|\mathcal{G}_{\delta,r}(\varkappa)\zeta(0)\|^2\right) \\
&\leq 4\left(\ell^2\mathfrak{p} + \|\zeta\|_{\mathscr{G}_J}^2 + \ell^2\left(\frac{\mathbb{K}\varkappa^{\beta-1}}{\Gamma(\beta)}\right)^2 E\|\zeta(0)\|^2\right) \\
&\leq 4\ell^2\left(\mathfrak{p} + \left(\frac{\mathbb{K}\varkappa^{\beta-1}}{\Gamma(\beta)}\right)^2 E\|\zeta(0)\|^2\right) + 4\|\zeta\|_{\mathscr{G}_J}^2 \\
&\leq 4\ell^2(\mathfrak{p} + M_1 E\|\zeta(0)\|^2) + 4\|\zeta\|_{\mathscr{G}_J}^2 = \mathfrak{p}',
\end{aligned} \tag{4}
$$

where $M_1 = \left(\frac{\mathbb{K}\varkappa^{\beta-1}}{\Gamma(\beta)}\right)^2$.

We define $\widetilde{\Psi} : \mathscr{G}_J'' \to \mathscr{G}_J''$ as follows:

$$
\widetilde{\Psi}w(\mathfrak{z}) = \begin{cases} 0,\ \mathfrak{z} \in (-\infty, 0], \\ \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ + \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa,\varrho,w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)d\varkappa \\ + \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa,\varrho,w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa), \mathfrak{z} \in \mathcal{E}, \end{cases} \tag{5}
$$

which demonstrates that $\widetilde{\Psi}$ has a unique fixed point. For greater convenience, we divide the proof into two steps.

Step 1: We claim that there exists $\mathfrak{p} > 0$ such that $\widetilde{\Psi}(\mathcal{B}_\mathfrak{p}) \subset \mathcal{B}_\mathfrak{p}$. If this is not true, then for all $\mathfrak{p} > 0$ there exists a function $w^\mathfrak{p}(\cdot) \in \mathcal{B}_\mathfrak{p}$ and $\widetilde{\Psi}(w^\mathfrak{p}) \notin \mathcal{B}_\mathfrak{p}$, that is, $E\|(\widetilde{\Psi}w^\mathfrak{p})(\mathfrak{z})\|^2 > \mathfrak{p}$ for some $\mathfrak{z} \in \mathcal{E}$.

From Lemma 3, (**H1**)–(**H5**), and Hölder's inequality, we obtain

$$
\begin{aligned}
\mathfrak{p} &\leq \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\|(\widetilde{\Psi}w^\mathfrak{p})(\mathfrak{z})\|^2 \\
&\leq 3\Big\{ \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \Big\|^2 \\
&\quad + \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, w_\varkappa^\mathfrak{p} + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa,\varrho,w_\varrho^\mathfrak{p} + \bar{\zeta}_\varrho)d\varrho\Big)d\varkappa \Big\|^2 \\
&\quad + \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, w_\varkappa^\mathfrak{p} + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa,\varrho,w_\varrho^\mathfrak{p} + \bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa) \Big\|^2 \Big\} \\
&= 3\{\mathbb{S}_1 + \mathbb{S}_2 + \mathbb{S}_3\},
\end{aligned}
$$

where

$$\begin{aligned}
\mathbb{S}_1 =& \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa\Big\|^2 \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2\|\mathbb{B}\|_\infty^2}{\Gamma^2(r)} E\left[\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\|z(\varkappa)\|d\varkappa\right]^2 \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2\|\mathbb{B}\|_\infty^2}{\Gamma^2(r)} \left[\left(\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{\frac{\mathfrak{q}(r-1)}{\mathfrak{q}-1}}d\varkappa\right)^{\frac{\mathfrak{q}-1}{\mathfrak{q}}}\left(E\int_0^{\mathfrak{z}} \|z(\varkappa)\|^{\mathfrak{q}}d\varkappa\right)^{\frac{1}{\mathfrak{q}}}\right]^2 \\
\leq& \frac{\vartheta^{2-2\beta+2r-\frac{2}{\mathfrak{q}}}}{\Gamma^2(r)}\mathbb{K}^2\|\mathbb{B}\|_\infty^2\|z\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\left(\frac{\mathfrak{q}-1}{\mathfrak{q}r-1}\right)^{\frac{2(\mathfrak{q}-1)}{\mathfrak{q}}}, \\
\mathbb{S}_2 =& \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)f\Big(\varkappa, w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa, \int_0^{\varkappa} g(\varkappa,\varrho,w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho)d\varrho\Big)d\varkappa\Big\|^2 \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2}{\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}d\varkappa\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}E\|f\Big(\varkappa, w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa, \int_0^{\varkappa} g(\varkappa,\varrho,w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho)d\varrho\Big)\|^2d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta+r}\mathbb{K}^2}{r\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\overline{\mathbb{K}}_f\Big(1+\|w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa\|^2_{\mathscr{G}_\jmath}+E\|\int_0^{\varkappa} g(\varkappa,\varrho,w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho)d\varrho\|^2\Big)d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta+r}\mathbb{K}^2}{r\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\overline{\mathbb{K}}_f\Big(1+\|w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa\|^2_{\mathscr{G}_\jmath}+\overline{m}_1\vartheta^2(1+\|w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho\|^2_{\mathscr{G}_\jmath})\Big)d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta+r}\mathbb{K}^2}{r\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\overline{\mathbb{K}}_f\Big(1+\mathfrak{p}'+\overline{m}_1\vartheta^2(1+\mathfrak{p}')\Big)d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\mathfrak{p}'+\overline{m}_1\vartheta^2(1+\mathfrak{p}')\Big), \\
\mathbb{S}_3 =& \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)h\Big(\varkappa, w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa, \int_0^{\varkappa} \widetilde{g}(\varkappa,\varrho,w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa)\Big\|^2 \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2Tr(\mathcal{Q})}{\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}E\|h\Big(\varkappa, w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa, \int_0^{\varkappa} g(\varkappa,\varrho,w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho)d\varrho\Big)\|^2d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2Tr(\mathcal{Q})}{\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}\overline{\mathbb{K}}_h\Big(1+\|w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa\|^2_{\mathscr{G}_\jmath}+E\|\int_0^{\varkappa} \widetilde{g}(\varkappa,\varrho,w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho)d\varrho\|^2\Big)d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2Tr(\mathcal{Q})}{\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}\overline{\mathbb{K}}_h\Big(1+\|w_\varkappa^{\mathfrak{p}}+\bar{\zeta}_\varkappa\|^2_{\mathscr{G}_\jmath}+\overline{m}_2\vartheta^2\big(1+\|w_\varrho^{\mathfrak{p}}+\bar{\zeta}_\varrho\|^2_{\mathscr{G}_\jmath}\big)\Big)d\varkappa \\
\leq& \frac{\vartheta^{2-2\beta}\mathbb{K}^2Tr(\mathcal{Q})}{\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}\overline{\mathbb{K}}_h\Big(1+\mathfrak{p}'+\overline{m}_2\vartheta^2(1+\mathfrak{p}')\Big)d\varkappa \\
\leq& \frac{\vartheta^{1-2\beta+2r}\mathbb{K}^2Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\mathfrak{p}'+\overline{m}_2\vartheta^2(1+\mathfrak{p}')\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathfrak{p} \leq& \frac{3\vartheta^{2-2\beta+2r-\frac{2}{\mathfrak{q}}}}{\Gamma^2(r)}\mathbb{K}^2\|\mathbb{B}\|_\infty^2\|z\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\left(\frac{\mathfrak{q}-1}{\mathfrak{q}r-1}\right)^{\frac{2(\mathfrak{q}-1)}{\mathfrak{q}}} \\
&+\frac{3\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\mathfrak{p}'+\overline{m}_1\vartheta^2(1+\mathfrak{p}')\Big) \\
&+\frac{3\vartheta^{1-2\beta+2r}\mathbb{K}^2Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\mathfrak{p}'+\overline{m}_2\vartheta^2(1+\mathfrak{p}')\Big). \qquad (6)
\end{aligned}$$

Now, by dividing (6) by $\mathfrak{p}$ and taking $\mathfrak{p} \to \infty$, we obtain

$$12\ell^2\left[\frac{\vartheta^{2-2\beta+2r}\mathbb{K}^2\overline{\mathbb{K}}_f}{\Gamma^2(r+1)}\left(1+\overline{m}_1\vartheta^2\right)+\frac{\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})\overline{\mathbb{K}}_h}{(2r-1)\Gamma^2(r)}\left(1+\overline{m}_2\vartheta^2\right)\right] \geq 1,$$

which contradicts our assumption. Thus, for some $\mathfrak{p} > 0$, $\widetilde{\Psi}(\mathcal{B}_{\mathfrak{p}}) \subset \mathcal{B}_{\mathfrak{p}}$.

Step 2: $\widetilde{\Psi}$ is a contraction on $\mathcal{B}_{\mathfrak{p}}$.

We take $w, \widehat{w} \in \mathcal{B}_{\mathfrak{p}}$ to obtain

$$\begin{aligned}
&E\|\widetilde{\Psi}w(\mathfrak{z})-\widetilde{\Psi}\widehat{w}(\mathfrak{z})\|^2 \\
&\quad\leq 2\sup_{\mathfrak{z}\in\mathcal{E}}\mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[f\Big(\varkappa,w_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa g(\varkappa,\varrho,w_\varrho+\bar{\zeta}_\varrho)d\varrho\Big) \\
&\qquad-f\Big(\varkappa,\widehat{w}_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa g(\varkappa,\varrho,\widehat{w}_\varrho+\bar{\zeta}_\varrho)d\varrho\Big)\Big]d\varkappa\Big\|^2 \\
&\qquad+2\sup_{\mathfrak{z}\in\mathcal{E}}\mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[h\Big(\varkappa,w_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa \widetilde{g}(\varkappa,\varrho,w_\varrho+\bar{\zeta}_\varrho)d\varrho\Big) \\
&\qquad-h\Big(\varkappa,\widehat{w}_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa \widetilde{g}(\varkappa,\varrho,\widehat{w}_\varrho+\bar{\zeta}_\varrho)d\varrho\Big)\Big]dW(\varkappa)\Big\|^2 \\
&\quad\leq \frac{2\mathbb{K}^2\vartheta^{2-2\beta}}{\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}d\varkappa\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}E\Big\|f\Big(\varkappa,w_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa g(\varkappa,\varrho,w_\varrho+\bar{\zeta}_\varrho)d\varrho\Big) \\
&\qquad-f\Big(\varkappa,\widehat{w}_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa g(\varkappa,\varrho,\widehat{w}_\varrho+\bar{\zeta}_\varrho)d\varrho\Big)\Big\|^2 d\varkappa \\
&\qquad+\frac{2\mathbb{K}^2\vartheta^{2-2\beta}}{\Gamma^2(r)}Tr(\mathcal{Q})\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}E\Big\|h\Big(\varkappa,w_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa \widetilde{g}(\varkappa,\varrho,w_\varrho+\bar{\zeta}_\varrho)d\varrho\Big) \\
&\qquad-h\Big(\varkappa,\widehat{w}_\varkappa+\bar{\zeta}_\varkappa,\int_0^\varkappa \widetilde{g}(\varkappa,\varrho,\widehat{w}_\varrho+\bar{\zeta}_\varrho)d\varrho\Big)\Big\|^2 d\varkappa \\
&\quad\leq \frac{2\mathbb{K}^2\vartheta^{2-2\beta+r}}{r\Gamma^2(r)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathbb{K}_f\Big(\|w_\varkappa-\widehat{w}_\varkappa\|^2_{\mathscr{G}_J}+m_1\vartheta^2\|w_\varrho-\widehat{w}_\varrho\|^2_{\mathscr{G}_J}\Big)d\varkappa \\
&\qquad+\frac{2\mathbb{K}^2\vartheta^{2-2\beta}}{\Gamma^2(r)}Tr(\mathcal{Q})\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{2(r-1)}\mathbb{K}_h\Big(\|w_\varkappa-\widehat{w}_\varkappa\|^2_{\mathscr{G}_J}+m_2\vartheta^2\|w_\varrho-\widehat{w}_\varrho\|^2_{\mathscr{G}_J}\Big)d\varkappa \\
&\quad\leq \left(\frac{2\mathbb{K}^2\vartheta^{2-2\beta+2r}}{\Gamma^2(r+1)}\mathbb{K}_f\Big(1+m_1\vartheta^2\Big)+\frac{2\mathbb{K}^2\vartheta^{1-2\beta+2r}}{(2r-1)\Gamma^2(r)}Tr(\mathcal{Q})\mathbb{K}_h\Big(1+m_2\vartheta^2\Big)\right) \\
&\qquad\times\|w_\varkappa-\widehat{w}_\varkappa\|^2_{\mathscr{G}_J} \\
&\quad\leq \left(\frac{2\mathbb{K}^2\vartheta^{2-2\beta+2r}}{\Gamma^2(r+1)}\mathbb{K}_f\Big(1+m_1\vartheta^2\Big)+\frac{2\mathbb{K}^2\vartheta^{1-2\beta+2r}}{(2r-1)\Gamma^2(r)}Tr(\mathcal{Q})\mathbb{K}_h\Big(1+m_2\vartheta^2\Big)\right) \\
&\qquad\times\Big(\ell^2\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2+\|w_0\|^2_{\mathscr{G}_J}+\|\widehat{w}_0\|^2_{\mathscr{G}_J}\Big) \\
&\quad\leq \ell^2\left(\frac{2\mathbb{K}^2\vartheta^{2-2\beta+2r}}{\Gamma^2(r+1)}\mathbb{K}_f\Big(1+m_1\vartheta^2\Big)+\frac{2\mathbb{K}^2\vartheta^{1-2\beta+2r}}{(2r-1)\Gamma^2(r)}Tr(\mathcal{Q})\mathbb{K}_h\Big(1+m_2\vartheta^2\Big)\right) \\
&\qquad\times\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2 \\
&\quad\leq \mathbb{Z}^*\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2,
\end{aligned}$$

where $\mathbb{Z}^* = 2\ell^2\left(\frac{\mathbb{K}^2\vartheta^{2-2\beta+2r}}{\Gamma^2(r+1)}\mathbb{K}_f\left(1+m_1\vartheta^2\right)+\frac{\mathbb{K}^2\vartheta^{1-2\beta+2r}}{(2r-1)\Gamma^2(r)}Tr(\mathcal{Q})\mathbb{K}_h\left(1+m_2\vartheta^2\right)\right) < 1$. Here, we have used the fact that $\|w_0\|^2_{\mathscr{G}_J} = 0$ and $\|\widehat{w}_0\|^2_{\mathscr{G}_J} = 0$. Taking the supremum over $\mathfrak{z}$, we obtain

$$\|\widetilde{\Psi}w - \widetilde{\Psi}\widehat{w}\|^2_\vartheta \le \mathbb{Z}^*\|w - \widehat{w}\|^2_\vartheta.$$

Thus, $\widetilde{\Psi}$ is a contraction on $\mathcal{B}_{\mathfrak{p}}$ and has a unique fixed point $w(\cdot) \in \mathcal{B}_{\mathfrak{p}}$, which is a mild solution of (1). This concludes the proof. □

4. Hilfer Fractional Neutral Stochastic Integrodifferential Systems with infinite Delay

In recent years, neutral differential systems have drawn a great deal of interest in applied mathematics. Several partial differential systems, including heat flow in materials, viscoelasticity, wave propagation, and a variety of natural developments, receive support from neutral systems with or without delay. The authors of [35] studied the optimal control and time-optimized control for a neutral integrodifferential evolution system.

Moreover, in [14], the authors investigated the existence and uniqueness of mild solutions for these equations by means of the Banach contraction mapping principle. Using fractional calculations and a fixed point technique, investigators have recently established the existence of mild solutions for Hilfer fractional neutral evolution systems in [13]. For more details on fractional neutral differential equation, see [32,47,48] and the references therein. To date, the existence of and optimal control results for Hilfer fractional stochastic integrodifferential equation with infinite delay have not been investigated. Motivated by the above facts, we consider a Hilfer fractional neutral stochastic delay integrodifferential system of the following form:

$$\begin{cases} D^{\delta,r}_{0^+}[y(\mathfrak{z}) - \eth(\mathfrak{z}, y_{\mathfrak{z}})] = Ay(\mathfrak{z}) + \mathbb{B}(\mathfrak{z})z(\mathfrak{z}) + f\left(\mathfrak{z}, y_{\mathfrak{z}}, \int_0^{\mathfrak{z}} g(\mathfrak{z},\varkappa,y_\varkappa)d\varkappa\right) \\ \qquad\qquad +h\left(\mathfrak{z}, y_{\mathfrak{z}}, \int_0^{\mathfrak{z}} \widetilde{g}(\mathfrak{z},\varkappa,y_\varkappa)d\varkappa\right)\frac{dW(\mathfrak{z})}{d\mathfrak{z}},\ \mathfrak{z} \in \mathcal{E}' = (0,\vartheta], \\ I^{(1-\delta)(1-r)}_{0^+}y(\mathfrak{z}) = \zeta(\mathfrak{z}) \in L^2(\Omega, \mathscr{G}_J),\ \mathfrak{z} \in (-\infty, 0], \end{cases} \tag{7}$$

where A is the infinitesimal generator of an analytic semigroup $\{\mathcal{G}(\mathfrak{z})\}_{\mathfrak{z}\ge 0}$ on $\mathcal{Y}$, A^γ is a fractional power, and $0 < \gamma \le 1$ as a closed linear operator on $\mathscr{D}(A^\gamma)$ along inverse $A^{-\gamma}$. The following are the properties of A^γ:

(i) Let $\mathscr{D}(A^\gamma)$ be a Hilbert space along $\|y\|_\gamma = \|A^\gamma y\|$ for $y \in \mathscr{D}(A^\gamma)$.
(ii) $\mathcal{G}(\mathfrak{z}) : \mathcal{Y} \to \mathcal{Y}_\gamma$ for $\mathfrak{z} \ge 0$.
(iii) $A^\gamma\mathcal{G}(\mathfrak{z})y = \mathcal{G}(\mathfrak{z})A^\gamma y$ for each $y \in \mathscr{D}(A^\gamma)$ and $\mathfrak{z} \ge 0$.
(iv) For every $\mathfrak{z} > 0$, $A^\gamma\mathcal{G}(\mathfrak{z})$ is bounded on $\mathcal{Y}$ and there exists $\mathbb{M}_\gamma > 0$ such that

$$\|A^\gamma\mathcal{G}(\mathfrak{z})\| \le \frac{\mathbb{M}_\gamma}{\mathfrak{z}^\gamma}.$$

Consider the following hypothesis:

(**H6**): $\eth : [0,\vartheta] \times \mathscr{G}_J \to \mathcal{Y}$ is a continuous function and there exist constants $\eta \in (0,1)$ and $\mathscr{T} > 0$ such that $\eth$ is $\mathcal{Y}_\eta$-valued and fulfills the following requirements:

$$E\|A^\eta\eth(\mathfrak{z}, y) - A^\eta\eth(\mathfrak{z}, \widehat{y})\|^2 \le \mathscr{T}\|y - \widehat{y}\|^2_{\mathscr{G}_J},\ y, \widehat{y} \in \mathscr{G}_J,\ \mathfrak{z} \in \mathscr{G}_J,$$
$$E\|A^\eta\eth(\mathfrak{z}, y)\|^2 \le \mathscr{T}(1 + \|y\|^2_{\mathscr{G}_J}),\ y \in \mathscr{G}_J,\ \mathfrak{z} \in \mathscr{G}_J.$$

For our convenience, $\|A^{-\eta}\| = P_0$, $\widetilde{P}_1 = \frac{r\mathcal{C}_{1-\eta}\Gamma(1+\eta)}{\Gamma(1+r\eta)}$.

Lemma 4. *For any $y \in \mathcal{Y}$, $\eta \in (0,1)$ and $\gamma \in (0,1]$, we have*

$$A\mathcal{N}_r(\mathfrak{z})y = A^{1-\eta}\mathcal{N}_r(\mathfrak{z})A^{\eta}y,$$
$$\|A^{\gamma}\mathcal{N}_r(\mathfrak{z})\| \le \frac{r\mathcal{C}_{\gamma}\Gamma(2-\gamma)}{\mathfrak{z}^{r\gamma}\Gamma(1+r(1-\gamma))}, \ 0 < \mathfrak{z} \le \vartheta.$$

Definition 6 ([21]). *A stochastic process $y : (-\infty, \vartheta] \to \mathcal{Y}$ is a mild solution of (7), provided that*

(**a**) *$y(\mathfrak{z})$ is measurable and $\mathfrak{I}_{\mathfrak{z}}$-adapted.*

(**b**) *$y(\mathfrak{z})$ is continuous on $[0, \vartheta]$ almost surely, and for each $\varkappa \in [0, \mathfrak{z})$, the function $(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)A\eth(\varkappa, y_{\varkappa})$ is integrable such that*

$$\begin{aligned} y(\mathfrak{z}) = {} & \mathcal{G}_{\delta,r}(\mathfrak{z})[\zeta(0) - \eth(0,\zeta)] + \eth(\mathfrak{z}, y_{\mathfrak{z}}) + \int_0^{\mathfrak{z}} \mathcal{S}_r(\mathfrak{z}-\varkappa)\big[A\eth(\varkappa, y_{\varkappa}) + \mathbb{B}(\varkappa)z(\varkappa)\big]d\varkappa \\ & + \int_0^{\mathfrak{z}} \mathcal{S}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, y_{\varkappa}, \int_0^{\varkappa} g(\varkappa, \varrho, y_{\varrho})d\varrho\Big)d\varkappa \\ & + \int_0^{\mathfrak{z}} \mathcal{S}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, y_{\varkappa}, \int_0^{\varkappa} \widetilde{g}(\varkappa, \varrho, y_{\varrho})d\varrho\Big)dW(\varkappa), \text{ for } \mathfrak{z} \in \mathcal{E}', \end{aligned}$$

is fulfilled. Because $\mathcal{S}_r(\mathfrak{z}) = \mathfrak{z}^{r-1}\mathcal{N}_r(\mathfrak{z})$, it is equivalent with

$$\begin{aligned} y(\mathfrak{z}) = {} & \mathcal{G}_{\delta,r}(\mathfrak{z})[\zeta(0) - \eth(0,\zeta)] + \eth(\mathfrak{z}, y_{\mathfrak{z}}) + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)A\eth(\varkappa, y_{\varkappa})d\varkappa \\ & + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ & + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, y_{\varkappa}, \int_0^{\varkappa} g(\varkappa, \varrho, y_{\varrho})d\varrho\Big)d\varkappa \\ & + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, y_{\varkappa}, \int_0^{\varkappa} \widetilde{g}(\varkappa, \varrho, y_{\varrho})d\varrho\Big)dW(\varkappa), \text{ for } \mathfrak{z} \in \mathcal{E}'. \end{aligned} \quad (8)$$

(**c**) *$I_{0^+}^{(1-\delta)(1-r)}y(\mathfrak{z}) = \zeta(\mathfrak{z}) \in L^2(\Omega, \mathscr{G}_{\jmath})$ on $(-\infty, 0]$ fulfilling $\|\zeta\|^2_{\mathscr{G}_{\jmath}} < \infty$.*

Theorem 2. *Assume that* (**H0**)–(**H6**) *are fulfilled. Next, system* (7) *has a unique mild solution provided that*

$$\begin{aligned} 24\ell^2 \Bigg[& \vartheta^{2(1-\beta)}P_0^2\mathscr{T} + \frac{\vartheta^{2-2\beta+2r\eta}}{(r\eta)^2}\widetilde{P}_1^2\mathscr{T} + \frac{\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1 + \overline{m}_1\vartheta^2\Big) \\ & + \frac{\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1 + \overline{m}_2\vartheta^2\Big)\Bigg] < 1. \end{aligned} \quad (9)$$

Proof. Define an operator $\wp : \mathscr{G}'_{\jmath} \to \mathscr{G}'_{\jmath}$ as

$$\wp y(\mathfrak{z}) = \begin{cases} \zeta(\mathfrak{z}), \quad \mathfrak{z} \in (-\infty, 0], \\ \mathcal{G}_{\delta,r}(\mathfrak{z})[\zeta(0) - \eth(0,\zeta)] + \eth(\mathfrak{z}, y_{\mathfrak{z}}) \\ + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)A\eth(\varkappa, y_{\varkappa})d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, y_{\varkappa}, \int_0^{\varkappa} g(\varkappa, \varrho, y_{\varrho})d\varrho\Big)d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, y_{\varkappa}, \int_0^{\varkappa} \widetilde{g}(\varkappa, \varrho, y_{\varrho})d\varrho\Big)dW(\varkappa), \text{for } \mathfrak{z} > 0. \end{cases} \quad (10)$$

For $\zeta \in \mathscr{G}_J$, we define $\bar{\zeta}$ as follows:

$$\bar{\zeta}(\mathfrak{z}) = \begin{cases} \zeta(\mathfrak{z}), & \mathfrak{z} \in (-\infty, 0], \\ \mathcal{G}_{\delta,r}\zeta(0), & \mathfrak{z} \in \mathcal{E}, \end{cases} \tag{11}$$

then, $\bar{\zeta} \in \mathscr{G}_J'$. Let $y(\mathfrak{z}) = w(\mathfrak{z}) + \bar{\zeta}(\mathfrak{z})$, $-\infty < \mathfrak{z} \leq \vartheta$. Clearly, y is satisfied from (7) if and only if w fulfills $w_0 = 0$ and

$$\begin{aligned} w(\mathfrak{z}) = &- \mathcal{G}_{\delta,r}(\mathfrak{z})\eth(0,\zeta) + \eth(\mathfrak{z}, w_{\mathfrak{z}} + \bar{\zeta}_{\mathfrak{z}}) + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varrho)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)A\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa)d\varkappa \\ &+ \int_0^{\mathfrak{z}} (\mathfrak{z} - \varrho)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ &+ \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^{\varkappa} g(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)d\varkappa \\ &+ \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^{\varkappa} \widetilde{g}(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa). \end{aligned}$$

Let $\mathscr{G}_J'' = \{w \in \mathscr{G}_J' : w_0 = 0 \in \mathscr{G}_J\}$. For any $w \in \mathscr{G}_J''$,

$$\begin{aligned} \|w\|_\vartheta &= \|w_0\|_{\mathscr{G}_J} + \sup_{0 \leq \varrho \leq \vartheta} \left(E\|w(\varkappa)\|^2\right)^{\frac{1}{2}} \\ &= \sup_{0 \leq \varkappa \leq \vartheta} \left(E\|w(\varkappa)\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, $(\mathscr{G}_J'', \|\cdot\|_\vartheta)$ is a Banach space. For some $\mathfrak{p} > 0$, we set $\mathcal{B}_\mathfrak{p} = \{w \in \mathscr{G}_J'' : \|w\|_\vartheta^2 \leq \mathfrak{p}\}$; then, $\mathcal{B}_\mathfrak{p} \subseteq \mathscr{G}_J''$ is uniformly bounded, $w \in \mathcal{B}_\mathfrak{p}$, and referring to Lemma 1,

$$\begin{aligned} \|w_{\mathfrak{z}} + \bar{\zeta}_{\mathfrak{z}}\|_{\mathscr{G}_J}^2 &\leq 2\left(\|w_{\mathfrak{z}}\|_{\mathscr{G}_J}^2 + \|\bar{\zeta}_{\mathfrak{z}}\|_{\mathscr{G}_J}^2\right) \\ &\leq 4\Big(\|w_0\|_{\mathscr{G}_J}^2 + \ell^2 \sup_{\varkappa \in [0,\mathfrak{z}]} \left(E\|w(\varkappa)\|^2\right) + \|\bar{\zeta}_0\|_{\mathscr{G}_J}^2 + \ell^2 \sup_{\varkappa \in [0,\mathfrak{z}]} \left(E\|\bar{\zeta}(\varkappa)\|^2\right)\Big) \\ &\leq 4\left(0 + \ell^2\|w\|_\vartheta^2 + \|\zeta\|_{\mathscr{G}_J}^2 + \ell^2 E\|\mathcal{G}_{\delta,r}(\varkappa)\zeta(0)\|^2\right) \\ &\leq 4\left(\ell^2\mathfrak{p} + \|\zeta\|_{\mathscr{G}_J}^2 + \ell^2\left(\frac{\mathbb{K}\varkappa^{\beta-1}}{\Gamma(\beta)}\right)^2 E\|\zeta(0)\|^2\right) \\ &\leq 4\ell^2\left(\mathfrak{p} + \left(\frac{\mathbb{K}\varkappa^{\beta-1}}{\Gamma(\beta)}\right)^2 E\|\zeta(0)\|^2\right) + 4\|\zeta\|_{\mathscr{G}_J}^2 \\ &\leq 4\ell^2(\mathfrak{p} + M_1 E\|\zeta(0)\|^2) + 4\|\zeta\|_{\mathscr{G}_J}^2 = \mathfrak{p}_1, \end{aligned} \tag{12}$$

where $M_1 = \left(\frac{\mathbb{K}\varkappa^{\beta-1}}{\Gamma(\beta)}\right)^2$.

We define $\widetilde{\wp} : \mathscr{G}_J'' \to \mathscr{G}_J''$ as follows:

$$\widetilde{\wp}w(\mathfrak{z}) = \begin{cases} 0, \quad \mathfrak{z} \in (-\infty, 0], \\ -\mathcal{G}_{\delta,r}(\mathfrak{z})\eth(0,\zeta) + \eth(\mathfrak{z}, w_{\mathfrak{z}} + \bar{\zeta}_{\mathfrak{z}}) + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)A\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa)d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^{\varkappa} g(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z} - \varkappa)h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^{\varkappa} \widetilde{g}(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa), \ \mathfrak{z} \in \mathcal{E}, \end{cases} \tag{13}$$

which demonstrates that $\widetilde{\wp}$ has a fixed point. For ease of understanding, we split the proof into two steps.

Step 1: We claim that there exists $\mathfrak{p} > 0$ such that $\widetilde{\wp}(\mathcal{B}_{\mathfrak{p}}) \subset \mathcal{B}_{\mathfrak{p}}$. If this is false, then for each $\mathfrak{p} > 0$ there exists a function $w^{\mathfrak{p}}(\cdot) \in \mathcal{B}_{\mathfrak{p}}$ and $\widetilde{\wp}(w^{\mathfrak{p}}) \notin \mathcal{B}_{\mathfrak{p}}$, that is, $E\|(\widetilde{\wp}w^{\mathfrak{p}})(\mathfrak{z})\|^2 > \mathfrak{p}$ for some $\mathfrak{z} \in \mathcal{E}$.

Furthermore, from Lemma 3, (**H1**), (**H2**), and Holder's inequality, we have

$$
\begin{aligned}
\mathfrak{p} &< \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\|(\widetilde{\wp}w)(\mathfrak{z})\|^2 \\
&\leq 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\|\mathcal{G}_{\delta,r}(\mathfrak{z})\eth(0,\zeta)\|^2 + 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\|\eth(\mathfrak{z}, w_{\mathfrak{z}} + \bar{\zeta}_{\mathfrak{z}})\|^2 \\
&\quad + 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)A\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa)d\varkappa\Big\|^2 \\
&\quad + 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa\Big\|^2 \\
&\quad + 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^{\varkappa} g(\varkappa,\varrho,w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)d\varkappa\Big\|^2 \\
&\quad + 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^{\varkappa} \widetilde{g}(\varkappa,\varrho,w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)dW(\varkappa)\Big\|^2 \\
&= \mathbb{S}_1 + \mathbb{S}_2 + \mathbb{S}_3 + \mathbb{S}_4 + \mathbb{S}_5 + \mathbb{S}_6,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{S}_1 &= 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\|\mathcal{G}_{\delta,r}(\mathfrak{z})\eth(0,\zeta)\|^2 \\
&\leq \frac{6\vartheta^{2(1-\beta)}\mathbb{K}^2\vartheta^{2(\beta-1)}\|A^{-\eta}\|^2}{\Gamma^2(\beta)}\mathscr{T}\big(1 + \|\zeta\|^2_{\mathscr{G}_\jmath}\big) \\
&\leq \frac{6\mathbb{K}^2 P_0^2}{\Gamma^2(\beta)}\mathscr{T}\big(1 + \|\zeta\|^2_{\mathscr{G}_\jmath}\big), \\
\mathbb{S}_2 &= 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\|\eth(\mathfrak{z}, w_{\mathfrak{z}} + \bar{\zeta}_{\mathfrak{z}})\|^2 \\
&\leq 6\vartheta^{2(1-\beta)}\|A^{-\eta}\|^2\mathscr{T}(1+\mathfrak{p}_1) \\
&\leq 6\vartheta^{2(1-\beta)}P_0^2\mathscr{T}(1+\mathfrak{p}_1), \\
\mathbb{S}_3 &= 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)A\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa)d\varkappa\Big\|^2 \\
&\leq 6\vartheta^{2(1-\beta)} E\Big\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varrho)^{r-1}A^{1-\eta}\mathcal{N}_r(\mathfrak{z}-\varkappa)A^{\eta}\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa)d\varkappa\Big\|^2 \\
&\leq 6\vartheta^{2(1-\beta)}\widetilde{P}_1^2\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r\eta-1}d\varkappa\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r\eta-1}E\|A^{\eta}\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa)\|^2 d\varkappa \\
&\leq \frac{6\vartheta^{2-2\beta+2r\eta}}{(r\eta)^2}\widetilde{P}_1^2\mathscr{T}(1+\mathfrak{p}_1), \\
\mathbb{S}_4 &= 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\bigg\|\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathbb{B}(\varkappa)z(\varkappa)d\varkappa\bigg\|^2 \\
&\leq \frac{6\vartheta^{2-2\beta}\mathbb{K}^2\|\mathbb{B}\|^2_\infty}{\Gamma^2(r)}E\left[\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\|z(\varkappa)\|d\varkappa\right]^2 \\
&\leq \frac{6\vartheta^{2-2\beta}\mathbb{K}^2\|\mathbb{B}\|^2_\infty}{\Gamma^2(r)}\left[\left(\int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{\frac{\mathfrak{q}(r-1)}{\mathfrak{q}-1}}d\varkappa\right)^{\frac{\mathfrak{q}-1}{\mathfrak{q}}}\left(E\int_0^{\mathfrak{z}}\|z(\varkappa)\|^{\mathfrak{q}}d\varkappa\right)^{\frac{1}{\mathfrak{q}}}\right]^2 \\
&\leq \frac{6\vartheta^{2-2\beta+2r-\frac{2}{\mathfrak{q}}}}{\Gamma^2(r)}\mathbb{K}^2\|\mathbb{B}\|^2_\infty\|z\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\left(\frac{\mathfrak{q}-1}{\mathfrak{q}r-1}\right)^{\frac{2(\mathfrak{q}-1)}{\mathfrak{q}}},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{S}_5 &= 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) f\Big(\varkappa, w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}, \int_0^{\varkappa} g(\varkappa,\varrho,w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho})d\varrho\Big)d\varkappa\Big\|^2 \\
&\le \frac{6\vartheta^{2-2\beta}\mathbb{K}^2}{\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}d\varkappa \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1} E\|f\Big(\varkappa, w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}, \int_0^{\varkappa} g(\varkappa,\varrho,w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho})d\varrho\Big)\|^2 d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta+r}\mathbb{K}^2}{r\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\overline{\mathbb{K}}_f\Big(1+\|w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}\|^2_{\mathscr{G}_J} + E\|\int_0^{\varkappa} g(\varkappa,\varrho,w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho})d\varrho\|^2\Big)d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta+r}\mathbb{K}^2}{r\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\overline{\mathbb{K}}_f\Big(1+\|w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}\|^2_{\mathscr{G}_J} + \overline{m}_1\vartheta^2(1+\|w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho}\|^2_{\mathscr{G}_J})\Big)d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta+r}\mathbb{K}^2}{r\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\overline{\mathbb{K}}_f\Big(1+\mathfrak{p}_1+\overline{m}_1\vartheta^2(1+\mathfrak{p}_1)\Big)d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\mathfrak{p}_1+\overline{m}_1\vartheta^2(1+\mathfrak{p}_1)\Big), \\
\mathbb{S}_6 &= 6\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)} E\Big\| \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa) h\Big(\varkappa, w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}, \int_0^{\varkappa} \widetilde{g}(\varkappa,\varrho,w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho})d\varrho\Big)dW(\varkappa)\Big\|^2 \\
&\le \frac{6\vartheta^{2-2\beta}\mathbb{K}^2 Tr(\mathcal{Q})}{\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{2(r-1)} E\|h\Big(\varkappa, w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}, \int_0^{\varkappa} g(\varkappa,\varrho,w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho})d\varrho\Big)\|^2 d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta}\mathbb{K}^2 Tr(\mathcal{Q})}{\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{2(r-1)}\overline{\mathbb{K}}_h\Big(1+\|w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}\|^2_{\mathscr{G}_J} + E\|\int_0^{\varkappa} \widetilde{g}(\varkappa,\varrho,w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho})d\varrho\|^2\Big)d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta}\mathbb{K}^2 Tr(\mathcal{Q})}{\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{2(r-1)}\overline{\mathbb{K}}_h\Big(1+\|w_{\varkappa}^{\mathfrak{p}}+\bar{\zeta}_{\varkappa}\|^2_{\mathscr{G}_J} + \overline{m}_2\vartheta^2(1+\|w_{\varrho}^{\mathfrak{p}}+\bar{\zeta}_{\varrho}\|^2_{\mathscr{G}_J})\Big)d\varkappa \\
&\le \frac{6\vartheta^{2-2\beta}\mathbb{K}^2 Tr(\mathcal{Q})}{\Gamma^2(r)} \int_0^{\mathfrak{z}} (\mathfrak{z}-\varkappa)^{2(r-1)}\overline{\mathbb{K}}_h\Big(1+\mathfrak{p}_1+\overline{m}_2\vartheta^2(1+\mathfrak{p}_1)\Big)d\varkappa \\
&\le \frac{6\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\mathfrak{p}_1+\overline{m}_2\vartheta^2(1+\mathfrak{p}_1)\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathfrak{p} &\le \frac{6\mathbb{K}^2 P_0^2}{\Gamma^2(\beta)}\mathscr{T}\big(1+\|\zeta\|^2_{\mathscr{G}_J}\big) + 6\vartheta^{2(1-\beta)}P_0^2\mathscr{T}(1+\mathfrak{p}_1) + \frac{6\vartheta^{2-2\beta+2r\eta}}{(r\eta)^2}\widetilde{P}_1^2\mathscr{T}(1+\mathfrak{p}_1) \\
&\quad + \frac{6\vartheta^{2-2\beta+2r-\frac{2}{\mathfrak{q}}}}{\Gamma^2(r)}\mathbb{K}^2\|\mathbb{B}\|^2_\infty\|z\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\left(\frac{\mathfrak{q}-1}{\mathfrak{q}r-1}\right)^{\frac{2(\mathfrak{q}-1)}{\mathfrak{q}}} \\
&\quad + \frac{6\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\mathfrak{p}_1+\overline{m}_1\vartheta^2(1+\mathfrak{p}_1)\Big) \\
&\quad + \frac{6\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\mathfrak{p}_1+\overline{m}_2\vartheta^2(1+\mathfrak{p}_1)\Big) \\
&= \frac{6\mathbb{K}^2 P_0^2}{\Gamma^2(\beta)}\mathscr{T}\big(1+\|\zeta\|^2_{\mathscr{G}_J}\big) + 6\vartheta^{2(1-\beta)}P_0^2\mathscr{T} + \frac{6\vartheta^{2-2\beta+2r\eta}}{(r\eta)^2}\widetilde{P}_1^2\mathscr{T} \\
&\quad + \frac{6\vartheta^{2-2\beta+2r-\frac{2}{\mathfrak{q}}}}{\Gamma^2(r)}\mathbb{K}^2\|\mathbb{B}\|^2_\infty\|z\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\left(\frac{\mathfrak{q}-1}{\mathfrak{q}r-1}\right)^{\frac{2(\mathfrak{q}-1)}{\mathfrak{q}}} \\
&\quad + \frac{6\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\overline{m}_1\vartheta^2\Big) + \frac{6\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\overline{m}_2\vartheta^2\Big) \\
&\quad + \Bigg[6\vartheta^{2(1-\beta)}P_0^2\mathscr{T} + \frac{6\vartheta^{2-2\beta+2r\eta}}{(r\eta)^2}\widetilde{P}_1^2\mathscr{T} + \frac{6\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\overline{m}_1\vartheta^2\Big) \\
&\quad + \frac{6\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\overline{m}_2\vartheta^2\Big)\Bigg]\mathfrak{p}_1. \qquad (14)
\end{aligned}
$$

Now, by dividing (14) by $\mathfrak{p}$ and taking $\mathfrak{p} \to \infty$, we obtain

$$24\ell^2\Bigg[\vartheta^{2(1-\beta)}P_0^2\mathscr{T} + \frac{\vartheta^{2-2\beta+2r\eta}}{(r\eta)^2}\widetilde{P}_1^2\mathscr{T} + \frac{\vartheta^{2-2\beta+2r}\mathbb{K}^2}{\Gamma^2(r+1)}\overline{\mathbb{K}}_f\Big(1+\overline{m}_1\vartheta^2\Big) + \frac{\vartheta^{1-2\beta+2r}\mathbb{K}^2 Tr(\mathcal{Q})}{(2r-1)\Gamma^2(r)}\overline{\mathbb{K}}_h\Big(1+\overline{m}_2\vartheta^2\Big)\Bigg] \geq 1,$$

which contradicts assumption (9). Hence, for some $\mathfrak{p} > 0$, $\widetilde{\wp}(\mathcal{B}_\mathfrak{p}) \subset \mathcal{B}_\mathfrak{p}$.

Step 2: $\widetilde{\wp}$ is a contraction on $\mathcal{B}_\mathfrak{p}$.

For each $w, \widehat{w} \in \mathcal{B}_\mathfrak{p}$, we have

$$\begin{aligned}
&E\|\widetilde{\wp}w(\mathfrak{z}) - \widetilde{\wp}\widehat{w}(\mathfrak{z})\|^2 \\
&= \sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)}E\Big\|\big[\eth(\mathfrak{z}, w_\mathfrak{z} + \bar{\zeta}_\mathfrak{z}) - \eth(\mathfrak{z}, \widehat{w}_\mathfrak{z} + \bar{\zeta}_\mathfrak{z})\big] \\
&\quad + \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\mathcal{U}\big[\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa) - \eth(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa)\big]d\varkappa \\
&\quad + \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big) \\
&\quad - f\Big(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, \widehat{w}_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)\Big]d\varkappa \\
&\quad + \int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big) \\
&\quad - h\Big(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, \widehat{w}_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)\Big]dW(\varkappa)\Big\|^2 \\
&\leq 4\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)}E\|\eth(\mathfrak{z}, w_\mathfrak{z} + \bar{\zeta}_\mathfrak{z}) - \eth(\mathfrak{z}, \widehat{w}_\mathfrak{z} + \bar{\zeta}_\mathfrak{z})\|^2 \\
&\quad + 4\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)A\big[\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa) - \eth(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa)\big]d\varkappa\Big\|^2 \\
&\quad + 4\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big) \\
&\quad - f\Big(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, \widehat{w}_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)\Big]d\varkappa\Big\|^2 \\
&\quad + 4\sup_{\mathfrak{z}\in\mathcal{E}} \mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big) \\
&\quad - h\Big(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, \widehat{w}_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)\Big]dW(\varkappa)\Big\|^2 \\
&\leq 4\vartheta^{2(1-\beta)}\|A^{-\eta}\|^2E\|A^\eta\eth(\mathfrak{z}, w_\mathfrak{z} + \bar{\zeta}_\mathfrak{z}) - A^\eta\eth(\mathfrak{z}, \widehat{u}_\mathfrak{z} + \bar{\zeta}_\mathfrak{z})\|^2 \\
&\quad + 4\vartheta^{2(1-\beta)}E\Big\|\int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}A^{1-\eta}\mathcal{N}_r(\mathfrak{z}-\varkappa)\big[A^\eta\eth(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa) - A^\eta\eth(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa)\big]d\varkappa\Big\|^2 \\
&\quad + 4\vartheta^{2(1-\beta)}E\Big\|\int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[f\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big) \\
&\quad - f\Big(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa g(\varkappa, \varrho, \widehat{w}_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)\Big]d\varkappa\Big\|^2 \\
&\quad + 4\vartheta^{2(1-\beta)}E\Big\|\int_0^\mathfrak{z} (\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[h\Big(\varkappa, w_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, w_\varrho + \bar{\zeta}_\varrho)d\varrho\Big) \\
&\quad - h\Big(\varkappa, \widehat{w}_\varkappa + \bar{\zeta}_\varkappa, \int_0^\varkappa \widetilde{g}(\varkappa, \varrho, \widehat{w}_\varrho + \bar{\zeta}_\varrho)d\varrho\Big)\Big]dW(\varkappa)\Big\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq 4\vartheta^{2-2\beta}P_0^2\mathscr{T}\|w_\mathfrak{z}-\widehat{w}_\mathfrak{z}\|^2_{\mathscr{G}_J}+\frac{4\vartheta^{2-2\beta}r^2\mathcal{C}^2_{1-\eta}\Gamma^2(1+\eta)}{\Gamma^2(1+r\eta)}\frac{\vartheta^{2r\eta}}{(r\eta)^2}\mathscr{T}\|w_\varkappa-\widehat{w}_\varkappa\|^2_{\mathscr{G}_J}\\
&\quad+\frac{4\vartheta^{2-2\beta+2r}\mathbb{K}^2\mathbb{K}_f(1+m_1\vartheta^2)}{\Gamma^2(r+1)}\|w_\varkappa-\widehat{w}_\varkappa\|^2_{\mathscr{G}_J}\\
&\quad+\frac{4\vartheta^{1-2\beta+2r}\mathbb{K}^2Tr(\mathcal{Q})\mathbb{K}_h(1+m_2\vartheta^2)}{(2r-1)\Gamma^2(r)}\|w_\varkappa-\widehat{w}_\varkappa\|^2_{\mathscr{G}_J}\\
&\leq\Bigg[4\vartheta^{2-2\beta}\mathscr{T}\left(P_0^2+\left(\frac{\mathcal{C}_{1-\eta}\Gamma(1+\eta)\vartheta^{r\eta}}{\Gamma(1+r\eta)\eta}\right)^2\right)+\frac{4\vartheta^{2-2\beta+2r}\mathbb{K}^2\mathbb{K}_f(1+m_1\vartheta^2)}{\Gamma^2(r+1)}\\
&\quad+\frac{4\vartheta^{1-2\beta+2r}\mathbb{K}^2Tr(\mathcal{Q})\mathbb{K}_h(1+m_2\vartheta^2)}{(2r-1)\Gamma^2(r)}\Bigg]\left\{\ell^2\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2+\|w_0\|^2_{\mathscr{G}_J}+\|\widehat{w}_0\|^2_{\mathscr{G}_J}\right\}\\
&\leq\Bigg[4\vartheta^{2-2\beta}\mathscr{T}\ell^2\left(P_0^2+\left(\frac{\mathcal{C}_{1-\eta}\Gamma(1+\eta)\vartheta^{r\eta}}{\Gamma(1+r\eta)\eta}\right)^2\right)+\frac{4\vartheta^{2-2\beta+2r}\mathbb{K}^2\ell^2\mathbb{K}_f(1+m_1\vartheta^2)}{\Gamma^2(r-1)}\\
&\quad+\frac{4\vartheta^{1-2\beta+2r}\mathbb{K}^2\ell^2Tr(\mathcal{Q})\mathbb{K}_h(1+m_2\vartheta^2)}{(2r-1)\Gamma^2(r)}\Bigg]\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2\\
&\leq\mathbb{M}^*\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2,
\end{aligned}
$$

which implies that

$$E\|\widetilde{\wp}w(\mathfrak{z})-\widetilde{\wp}\widehat{w}(\mathfrak{z})\|^2\leq\mathbb{M}^*\sup_{\varkappa\in\mathcal{E}}E\|w(\varkappa)-\widehat{w}(\varkappa)\|^2,$$

where

$$
\begin{aligned}
\mathbb{M}^*&=4\vartheta^{2-2\beta}\mathscr{T}\ell^2\left(P_0^2+\left(\frac{\mathcal{C}_{1-\eta}\Gamma(1+\eta)\vartheta^{r\eta}}{\Gamma(1+r\eta)\eta}\right)^2\right)\\
&\quad+\frac{4\vartheta^{2-2\beta+2r}\mathbb{K}^2\ell^2\mathbb{K}_f(1+m_1\vartheta^2)}{\Gamma^2(r-1)}+\frac{4\vartheta^{1-2\beta+2r}\mathbb{K}^2\ell^2Tr(\mathcal{Q})\mathbb{K}_h(1+m_2\vartheta^2)}{(2r-1)\Gamma^2(r)}<1.
\end{aligned}
$$

Here, we have used the fact that $\|w_0\|^2=0$ and $\|\widehat{w}_0\|^2=0$. Taking the supremum over $\mathfrak{z}$, we obtain $\|\widetilde{\wp}w-\widetilde{\wp}\widehat{w}\|^2_\vartheta\leq\mathbb{M}^*\|w-\widehat{w}\|^2_\vartheta$. Thus, $\widetilde{\wp}$ is a contradiction. It follows that $\widetilde{\wp}$ has a unique fixed point $w(\cdot)\in\mathcal{B}_\mathfrak{p}$, which is a mild solution of Equation (7). This completes the proof. □

5. Existence of Optimal Controls

Consider the Lagrange problem (LP).
Find a control $z^0\in\mathbb{A}_{ad}$ such that

$$\mathscr{L}(z^0)\leq\mathscr{L}(z),\ \forall\, z\in\mathbb{A}_{ad}$$

where

$$\mathscr{L}(z)=E\left\{\int_0^b\mathscr{M}(\mathfrak{z},y^z_\mathfrak{z},y^z(\mathfrak{z}),z(\mathfrak{z}))d\mathfrak{z}\right\},$$

and y^z is the mild solution of system (7) related to the control $z\in\mathbb{A}_{ad}$. We form the following hypothesis to illustrate the existence of a solution for problem (LP).

(**H7**):(**i**) The functional $\mathscr{M}:\mathcal{E}\times\mathscr{G}_J\times\mathcal{Y}\times\mathcal{K}\to\mathbb{R}\cup\{\infty\}$ is Borel measurable.

(**ii**) The sequentially lower semicontinuous functional $\mathscr{M}(\mathfrak{z},\cdot,\cdot,\cdot)$ on $\mathscr{G}_J\times\mathcal{Y}\times\mathcal{K}$ for almost all $\mathfrak{z}\in\mathcal{E}$.

(**iii**) $\mathscr{M}(\mathfrak{z}, y, \hat{y}, \cdot)$ is convex on $\mathcal{K}$ for each $y \in \mathscr{G}_J$, $\hat{y} \in \mathcal{Y}$ and almost all $\mathfrak{z} \in \mathcal{E}$.

(**iv**) There exist constants $\widetilde{a}, \widetilde{b} \geq 0, \widetilde{c} > 0$, $\mathfrak{g}$ is non-negative, and $\mathfrak{g} \in L^1(\mathcal{E}, \mathbb{R})$ such that

$$\mathfrak{g}(\mathfrak{z}) + \widetilde{a}\|y\|^2_{\mathscr{G}_J} + \widetilde{b}\|\hat{y}\|^2 + \widetilde{c}\|z\|^{\mathfrak{q}}_{\mathcal{K}} \leq \mathscr{M}(\mathfrak{z}, y, \hat{y}, z).$$

Theorem 3. *Assume that* (**H7**) *and Theorem 2 are true and $\mathbb{B}$ is a strongly continuous operator. The Lagrange Problem (LP) then accepts at least one optimal pair, i.e., there is a control $z^0 \in \mathbb{A}_{ad}$ such that*

$$\mathscr{L}(z^0) = E\left\{ \int_0^{\vartheta} \mathscr{M}(\mathfrak{z}, y^0_{\mathfrak{z}}, y^0(\mathfrak{z}), z^0(\mathfrak{z}))d\mathfrak{z} \right\} \leq \mathscr{L}(z), \ \forall\, z \in \mathbb{A}_{ad}.$$

Proof. Provided that $\inf\{\mathscr{L}(z) \mid z \in \mathbb{A}_{ad}\} = +\infty$, there is nothing to verify. Without loss of generality, we conclude that infimum of $\{\mathscr{L}(z) \mid z \in \mathbb{A}_{ad}\} = \varpi < +\infty$. Using (**H7**), we have $\varpi > -\infty$. By definition of infimum, there is a minimizing sequence feasible pair $\{(y^{\check{n}}, z^{\check{n}})\} \subset U_{ad}$, where $U_{ad} = \{(y, z) : y$ is a mild solution of (7) relating to $z \in \mathbb{A}_{ad}\}$ such that $\mathscr{L}(y^{\check{n}}, z^{\check{n}}) \to \varpi$, as $\check{n} \to +\infty$. Then, $\{z^{\check{n}}\} \subseteq \mathbb{A}_{ad}$, $\check{n} = 1, 2, \cdots, \{z^{\check{n}}\}$ is a bounded subset of the separable reflexive Hilbert space $L^{\mathfrak{q}}(\mathcal{E}, \mathcal{K})$, and there exists a subsequence, relabeled as $\{z^{\check{n}}\}$ and $z^0 \in L^{\mathfrak{q}}(\mathcal{E}, \mathcal{K})$, such that $z^{\check{n}} \to z^0$ weakly in $L^{\mathfrak{q}}(\mathcal{E}, \mathcal{K})$. Because $\mathbb{A}_{ad}$ is closed convex, the Marzur Lemma state that $z^0 \in \mathbb{A}_{ad}$.

Consider that $\{y^{\check{n}}\}$ is the sequence of solutions of (7) corresponding to $\{z^{\check{n}}\}$, that is,

$$y^{\check{n}}(\mathfrak{z}) = \begin{cases} \mathcal{G}_{\delta,r}(\mathfrak{z})[\zeta(0) - \eth(0, \zeta)] + \eth(\mathfrak{z}, y^{\check{n}}_{\mathfrak{z}}) + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1} \mathscr{N}_r(\mathfrak{z} - \varkappa) A \eth(\varkappa, y^{\check{n}}_{\varkappa}) d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1} \mathcal{N}_r(\mathfrak{z} - \varkappa) \mathbb{B}(\varkappa) z^{\check{n}}(\varkappa) d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1} \mathcal{N}_r(\mathfrak{z} - \varkappa) f\Big(\varkappa, y^{\check{n}}_{\varkappa}, \int_0^{\varkappa} g(\varkappa, \varrho, y^{\check{n}}_{\varrho}) d\varrho\Big) d\varkappa \\ + \int_0^{\mathfrak{z}} (\mathfrak{z} - \varkappa)^{r-1} \mathcal{N}_r(\mathfrak{z} - \varkappa) h\Big(\varkappa, y^{\check{n}}_{\varkappa}, \int_0^{\varkappa} \widetilde{g}(\varkappa, \varrho, y^{\check{n}}_{\varrho}) d\varrho\Big) dW(\varkappa), \quad \text{for } \mathfrak{z} > 0, \\ \zeta(\mathfrak{z}), \quad \mathfrak{z} \in (-\infty, 0]. \end{cases}$$

By referring Theorem 2, it is easy to see that there exists a $\tau > 0$ such that

$$\|y^{\check{n}}\|^2_{\vartheta} \leq \tau, \ \check{n} = 0, 1, 2, \cdots.$$

Let $y^{\check{n}}(\mathfrak{z}) = w^{\check{n}}(\mathfrak{z}) + \bar{\zeta}(\mathfrak{z})$, where $w^{\check{n}} \in \mathscr{G}''_J$ and $\bar{\zeta} : (-\infty, \vartheta] \to \mathcal{Y}$ are the function provided by (11). For $\mathfrak{z} \in \mathcal{E}$, we can obtain

$$
\begin{aligned}
&E\|w^{\check{n}}(\mathfrak{z})-w^0(\mathfrak{z})\|^2\\
&\leq 5\sup_{\mathfrak{z}\in\mathscr{E}}\mathfrak{z}^{2(1-\beta)}E\|\eth(\mathfrak{z},w^{\check{n}}_{\mathfrak{z}}+\bar{\zeta}_{\mathfrak{z}})-\eth(\mathfrak{z},w^0_{\mathfrak{z}}+\bar{\zeta}_{\mathfrak{z}})\|^2\\
&\quad+5\sup_{\mathfrak{z}\in\mathscr{E}}\mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\big[A\eth(\varkappa,w^{\check{n}}_{\mathfrak{z}}+\bar{\zeta}_{\mathfrak{z}})-A\eth(\varkappa,w^0_{\mathfrak{z}}+\bar{\zeta}_{\mathfrak{z}})\big]d\varkappa\Big\|^2\\
&\quad+5\sup_{\mathfrak{z}\in\mathscr{E}}\mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)[\mathbb{B}(\varkappa)z^{\check{n}}(\varkappa)-\mathbb{B}(\varkappa)z^0(\varkappa)]d\varkappa\Big\|^2\\
&\quad+5\sup_{\mathfrak{z}\in\mathscr{E}}\mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[f\Big(\varkappa,w^{\check{n}}_{\varkappa}+\bar{\zeta}_{\varkappa},\int_0^{\varkappa}g(\varkappa,\varrho,w^{\check{n}}_{\varrho}+\bar{\zeta}_{\varrho})d\varrho\Big)\\
&\quad-f\Big(\varkappa,w^{\check{n}}_{\varkappa}+\bar{\zeta}_{\varkappa},\int_0^{\varkappa}g(\varkappa,w^0_{\varrho}+\bar{\zeta}_{\varrho})d\varrho\Big)\Big]d\varkappa\Big\|^2\\
&\quad+5\sup_{\mathfrak{z}\in\mathscr{E}}\mathfrak{z}^{2(1-\beta)}E\Big\|\int_0^{\mathfrak{z}}(\mathfrak{z}-\varkappa)^{r-1}\mathcal{N}_r(\mathfrak{z}-\varkappa)\Big[h\Big(\varkappa,w^{\check{n}}_{\varkappa}+\bar{\zeta}_{\varkappa},\int_0^{\varkappa}\widetilde{g}(\varkappa,\varrho,w^{\check{n}}_{\varrho}+\bar{\zeta}_{\varrho})d\varrho\Big)\\
&\quad-h\Big(\varkappa,w^{\check{n}}_{\varkappa}+\bar{\zeta}_{\varkappa},\int_0^{\varkappa}\widetilde{g}(\varkappa,w^0_{\varrho}+\bar{\zeta}_{\varrho})d\varrho\Big)\Big]dW(\varkappa)\Big\|^2\\
&\leq 5\vartheta^{2-2\beta}P_0^2\mathscr{T}\|w^{\check{n}}_{\mathfrak{z}}-w^0_{\mathfrak{z}}\|^2_{\mathscr{G}_\jmath}+\frac{5\vartheta^{2-2\beta+2r\eta}C^2_{1-\eta}\Gamma^2(1+\eta)}{\eta^2\Gamma^2(1+r\eta)}\mathscr{T}\|w^{\check{n}}_{\varkappa}-w^0_{\varkappa}\|^2_{\mathscr{G}_\jmath}\\
&\quad+\frac{5\vartheta^{2-2\beta+2r-\frac{2}{\mathfrak{q}}}\mathbb{K}^2}{\Gamma^2(r)}\left(\frac{\mathfrak{q}-1}{\mathfrak{q}r-1}\right)^{\frac{2(\mathfrak{q}-1)}{\mathfrak{q}}}\left(E\int_0^{\mathfrak{z}}\|\mathbb{B}(\varkappa)z^{\check{n}}(\varkappa)-\mathbb{B}(\varkappa)z^0(\varkappa)\|^{\mathfrak{q}}d\varkappa\right)^{\frac{2}{\mathfrak{q}}}\\
&\quad+\frac{5\vartheta^{2-2\beta+2r}\mathbb{K}^2\mathbb{K}_f(1+m_1\vartheta^2)}{\Gamma^2(r+1)}\|w^{\check{n}}_{\varkappa}-w^0_{\varkappa}\|^2\\
&\quad+\frac{5\vartheta^{1-2\beta+2r}\mathbb{K}^2Tr(\mathcal{Q})\mathbb{K}_h(1+m_2\vartheta^2)}{(2r-1)\Gamma^2(r)}\|w^{\check{n}}_{\varkappa}-w^0_{\varkappa}\|^2,
\end{aligned}
$$

which suggests that there exists $\mathscr{V}^*>0$. Thus, we obtain

$$
\sup_{\varkappa\in\mathscr{E}}E\|w^{\check{n}}(\varkappa)-w^0(\varkappa)\|^2\leq\mathscr{V}^*\|\mathbb{B}z^{\check{n}}-\mathbb{B}z^0\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})},\ \text{for } \mathfrak{z}\in\mathcal{E}. \tag{15}
$$

Hence, $\mathbb{B}$ is strongly continuous, and we have

$$
\|\mathbb{B}z^{\check{n}}-\mathbb{B}z^0\|^2_{L^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})}\xrightarrow{s}0 \text{ as } \check{n}\to\infty. \tag{16}
$$

Then, we have

$$
\|w^{\check{n}}-w^0\|^2_{\vartheta}\xrightarrow{s}0 \text{ as } \check{n}\to\infty,
$$

which is equivalent to

$$
\|y^{\check{n}}-y^0\|^2_{\vartheta}\xrightarrow{s}0 \text{ as } \check{n}\to\infty.
$$

Therefore,

$$
y^{\check{n}}\xrightarrow{s}y^0 \text{ in } \mathscr{G}'_\jmath \text{ as } \check{n}\to\infty.
$$

We can deduce that

$$
(y_{\mathfrak{z}}\times y,z)\to E\Big\{\int_0^{\vartheta}\mathscr{M}(\mathfrak{z},y_{\mathfrak{z}},y(\mathfrak{z}),z(\mathfrak{z}))d\mathfrak{z}\Big\}
$$

is sequentially lower semicontinuous in the strong topology of $L^1(\mathcal{E},\mathscr{G}_\jmath\times\mathcal{Y})$ and weak topology of $L^{\mathfrak{q}}(\mathcal{E},\mathcal{K})\subset L^1(\mathcal{E},\mathcal{K})$ from Balder's theorem in [49]. Hence, $\mathscr{L}$ is weakly lower semicontinuous on $L^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})$ and **(H7)**, $\mathscr{L} > -\infty$, and $\mathscr{L}$ succeeds its mimimum at $z^0 \in \mathbb{A}_{ad}$, i.e.,

$$\begin{aligned}\varpi = \lim_{\check{n}\to\infty} E\bigg\{\int_0^{\vartheta} \mathscr{M}(\mathfrak{z}, y^{\check{n}}_{\mathfrak{z}}, y^{\check{n}}(\mathfrak{z}), z^{\check{n}}(\mathfrak{z}))d\mathfrak{z}\bigg\} \\ \geq E\bigg\{\int_0^{\vartheta} \mathscr{M}(\mathfrak{z}, y^0_{\mathfrak{z}}, y^0(\mathfrak{z}), z^0(\mathfrak{z}))d\mathfrak{z}\bigg\} = \mathscr{L}(z^0) \geq \varpi,\end{aligned}$$

and the proof is completed. $\square$

6. Example

Consider the subsequent Hilfer fractional control system:

$$\begin{cases} D_{0^+}^{\delta,\frac{2}{3}}\Big[y(\mathfrak{z},\varsigma) - \int_{-\infty}^{0} b(\mathfrak{z},\varsigma)y(\mathfrak{z},\varsigma)d\varsigma\Big] = \frac{\partial^2}{\partial\varsigma^2}y(\mathfrak{z},\varsigma) + \int_{[0,1]}\mathscr{F}(\varsigma,\varkappa)z(\varkappa,\mathfrak{z})d\varkappa \\ +\Im\Big(\mathfrak{z}, \int_{-\infty}^{\mathfrak{z}}\Im_1(\varkappa-\mathfrak{z})y(\varkappa,\varsigma)d\varkappa, \int_0^{\mathfrak{z}}\int_{-\infty}^{0}\Im_2(\varkappa,\varsigma,\varepsilon-\varkappa)y(\varepsilon,\varsigma)d\varepsilon d\varkappa\Big) \\ +\aleph\Big(\mathfrak{z}, \int_{-\infty}^{\mathfrak{z}}\Im_1(\varkappa-\mathfrak{z})y(\varkappa,\varsigma)d\varkappa, \int_0^{\mathfrak{z}}\int_{-\infty}^{0}\Im_3(\varkappa,\varsigma,\varepsilon-\varkappa)y(\varepsilon,\varsigma)d\varepsilon d\varkappa\Big)\frac{dW(\mathfrak{z})}{d\mathfrak{z}}, \mathfrak{z}\in\mathcal{E}', \\ I^{(1-\delta)\frac{1}{3}}[y(\mathfrak{z},\varsigma)]|_{\mathfrak{z}=0} = y_0(\varsigma),\ 0\leq\varsigma\leq\pi, \\ y(\mathfrak{z},0) = y(\mathfrak{z},\pi) = 0,\ \mathfrak{z}\geq 0, \\ y(\mathfrak{z},\varsigma) = \phi(\mathfrak{z},\varsigma),\ \varsigma\in[0,\pi],\ -\infty<\mathfrak{z}\leq 0, \end{cases} \tag{17}$$

where $D_{0^+}^{\delta,\frac{2}{3}}$ is the Hilfer fractional derivative, $r = \frac{2}{3}$, $\delta \in [0,1]$, and $b(\mathfrak{z},\varsigma)$ represents the neutral function, which is discussed further below. On the filtered probability space $(\Omega, \mathfrak{J}, \widehat{\mathbb{P}})$, $W(\mathfrak{z})$ is a one-dimensional standard Wiener process in $\mathcal{Y}$. The functions $\phi(\mathfrak{z},\varsigma), \Im, \aleph, \Im_2$ and $\Im_3$ are continuous.

Consider $\mathcal{K} = \mathcal{Y} = L^2([0,\pi])$. The operator $A : \mathscr{D}(A) \subset \mathcal{Y}$ into $\mathcal{Y}$ is described by $A\mathfrak{u} = \mathfrak{u}''$, $\mathfrak{u} \in \mathscr{D}(A)$, where

$$\mathscr{D}(A) = \big\{\mathfrak{u}\in\mathcal{Y} : \mathfrak{u}, \mathfrak{u}' \text{ are absolutely continuous},\ \mathfrak{u}''\in\mathcal{Y},\ \mathfrak{u}(0) = \mathfrak{u}(\pi) = 0\big\}.$$

Then, A generates a strongly continuous semigroup $\mathcal{G}(\mathfrak{z})_{\mathfrak{z}\geq 0}$ which is compact, analytic, and self-adjoint. Further, A has a discrete spectrum, the eigenvalues are $-h^2, h \in \mathbb{N}$, and the corresponding normalized eigenvectors are

$$\omega_h(\varsigma) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(h\varsigma),\ h = 1, 2, \cdots.$$

Now, we consider the following assumptions:

(i) Provided that $\mathfrak{u} \in \mathscr{D}(A)$, then $A\mathfrak{u} = \sum_{h=1}^{\infty} h^2\langle\mathfrak{u}, \omega_h\rangle\omega_h$.

(ii) For each $\mathfrak{u} \in \mathcal{Y}$, $A^{-1/2} = \sum_{h=1}^{\infty}\frac{1}{h}\langle\mathfrak{u},\omega_h\rangle\omega_h$. In particular, $\|\mathcal{U}^{-1/2}\| = 1$.

(iii) The operator $A^{1/2}$ is presented by $A^{1/2}\mathfrak{u} = \sum_{h=1}^{\infty} h\langle\mathfrak{u},\omega_h\rangle\omega_h$ on the space $\mathscr{D}(A^{1/2}) = \{\mathfrak{u}\in\mathcal{Y},\ \sum_{h=1}^{\infty} h\langle\mathfrak{u},\omega_h\rangle\omega_h \in\mathcal{Y}\}$.

Consider $\jmath(\varkappa) = e^{2\varkappa},\ \varkappa < 0$, then $\ell = \int_{-\infty}^{0}\jmath(\varkappa)d\varkappa = \frac{1}{2}$. Assume that $\mathscr{G}_\jmath$ is a phase space endowed with the norm

$$\|\zeta\|_{\mathscr{G}_\jmath} = \int_{-\infty}^{0}\jmath(\varkappa)\sup_{\varkappa\leq\eta\leq 0}\big(E\|\zeta(\eta)\|^2\big)^{\frac{1}{2}}d\varkappa.$$

Then, $(\mathscr{G}_\jmath, \|\cdot\|_{\mathscr{G}_\jmath})$ is a Banach space.

For $(\mathfrak{z},\zeta) \in [0,\vartheta] \times \mathscr{G}_J$, where $\zeta(\theta,\varsigma) = \phi(\theta,\varsigma)$, $(\theta,\varsigma) \in (-\infty,0] \times [0,\pi]$, we consider

$$\begin{aligned}
y(\mathfrak{z})(\varsigma) &= y(\mathfrak{z},\varsigma),\\
g(\mathfrak{z},\zeta)(\varsigma) &= \int_{-\infty}^{0} \Im_2(\mathfrak{z},\varsigma,\varkappa)\zeta(\varkappa)(\varsigma)d\varkappa,\\
f\left(\mathfrak{z},\zeta,\int_0^{\mathfrak{z}} g(\varkappa,\zeta)d\varkappa\right)(\varsigma) &= \Im\left(\int_{-\infty}^{0}\Im_1(\varkappa)\zeta(\varkappa)(\varsigma)d\varkappa, \int_0^{\mathfrak{z}} g(\varkappa,\zeta)(\varsigma)d\varkappa\right),\\
\widetilde{g}(\mathfrak{z},\zeta)(\varsigma) &= \int_{-\infty}^{0} \Im_3(\mathfrak{z},\varsigma,\varkappa)\zeta(\varkappa)(\varsigma)d\varkappa,\\
h\left(\mathfrak{z},\zeta,\int_0^{\mathfrak{z}} \widetilde{g}(\varkappa,\zeta)d\varkappa\right)(\varsigma) &= \aleph\left(\int_{-\infty}^{0}\Im_1(\varkappa)\zeta(\varkappa)(\varsigma)d\varkappa, \int_0^{\mathfrak{z}} \widetilde{g}(\varkappa,\zeta)(\varsigma)d\varkappa\right),\\
\eth(\mathfrak{z},\zeta)(\varsigma) &= \int_{-\infty}^{0} b(\mathfrak{z},\varsigma)\zeta(\mathfrak{z},\varsigma)d\varsigma.
\end{aligned}$$

It is clear that the functions $g, f, \widetilde{g}, h$, and $\eth$ satisfy the assumptions (**H1**)–(**H3**) and (**H6**).

Now, we define the function $z : \mathcal{G}y([0,\pi]) \to \mathbb{R}$ as control, such that $z \in L^2(\mathcal{G}y([0,\pi]))$, which means that $\mathfrak{z} \to z(\mathfrak{z})$ is measurable. The set $\mathbb{A} = \{z \in \mathcal{K} : \|z\|_{\mathcal{K}} \leq \tilde{\mu}\}$, where $\tilde{\mu} \in L^2(\mathcal{E},\mathbb{R}^+)$. We limit the admissible control $\mathbb{A}_{ad}$ to be all $z \in L^2(\mathcal{G}y([0,\pi])) \ni \|z(\cdot,\mathfrak{z})\| \leq \tilde{\mu}(\mathfrak{z})$ a.e. $\mathfrak{z} \in \mathcal{E}$.

We can describe $\mathbb{B}(\mathfrak{z})z(\mathfrak{z})(\varsigma) = \int_{[0,1]} \mathscr{F}(\varsigma,\varkappa)z(\varkappa,\mathfrak{z})d\varkappa$ and assume the following:

(**i**) $\mathscr{F}$ is a continuous function.

(**ii**) $z \in L^2([0,\pi] \times \mathcal{E})$ and $\mathscr{M} : \mathcal{E} \times \mathscr{G}_J \times \mathcal{Y} \times \mathcal{K} \to \mathbb{R} \cup \{\infty\}$ are defined by

$$\begin{aligned}
&\mathscr{M}(\mathfrak{z}, y_{\mathfrak{z}}^{z}, y^{z}(\mathfrak{z}), z(\mathfrak{z}))(\varsigma)\\
&\quad= \int_{[0,\pi]}\int_{-\infty}^{0} |y^z(\mathfrak{z}+\varkappa,\varsigma)|^2 d\varkappa d\varsigma + \int_{[0,\pi]} |y^z(\mathfrak{z},\varsigma)|^2 d\varsigma + \int_{[0,\pi]} |z(\varsigma,\mathfrak{z})|^2 d\varsigma.
\end{aligned}$$

Then, the system (17) can be written in the form of (7). It is clear to see that all the requirements of Theorem 3 are fulfilled. Therefore, there exists an admissible control pair (y,z) such that the associated cost functional

$$\mathscr{L}(z) = E\left\{\int_0^{\vartheta} \mathscr{M}(\mathfrak{z}, y_{\mathfrak{z}}^{z}, y^{z}(\mathfrak{z}), z(\mathfrak{z}))d\mathfrak{z}\right\}$$

achieves its minimum.

7. Conclusions

This manuscript has studied the optimal control problem for Hilfer fractional neutral stochastic integrodifferential systems with infinite delay. We have examined the existence of mild solutions for the Hilfer fractional stochastic integrodifferential system with infinite delay by applying fractional calculus, semigroups, stochastic analysis theory, and Banach fixed point theorem. In addition, we have established the existence of mild solutions of the Hilfer fractional neutral stochastic delay integrodifferential system. Moreover, the existence of optimal control for the corresponding system has been discussed. Finally, we offer an example to demonstrate our results. Our future work shall focus on examining the optimal control for Sobolev-type Hilfer fractional stochastic integrodifferential inclusions with finite delay.

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