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Generalized Hermite-Hadamard Inequalities on Discrete Time Scales

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Abstract: This paper is concerned with some new Hermite-Hadamard inequalities on two types of time scales, $\mathbb{Z}$ and $\mathbb{N}_{c,h}$. Based on the substitution rules, we first prove the discrete Hermite-Hadamard inequalities on $\mathbb{Z}$ relating to the midpoint $\frac{a+b}{2}$ and extend them to discrete fractional forms. In addition, by using traditional methods, we prove discrete Hermite-Hadamard inequalities on $\mathbb{N}_{c,h}$ and explore the corresponding fractional inequalities involving the nabla h-fractional sums. Finally, two examples are given to illustrate the obtained results.

Keywords: generalized Hermite-Hadamard inequalities; discrete fractional calculus; convex functions; time scales

MSC: 26B25; 26D10; 26A33; 26D15

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1. Introduction

The convexity of functions has been a continual research topic. Many mathematicians have devoted themselves to introducing different forms of convexity such as r-convex, m-convex, quasi-convex, etc. (see [1]). At the same time, as a generalization of convexity on $\mathbb{R}$, a few researchers have studied the convexity on different sets. In [2], Atıcı and Yaldız defined the convexity of a real function on any time scale $\mathbb{T}$.

The convexity for a real function f on $\mathbb{T}$ means that for any $x, y \in \mathbb{T}$ with $x < y$,

$$f(\kappa x + (1-\kappa)y) \leq \kappa f(x) + (1-\kappa) f(y),$$

where $\kappa \in \mathbb{T}_{[x,y]}$ and $\mathbb{T}_{[x,y]} = \left\{ s | s = \frac{b-t}{b-a}, \text{ for } t \in [x,y] \cap \mathbb{T} \right\}$. Obviously, for $\mathbb{T} = \mathbb{R}$, the convex function f on $\mathbb{T}$ reduces to an ordinary convex function.

As an important direction for the theory of convex analysis, the Hermite-Hadamard (H-H) inequality has attracted much attention and has been generalized to many forms in recent decades (see, e.g., [3,4]). The classical H-H inequality is stated as follows: Let f be a real convex function defined on $[a,b]$, where $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(\tau) d\tau \leq \frac{f(a)+f(b)}{2}.$$

The H-H inequality also plays a meaningful role in fractional calculus. For instance, in [5], Sarikaya et al. first established the H-H inequality for Riemann-Liouville fractional integral operators: Let $f : [a,b] \to \mathbb{R}$ be a convex function; then, for $\alpha > 0$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[{}^{RL}\mathcal{I}^{\alpha}_{a+} f(b) + {}^{RL}\mathcal{I}^{\alpha}_{b-} f(a) \right] \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

In [6], Sarikaya et al. proposed an alternative form to the H-H inequalities for Riemann-Liouville fractional integral operators based on the midpoint of a and b: Assume f is a convex function defined on $[a,b]$; then, for $\alpha > 0$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[{}^{RL}\mathcal{I}^{\alpha}_{(\frac{a+b}{2})+}f(b) + {}^{RL}\mathcal{I}^{\alpha}_{(\frac{a+b}{2})-}f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

In 2021, Fernandez and Mohammed [7] proved the H-H inequality relating to Atangana-Baleanu fractional operators defined using Mittag-Leffler kernels. Later, Sahoo et al. [8] introduced the H-H inequalities related to k-Riemann-Liouville fractional operators for different kinds of convex functions. For more recent results which generalize the classical H-H inequality via different fractional operators, we refer the reader to the papers [9–13] and references therein.

On the other hand, the H-H inequality has been also extended to discrete calculus and even discrete fractional calculus. In [2], Atıcı and Yaldız proved discrete H-H inequalities on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left(\int_a^b f(\tau)\Delta\tau + \int_a^b f(\tau)\nabla\tau\right) \leq \frac{f(a)+f(b)}{2}.$$

They also improved it to be discrete fractional H-H inequalities on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\epsilon)}{2\eta(b-a)}\left[\Delta^{-\epsilon}_{b-1}f(\tau)|_{\tau=a-\epsilon} + \nabla^{-\epsilon}_{a}f(\tau)|_{\tau=b}\right] \leq \frac{f(a)+f(b)}{2}, \tag{3}$$

where $\epsilon > 0$ and

$$\eta = \int_{\mathbb{T}_{[a,b]}} ((b-a)\tau + (\epsilon-1))^{(\alpha-1)}\Delta\tau,$$

which can be seen as a discrete counterpart of (1). Recently, Mohammed et al. [14] proved some Hermite-Hadamard and Opial inequalities using the integration by parts and chain rule formulas on time scales. For more of the H-H inequalities on time scales, one can see [15–21] and the references therein.

So far, the Hermite-Hadamard inequalities for the midpoint on the time scale have rarely been studied. The goal of this article is to prove the generalized H-H inequalities for discrete convex functions on $\mathbb{Z}$ and $\mathbb{N}_{c,h}$ and establish two H-H inequalities involving the nabla fractional sum operators, as similar forms to discrete versions of (2).

The whole study work has been arranged as follows: In Section 2, we recall some basic definitions and theorems. In Section 3, we establish some discrete H-H inequalities on the time scales $\mathbb{Z}$ and $\mathbb{N}_{c,h}$, respectively. In addition, they are extended to discrete fractional forms. Furthermore, there are two examples of the H-H inequalities in Section 4. Finally, the conclusion of this paper is given in Section 5.

2. Preliminaries

In this section, we recall some definitions and results used in the remaining sections. Let $\mathbb{T}$ be any time scale. The sets considered in this article are $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$, $\mathbb{N}_{a,h} = \{a,\ a+h,\ a+2h,\dots\}$ and ${}_{b,h}\mathbb{N} = \{\dots, b-2h,\ b-h,\ b\}$. For $h=1$, we write $\mathbb{N}_{a,h}$ and ${}_{b,h}\mathbb{N}$ to denote $\mathbb{N}_a$ and ${}_b\mathbb{N}$, respectively.

For $\tau \in \mathbb{T}$, we denote the forward jump and the backward jump operators by $\sigma(\tau)$ and $\rho(\tau)$, respectively. The forward graininess and the backward graininess operators are given by $\mu(\tau) = \sigma(\tau) - \tau$ and $v(\tau) = \tau - \rho(\tau)$, respectively. If $\mathbb{T} = \mathbb{N}_{c,h}$, we denote $\sigma(\tau)$ and $\rho(\tau)$ by $\sigma_h(\tau) = \tau + h$ and $\rho_h(\tau) = \tau - h$ with $h > 0$, respectively. Let $f(t)$ be a function defined on $\mathbb{T}$; then, for $k_1,\ k_2 \in \mathbb{T}$ and $k_1 > k_2$, $\sum_{l\in[k_1,k_2]\cap\mathbb{T}} f(l) = 0$, i.e., empty sums are taken to be 0.

Assume t and $\epsilon \in \mathbb{R}$ are arbitrary real numbers; then, the rising and falling h-factorial functions are given by

$$t_h^{\overline{\epsilon}} = h^{\epsilon} \frac{\Gamma\left(\frac{t}{h} + \epsilon\right)}{\Gamma\left(\frac{t}{h}\right)}, \quad \frac{t}{h}, \frac{t}{h} + \epsilon \notin {}_0\mathbb{N},$$

and

$$t_h^{(\epsilon)} = h^{\epsilon} \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - \epsilon\right)}, \quad \frac{t}{h} + 1, \frac{t}{h} + 1 - \epsilon \notin {}_0\mathbb{N},$$

where the gamma function is defined by $\Gamma(s) = \int_0^{\infty} \zeta^{s-1} e^{-\zeta} d\zeta$. In particular, for $h = 1$, we obtain $t^{\overline{\epsilon}} = \frac{\Gamma(t+\epsilon)}{\Gamma(t)}$, $t^{(\epsilon)} = \frac{\Gamma(t+1)}{\Gamma(t-\epsilon+1)}$.

Next, we recall the differences and sums of f on $\mathbb{T}$.

Definition 1 ([22]). *Let real function f defined on $\mathbb{T}$ be given. Then, for $\tau \in \mathbb{T}$, the nabla and delta differences of f are given by*

$$f^{\nabla}(\tau) = h\frac{f(\sigma(\tau)) - f(\tau)}{\mu(\tau)}, \tag{4}$$

$$f^{\Delta}(\tau) = h\frac{f(\tau) - f(\rho(\tau))}{v(\tau)}. \tag{5}$$

The nabla and delta sums of f on $\mathbb{T}$ are given by

$$\int_a^b f(s)\nabla s = \sum_{k \in (a,b] \cap \mathbb{T}} f(k)v(k), \tag{6}$$

$$\int_a^b f(s)\Delta s = \sum_{k \in [a,b) \cap \mathbb{T}} f(k)\mu(k), \tag{7}$$

where $a, b, c \in \mathbb{R}$ and $a, b \in \mathbb{T}$.

Especially for $\mathbb{T} = \mathbb{N}_{c,h}$, we have

$$\int_a^b f(s)\nabla_h s = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} f(kh)h \quad \text{and} \quad \int_a^b f(s)\Delta_h s = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h,$$

where $a, b \in \mathbb{N}_{c,h}$.

Definition 2 (*The nabla h-fractional sums [23]*)**.** *Let real function f and $\alpha > 0$ be given. Then, the nabla left and right h-fractional sums are defined as*

$$\left({}_a\nabla_h^{-\alpha} f\right)(t) = \frac{h}{\Gamma(\alpha)} \sum_{k=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho_h(kh))_h^{\overline{\alpha-1}} f(kh), \quad t \in \mathbb{N}_{a+h,h}, \tag{8}$$

$$\left({}_h\nabla_b^{-\alpha} f\right)(t) = \frac{h}{\Gamma(\alpha)} \sum_{k=\frac{t}{h}}^{\frac{b}{h}-1} (kh - \rho_h(t))_h^{\overline{\alpha-1}} f(kh), \quad t \in {}_{b-h,h}\mathbb{N} \tag{9}$$

Definition 3 (*The delta h-fractional sums [24]*)**.** *Let real function f and $\alpha > 0$ be given. Then, the delta left and right h-fractional sums are defined as*

$$\left({}_a\Delta_h^{-\alpha} f\right)(t) = \frac{h}{\Gamma(\alpha)} \sum_{l=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t - \sigma_h(lh))_h^{(\alpha-1)} f(lh), \quad t \in \mathbb{N}_{a+\alpha h,h}, \tag{10}$$

$$\left({}_{h}\Delta_{b}^{-\alpha}f\right)(t) = \frac{h}{\Gamma(\alpha)} \sum_{l=\frac{t}{h}+\alpha}^{\frac{b}{h}} (lh - \sigma_h(t))_h^{(\alpha-1)} f(lh),\ t \in {}_{b-\alpha h,h}\mathbb{N}. \tag{11}$$

If $h = 1$, we denote $\left({}_{a}\nabla_{h}^{-\alpha}f\right)(\cdot)$,$\left({}_{h}\nabla_{b}^{-\alpha}f\right)(\cdot)$, $\left({}_{a}\Delta_{h}^{-\alpha}f\right)(\cdot)$ and $\left({}_{h}\Delta_{b}^{-\alpha}f\right)(\cdot)$ by $\left({}_{a}\nabla^{-\alpha}f\right)(\cdot)$, $\left(\nabla_{b}^{-\alpha}f\right)(\cdot)$, $\left({}_{a}\Delta^{-\alpha}f\right)(\cdot)$ and $\left(\Delta_{b}^{-\alpha}f\right)(\cdot)$, respectively.

Remark 1. *There is an equality between the delta right fractional sum and the nabla right fractional sum*

$$\left(\Delta_{b-1}^{-\alpha}f\right)(t)|_{t=a-\alpha} = \left(\nabla_{b}^{-\alpha}f\right)(t)|_{t=a}. \tag{12}$$

In fact,

$$\begin{aligned}
\left(\Delta_{b-1}^{-\alpha}f\right)(t)|_{t=a-\alpha} =& \frac{1}{\Gamma(\alpha)}\left[\sum_{k=t+\alpha}^{b-1}(k-\sigma(t))^{(\alpha-1)}f(k)\right]_{t=a-\alpha} \\
=& \frac{1}{\Gamma(\alpha)}\left[\sum_{k=t+\alpha}^{b-1}(k-\alpha-\rho(t))^{\overline{\alpha-1}}f(k)\right]_{t=a-\alpha} \\
=& \frac{1}{\Gamma(\alpha)}\sum_{k=a}^{b-1}(k-\rho(a))^{\overline{\alpha-1}}f(k) \\
=& \frac{1}{\Gamma(\alpha)}\left[\sum_{k=t}^{b-1}(k-\rho(t))^{\overline{\alpha-1}}f(k)\right]_{t=a} \\
=& \left(\nabla_{b}^{-\alpha}f\right)(t)|_{t=a},
\end{aligned}$$

where we use that

$$(k+\alpha-\sigma(t))^{(\alpha-1)} = (k-\rho(t))^{\overline{\alpha-1}}.$$

The following substitution rules on time scale $\mathbb{T}$ are necessary for the proof of discrete H-H inequalities on $\mathbb{Z}$.

Theorem 1 ([22])**.** *Let $\Phi : \mathbb{T} \to \mathbb{R}$ be strictly increasing and differentiable with rd-continuous derivative. If $\hat{\mathbb{T}} := \Phi(\mathbb{T})$ is a time scale and $F : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then for $a, b \in \mathbb{T}$,*

$$\int_a^b F(\tau)\Phi^{\Delta}(\tau)\Delta\tau = \int_{\Phi(a)}^{\Phi(b)} (F \circ \Phi^{-1})(\tau)\hat{\Delta}\tau, \tag{13}$$

where $\hat{\Delta}$ is derivative operator on the time scale $\hat{\mathbb{T}}$.

Theorem 2 ([25])**.** *Let $\Phi : \mathbb{T} \to \mathbb{R}$ be strictly decreasing and differentiable with rd-continuous derivative. If $\hat{\mathbb{T}} := \Phi(\mathbb{T})$ is a time scale and $F : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then for $a, b \in \mathbb{T}$*

$$\int_a^b F(\tau)(-\Phi^{\Delta})(\tau)\Delta\tau = \int_{\Phi(b)}^{\Phi(a)} (F \circ \Phi^{-1})(\tau)\hat{\nabla}\tau, \tag{14}$$

where $\hat{\nabla}$ is derivative operator on the time scale $\hat{\mathbb{T}}$.

3. Main Results

In this part, we construct and establish some generalized H-H inequalities for convex functions defined on $\mathbb{Z}$ and $\mathbb{N}_{c,h}$ using two methods.

3.1. Generalized Hermite-Hadamard Inequalities on Discrete Time Scale $\mathbb{Z}$

In this subsection, two generalized H-H inequalities on $\mathbb{Z}$ are deduced by using the substitution rules shown in Theorems 1 and 2. Here, the notations are used:

$$\mathbb{T}_{[\frac{a+b}{2},b]} = \left\{\kappa \middle| \kappa = \frac{2(b-s)}{b-a} \quad \text{for} \quad s \in \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}}\right\},$$

and

$$\mathbb{T}_{[a,\frac{a+b}{2}]} = \left\{\kappa \middle| \kappa = \frac{2(s-b)}{b-a} \quad \text{for} \quad s \in \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}}\right\}.$$

First, we prove the discrete H-H inequalities on $\mathbb{Z}$ relating to the midpoint $\frac{a+b}{2}$.

Theorem 3. *Let $a, b \in \mathbb{Z}$ with $a < b$ and $f : \mathbb{Z} \to \mathbb{R}$ defined on $[a,b]_{\mathbb{Z}}$ be a convex function. If $\frac{a+b}{2} \in \mathbb{Z}$, then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau + \int_{\frac{a+b}{2}}^b f(\tau)\nabla\tau\right] \leq \frac{f(a)+f(b)}{2}. \tag{15}$$

Proof. Let $\kappa \in \mathbb{T}_{[\frac{a+b}{2},b]} \setminus \{0,1\}$ be fixed. We define

$$x = \frac{\kappa}{2}a + \frac{2-\kappa}{2}b, \quad y = \frac{2-\kappa}{2}a + \frac{\kappa}{2}b$$

are in $[a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2}$. Since f is convex on $[x,y]_{\mathbb{Z}}$, we can deduce

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right) + f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\right] \\ &\leq \frac{1}{2}\left[\frac{\kappa}{2}f(a) + \frac{2-\kappa}{2}f(b) + \frac{2-\kappa}{2}f(a) + \frac{\kappa}{2}f(b)\right] = \frac{f(a)+f(b)}{2}. \end{aligned} \tag{16}$$

Integrating the above inequalities with respect to κ over $\mathbb{T}_{[\frac{a+b}{2},b]}$, then we have

$$\begin{aligned} &\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} f\left(\frac{a+b}{2}\right)\hat{\Delta}\kappa \\ \leq &\frac{1}{2}\left[\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right)\hat{\Delta}\kappa + \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa\right] \\ \leq &\frac{1}{2}\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} f(a) + f(b)\hat{\Delta}\kappa. \end{aligned} \tag{17}$$

Hence,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}[h_1 + h_2] \leq \frac{1}{2}[f(a) + f(b)], \tag{18}$$

where

$$h_1 = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right)\hat{\Delta}\kappa, \quad h_2 = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa.$$

Here, we calculate h_1 and h_2, separately.
First, we prove that

$$h_1 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(\tau)\nabla\tau. \tag{19}$$

Let $\Phi_1(\tau) : \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}} \to \mathbb{T}_{[\frac{a+b}{2}, b]}$ be defined by $\Phi_1(\tau) = \frac{2(b-\tau)}{b-a}$ with $\tau \in [\frac{a+b}{2}, b]_{\mathbb{Z}}$. Then, $\Phi_1(\tau)$ is decreasing:

$$\Phi_1^{-1}(\tau) = \frac{\tau}{2}a + \frac{2-\tau}{2}b \text{ and } \left(-\Phi_1^{\nabla}\right)(\tau) = \frac{2}{b-a}.$$

Making use of Theorem 2, we obtain

$$\begin{aligned} h_1 &= \int_{T_{[\frac{a+b}{2}, b]}} f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right)\hat{\Delta}\kappa = \int_{0=\Phi_1(b)}^{1=\Phi_1\left(\frac{a+b}{2}\right)} \left(f \circ \Phi_1^{-1}\right)(\kappa)\hat{\Delta}\kappa \\ &= \int_{\frac{a+b}{2}}^{b} f(\tau)\left(-\Phi_1^{\nabla}\right)(\tau)\nabla\tau = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau. \end{aligned} \tag{20}$$

Next, we claim

$$h_2 = \frac{2}{b-a}\int_b^{\frac{a+b}{2}} f(\tau)\Delta\tau. \tag{21}$$

Assume $\kappa = \frac{2(b-\tau)}{b-a}$ with $\tau \in \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}}$. Setting $\hat{\tau} = a + b - \tau$ and $\hat{\kappa} = \frac{2(\hat{\tau}-a)}{b-a}$, we have $\hat{\tau} \in \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}}$ and $\hat{\kappa} \in \mathbb{T}_{[a, \frac{a+b}{2}]}$. Then

$$h_2 = \int_{\mathbb{T}_{[\frac{a+b}{2}, b]}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa = \int_{\mathbb{T}_{[a, \frac{a+b}{2}]}} f\left(\frac{2-\hat{\kappa}}{2}a + \frac{\hat{\kappa}}{2}b\right)\tilde{\Delta}\hat{\kappa}. \tag{22}$$

Let $\Phi_2(\tau) : \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}} \to \mathbb{T}_{[a, \frac{a+b}{2}]}$ be a map defined as $\Phi_2(\tau) = \frac{2(\tau-a)}{b-a}$. Then, $\Phi_2(\tau)$ is increasing:

$$\Phi_2^{-1}(\tau) = \frac{2-\tau}{2}a + \frac{\tau}{2}b \quad \text{and} \quad \Phi_2^{\Delta}(\tau) = \frac{2}{b-a}.$$

Hence, using Theorem 1, we have

$$\begin{aligned} h_2 &= \int_{0=\Phi_2(a)}^{1=\Phi_2(\frac{a+b}{2})} \left(f \circ \Phi_2^{-1}\right)(\hat{\kappa})\tilde{\Delta}\hat{\kappa} = \int_a^{\frac{a+b}{2}} f(\tau)\Phi_2^{\Delta}(\tau)\Delta\tau \\ &= \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau. \end{aligned} \tag{23}$$

Inserting h_1 and h_2 into (18), we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \le \frac{f(a)+f(b)}{2},$$

which completes the proof of the theorem. □

Next, the above H-H inequalities are extended to discrete fractional forms involving the nabla h-fractional sums on $\mathbb{Z}$.

Theorem 4. *Let $a, b \in \mathbb{Z}$ with $a < b$ and $f : \mathbb{Z} \to \mathbb{R}$ defined on $[a, b]_{\mathbb{Z}}$ be a convex function. If $\frac{a+b}{2} \in \mathbb{Z}$, then for $\epsilon > 0$, we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\epsilon)}{\Omega(b-a)}\left[{}_{\frac{a+b}{2}}\nabla^{-\epsilon} f(b) + \nabla^{-\epsilon}_{\frac{a+b}{2}} f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{24}$$

where

$$\Omega = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} \hat{\Delta}\kappa.$$

Proof. Let $\kappa \in \mathbb{T}_{[\frac{a+b}{2},b]} \setminus \{0,1\}$ be fixed. We define

$$x = \frac{\kappa}{2}a + \frac{2-\kappa}{2}b, \quad y = \frac{2-\kappa}{2}a + \frac{\kappa}{2}b$$

are in $[a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2}$. Since f is convex on $[x,y]_{\mathbb{Z}}$, we can deduce

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leq& \frac{1}{2}\left[f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right) + f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\right] \\ \leq& \frac{1}{2}\left[\frac{\kappa}{2}f(a) + \frac{2-\kappa}{2}f(b) + \frac{2-\kappa}{2}f(a) + \frac{\kappa}{2}f(b)\right] = \frac{f(a)+f(b)}{2}. \end{aligned} \tag{25}$$

Multiplying each term by $\left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}}$ and integrating with respect to κ on $\mathbb{T}_{[\frac{a+b}{2},b]}$, then

$$\begin{aligned} &\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{a+b}{2}\right)\hat{\Delta}\kappa \\ \leq& \frac{1}{2}\left[\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right)\hat{\Delta}\kappa \right. \\ &\left. + \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa\right] \\ \leq& \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} \frac{f(a)+f(b)}{2}\hat{\Delta}\kappa, \end{aligned} \tag{26}$$

that is,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leq& \frac{1}{2\Omega}\left[\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right)\hat{\Delta}\kappa \right. \\ &\left. + \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa\right] \\ \leq& \frac{f(a)+f(b)}{2}, \end{aligned} \tag{27}$$

where

$$\Omega = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} \hat{\Delta}\kappa.$$

Define

$$h_1 = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right)\hat{\Delta}\kappa,$$

$$h_2 = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa.$$

First, we claim that

$$h_1 = \frac{2\Gamma(\epsilon)}{b-a} \, {}_{\frac{a+b}{2}}\nabla^{-\epsilon} f(b). \tag{28}$$

Let $\Phi_1(\tau) : \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}} \to \mathbb{T}_{[\frac{a+b}{2}, b]}$ be defined as $\Phi_1(\tau) = \frac{2(b-\tau)}{b-a}$ with $\tau \in [\frac{a+b}{2}, b]_{\mathbb{Z}}$. Then, $\Phi_1(\tau)$ is decreasing:

$$\Phi_1^{-1}(\tau) = \frac{\tau}{2}a + \frac{2-\tau}{2}b \text{ and } \left(-\Phi_1^{\nabla}\right)(\tau) = \frac{2}{b-a}.$$

In addition, let $g_1(\tau) = (b-\tau+1)^{\overline{\epsilon-1}}$ and $F_1(\tau) = g_1(\tau)f(\tau)$. Then, we obtain

$$\begin{aligned} F_1(\Phi_1^{-1}(\tau)) &= g_1(\Phi_1^{-1}(\tau)) f(\Phi_1(\tau)^{-1}) \\ &= \left(b - \frac{\tau}{2}a - \frac{2-\tau}{2}b + 1\right)^{\overline{\epsilon-1}} f\left(\frac{\tau}{2}a + \frac{2-\tau}{2}b\right) \\ &= \left(\frac{b-a}{2}\tau + 1\right)^{\overline{\epsilon-1}} f\left(\frac{\tau}{2}a + \frac{2-\tau}{2}b\right). \end{aligned} \tag{29}$$

Hence,

$$h_1 = \int_{\mathbb{T}_{[\frac{a+b}{2}, b]}} F_1(\Phi_1^{-1}(\kappa)) \hat{\Delta}\kappa. \tag{30}$$

Making use of Theorem 2, we obtain

$$\begin{aligned} h_1 &= \int_{0=\Phi_1(b)}^{1=\Phi_1\left(\frac{a+b}{2}\right)} \left(F_1 \circ \Phi_1^{-1}\right)(\kappa)\hat{\Delta}\kappa = \int_{\frac{a+b}{2}}^{b} F_1(\tau)\left(-\Phi_1^{\nabla}\right)(\tau)\nabla\tau \\ &= \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} (b-\tau+1)^{\overline{\epsilon-1}} f(\tau)\nabla\tau = \frac{2\Gamma(\epsilon)}{b-a} \, {}_{\frac{a+b}{2}}\nabla^{-\epsilon} f(b). \end{aligned} \tag{31}$$

On the other hand, we assert that

$$h_2 = \frac{2\Gamma(\epsilon)}{b-a} \nabla_{\frac{a+b}{2}}^{-\epsilon}(a). \tag{32}$$

Assume $\kappa = \frac{2(b-\tau)}{b-a}$ for $\tau \in \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}}$. Setting $\hat{\tau} = a+b-\tau$ and $\hat{\kappa} = \frac{2(\hat{\tau}-a)}{b-a}$, we have $\hat{\tau} \in \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}}$ and $\hat{\kappa} \in \mathbb{T}_{[a, \frac{a+b}{2}]}$. Then

$$\begin{aligned} h_2 &= \int_{\mathbb{T}_{[\frac{a+b}{2}, b]}} \left(\frac{b-a}{2}\kappa + 1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\hat{\Delta}\kappa \\ &= \int_{\mathbb{T}_{[a, \frac{a+b}{2}]}} \left(\frac{b-a}{2}\hat{\kappa} + 1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\hat{\kappa}}{2}a + \frac{\hat{\kappa}}{2}b\right)\tilde{\Delta}\hat{\kappa}. \end{aligned} \tag{33}$$

Let $\Phi_2(\tau) : \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}} \to \mathbb{T}_{[a, \frac{a+b}{2}]}$ be a map defined as $\Phi_2(\tau) = \frac{2(\tau-a)}{b-a}$. Then, $\Phi_2(\tau)$ is increasing:

$$\Phi_2^{-1}(\tau) = \frac{2-\tau}{2}a + \frac{\tau}{2}b \quad \text{and} \quad \Phi_2^{\Delta}(\tau) = \frac{2}{b-a}.$$

In addition, we define $g_2(\tau) = (\tau-a+1)^{\overline{\epsilon-1}}$ and $F_2(\tau) = g_2(\tau)f(\tau)$. Then, we obtain

$$F_2(\Phi_2^{-1}(\tau)) = \left(\frac{b-a}{2}\tau - a + 1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\tau}{2}a + \frac{\tau}{2}b\right). \tag{34}$$

Hence, using Theorem 1, we have

$$\begin{aligned}
h_2 &= \int_{\mathbb{T}_{[a,\frac{a+b}{2}]}} \left(\frac{b-a}{2}\kappa+1\right)^{\overline{\epsilon-1}} f\left(\frac{2-\kappa}{2}a+\frac{\kappa}{2}b\right)\tilde{\Delta}\kappa \\
&= \int_{0=\Phi_2(a)}^{1=\Phi_2\left(\frac{a+b}{2}\right)} F_2\circ\Phi_2^{-1}(\kappa)\tilde{\Delta}\kappa = \int_a^{\frac{a+b}{2}} F_2(\tau)\Phi_2^{\Delta}(\tau)\Delta\tau \qquad (35)\\
&= \frac{2}{b-a}\int_a^{\frac{a+b}{2}}(\tau-a+1)^{\overline{\epsilon-1}}f(\tau)\Delta\tau = \frac{2\Gamma(\epsilon)}{b-a}\nabla^{-\epsilon}_{\frac{a+b}{2}}f(a).
\end{aligned}$$

Thus, we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\epsilon)}{\Omega(b-a)}\left[{}_{\frac{a+b}{2}}\nabla^{-\epsilon}f(b)+\nabla^{-\epsilon}_{\frac{a+b}{2}}(a)\right] \le \frac{f(a)+f(b)}{2},$$

proving the theorem. □

Remark 2. *Note that, concerning the discrete fractional H-H inequalities on $\mathbb{Z}$, we obtain*

1. *For $\epsilon=1$, Theorem 4 reduces to Theorem 3.*
2. *From Remark 1, the following inequalities for the nabla left and delta right fractional sums are equivalent to inequalities (24):*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\epsilon)}{2\Omega(b-a)}\left[{}_{\frac{a+b}{2}}\nabla^{-\epsilon}f(b)+\ \nabla^{-\epsilon}_{\frac{a+b}{2}-1}f(a-\epsilon)\right] \le \frac{f(a)+f(b)}{2},$$

which can be seen as the midpoint type of the inequalities (3).

3.2. Generalized Hermite-Hadamard Inequalities on Discrete Time Scale $\mathbb{N}_{c,h}$

Here, we use the definitions of h-sum operators to prove the discrete H-H inequalities on $\mathbb{N}_{c,h}$. Denote $\mathbb{T}_{[\frac{a+b}{2},b]}$ by

$$\mathbb{T}_{[\frac{a+b}{2},b]} = \left\{\kappa|\kappa=\frac{2(b-s)}{b-a} \quad \text{for} \quad s\in\left[\frac{a+b}{2},b\right]_{\mathbb{N}_{c,h}}\right\}.$$

Theorem 5. *Let $a,b\in\mathbb{N}_{c,h}$ with $a<b$ and $f:\mathbb{N}_{c,h}\to\mathbb{R}$ defined on $[a,b]_{\mathbb{N}_{c,h}}$ be a convex function. If $\frac{a+b}{2}\in\mathbb{N}_{c,h}$, then inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(\tau)\Delta_h\tau + \int_{\frac{a+b}{2}}^b f(\tau)\nabla_h\tau\right] \le \frac{f(a)+f(b)}{2} \qquad (36)$$

hold.

Proof. Fix $\kappa=\frac{2(b-\tau)}{b-a}\in\mathbb{T}_{[\frac{a+b}{2},b]}\setminus\{0,1\}$ with $\tau\in\left[\frac{a+b}{2},b\right]_{\mathbb{N}_{c,h}}$. Suppose

$$x=\frac{\kappa}{2}a+\frac{2-\kappa}{2}b,\ y=\frac{2-\kappa}{2}a+\frac{\kappa}{2}b,$$

then $x,y\in[a,b]_{\mathbb{N}_{c,h}}$ and $\frac{x+y}{2}=\frac{a+b}{2}$. Due to the convexity of f, we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(\frac{\kappa}{2}a+\frac{2-\kappa}{2}b\right)+f\left(\frac{2-\kappa}{2}a+\frac{\kappa}{2}b\right)\right] \le \frac{f(a)+f(b)}{2}. \qquad (37)$$

Setting $\tau = \frac{a+b}{2} + kh$ with $k \in \mathbb{N}$, then $\kappa = 1 - \frac{2kh}{b-a}$ and

$$f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right) = f\left(\frac{a+b}{2} + kh\right), \quad f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right) = f\left(\frac{a+b}{2} - kh\right). \tag{38}$$

Therefore,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{a+b}{2} + kh\right) + f\left(\frac{a+b}{2} - kh\right)\right] \leq \frac{f(a)+f(b)}{2}. \tag{39}$$

Taking sum over k from 1 to $\frac{b-a}{2h}$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[\sum_{k=1}^{\frac{b-a}{2h}} f\left(\frac{a+b}{2} + kh\right) + \sum_{k=1}^{\frac{b-a}{2h}} f\left(\frac{a+b}{2} - kh\right)\right] \leq \frac{f(a)+f(b)}{2}. \tag{40}$$

Define

$$h_1 = h\sum_{k=1}^{\frac{b-a}{2h}} f\left(\frac{a+b}{2} + kh\right), \quad h_2 = h\sum_{k=1}^{\frac{b-a}{2h}} f\left(\frac{a+b}{2} - kh\right).$$

Next, we calculate h_1 and h_2, separately.
For h_1, setting $l = k + \frac{a+b}{2h}$, we obtain

$$h_1 = h \sum_{l=\frac{a+b}{2h}+1}^{\frac{b}{h}} f(lh) = \int_{\frac{a+b}{2}}^{b} f(\tau)\nabla_h \tau. \tag{41}$$

For h_2, setting $j = \frac{a+b}{2h} - k$, we obtain

$$h_2 = h \sum_{j=\frac{a}{h}}^{\frac{a+b}{2h}-1} f(jh) = \int_{a}^{\frac{a+b}{2}} f(\tau)\Delta_h \tau. \tag{42}$$

Hence,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}} f(\tau)\Delta_h\tau + \int_{\frac{a+b}{2}}^{b} f(\tau)\nabla_h\tau\right] \leq \frac{f(a)+f(b)}{2},$$

which are the desired inequalities. □

Remark 3. *Note that the inequalities (36) with $h = 1$ and $c \in \mathbb{Z}$ deduce to the inequalities (15) in Theorem 3.*

The next result is important to prove the discrete fractional H-H inequalities on $\mathbb{N}_{c,h}$.

Lemma 1. *Let a real function f be defined on $\mathbb{N}_{c,h}$ and $\epsilon > 0$ be given. Suppose $a, b \in \mathbb{N}_{c,h}$, $a < b$ and $\frac{a+b}{2} \in \mathbb{N}_{c,h}$. Then we obtain*

$$_{\frac{a+b}{2}}\nabla_h^{-\epsilon}(Qf)(\tau)|_{\tau=b} = {}_h\nabla_{\frac{a+b}{2}}^{-\epsilon} f(\tau)|_{\tau=a}, \tag{43}$$

where $(Qf)(\tau) = f(a+b-\tau)$.

Proof. Using Definition 2, we can show that

$$ {}_{\frac{a+b}{2}}\nabla_h^{-\epsilon}(Qf)(\tau)|_{\tau=b} = \frac{h}{\Gamma(\epsilon)} \sum_{l=\frac{a+b}{2h}+1}^{\frac{b}{h}} (b-\rho_h(lh))_h^{\overline{\epsilon-1}} f(a+b-lh). \tag{44} $$

Then, we set $i = \frac{a+b-l}{lh}$ to obtain

$$ \begin{aligned} {}_{\frac{a+b}{2}}\nabla_h^{-\epsilon}(Qf)(\tau)|_{t=b} &= \frac{h}{\Gamma(\epsilon)} \sum_{i=\frac{a+b}{2h}-1}^{\frac{a}{h}} (ih-\rho_h(a))_h^{\overline{\epsilon-1}} f(ih) \\ &= \frac{1}{\Gamma(\epsilon)} \int_a^{\frac{a+b}{2}} (s-\rho_h(a))_h^{\overline{\epsilon-1}} f(s)\Delta_h s \\ &= {}_h\nabla_{\frac{a+b}{2}}^{-\epsilon} f(\tau)|_{\tau=a}. \end{aligned} \tag{45} $$

Thus, Lemma 1 is proved. □

From Lemma 1, the following theorem can be proved.

Theorem 6. *Let $a, b \in \mathbb{N}_{c,h}$ with $a < b$ and $f : \mathbb{N}_{c,h} \to \mathbb{R}$ defined on $[a,b]_{\mathbb{N}_{c,h}}$ be a convex function. If $\frac{a+b}{2} \in \mathbb{N}_{c,h}$, then for $\epsilon > 0$, the inequalities*

$$ f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\epsilon)}{2\Omega}\left[{}_{\frac{a+b}{2}}\nabla_h^{-\epsilon} f(b) + {}_h\nabla_{\frac{a+b}{2}}^{-\epsilon} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{46} $$

hold, where

$$ \Omega = \int_{\frac{a+b}{2}}^{b} (b-\rho_h(\tau))_h^{\overline{\epsilon-1}} \nabla_h \tau. $$

Proof. Fix $\kappa = \frac{2(b-\tau)}{b-a} \in \mathbb{T}_{[\frac{a+b}{2}, b]} \setminus \{0,1\}$ with $\tau \in \left[\frac{a+b}{2}, b\right]_{\mathbb{N}_{c,h}}$. Suppose

$$ x = \frac{\kappa}{2}a + \frac{2-\kappa}{2}b,\ y = \frac{2-\kappa}{2}a + \frac{\kappa}{2}b, $$

then $x, y \in [a,b]_{\mathbb{N}_{c,h}}$ and $\frac{x+y}{2} = \frac{a+b}{2}$. Due to the convexity of f, we obtain

$$ f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right) + f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right)\right] \leq \frac{f(a)+f(b)}{2}. \tag{47} $$

Setting $\tau = \frac{a+b}{2} + kh$, then $\kappa = 1 - \frac{2kh}{b-a}$ and

$$ f\left(\frac{\kappa}{2}a + \frac{2-\kappa}{2}b\right) = f\left(\frac{a+b}{2} + kh\right), \quad f\left(\frac{2-\kappa}{2}a + \frac{\kappa}{2}b\right) = f\left(\frac{a+b}{2} - kh\right). \tag{48} $$

Therefore,

$$ f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{a+b}{2} + kh\right) + f\left(\frac{a+b}{2} - kh\right)\right] \leq \frac{f(a)+f(b)}{2}. \tag{49} $$

Multiplying each term by $\left(\tau-\rho_h(kh+\frac{a+b}{2})\right)_h^{\overline{\epsilon-1}}$ and taking sum over k from 1 to $\frac{2\tau-a-b}{2h}$ with $\tau\in\left[\frac{a+b}{2},b\right]_{\mathbb{N}_{c,h}}$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\frac{2\tau-a-b}{2h}}\left(\tau-\rho_h\left(kh+\frac{a+b}{2}\right)\right)_h^{\overline{\epsilon-1}}f\left(\frac{a+b}{2}\right)\\
&\leq\frac{1}{2}\left[\sum_{k=1}^{\frac{2\tau-a-b}{2h}}\left(\tau-\rho_h\left(kh+\frac{a+b}{2}\right)\right)_h^{\overline{\epsilon-1}}f\left(\frac{a+b}{2}+kh\right)\right.\\
&\quad\left.+\sum_{k=1}^{\frac{2\tau-a-b}{2h}}\left(\tau-\rho_h\left(kh+\frac{a+b}{2}\right)\right)_h^{\overline{\epsilon-1}}f\left(\frac{a+b}{2}-kh\right)\right]\\
&\leq\sum_{k=1}^{\frac{2\tau-a-b}{2h}}\left(\tau-\rho_h\left(kh+\frac{a+b}{2}\right)\right)_h^{\overline{\epsilon-1}}\frac{f(a)+f(b)}{2}.
\end{aligned}\tag{50}$$

Setting $l=k+\frac{a+b}{2h}$ and $t=b$, we obtain

$$\begin{aligned}
&\sum_{l=\frac{a+b}{2h}+1}^{\frac{b}{h}}(b-\rho_h(lh))_h^{\overline{\epsilon-1}}f\left(\frac{a+b}{2}\right)\\
&\leq\frac{1}{2}\left[\sum_{l=\frac{a+b}{2h}+1}^{\frac{\tau}{h}}(\tau-\rho_h(lh))_h^{\overline{\epsilon-1}}f(lh)\right.\\
&\quad\left.+\sum_{l=\frac{a+b}{2h}+1}^{\frac{\tau}{h}}(\tau-\rho_h(lh))_h^{\overline{\epsilon-1}}f(a+b-lh)\right]_{\tau=b}\\
&\leq\sum_{l=\frac{a+b}{2h}+1}^{\frac{b}{h}}(b-\rho_h(lh))_h^{\overline{\epsilon-1}}\frac{f(a)+f(b)}{2},
\end{aligned}\tag{51}$$

that is,

$$\Omega f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\epsilon)}{2}\left[{}_{\frac{a+b}{2}}\nabla_h^{-\epsilon}f(\tau)|_{\tau=b}+{}_{\frac{a+b}{2}}\nabla_h^{-\epsilon}(Qf)(\tau)|_{\tau=b}\right]\leq\Omega\frac{f(a)+f(b)}{2},\tag{52}$$

where

$$\Omega=\int_{\frac{a+b}{2}}^{b}(b-\rho_h(\tau))_h^{\overline{\epsilon-1}}\nabla_h\tau.$$

Then, with the help of Lemma 1, we can obtain

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\epsilon)}{2\Omega}\left[{}_{\frac{a+b}{2}}\nabla_h^{-\epsilon}f(\tau)|_{\tau=b}+{}_h\nabla_{\frac{a+b}{2}}^{-\epsilon}f(\tau)|_{\tau=a}\right]\leq\frac{f(a)+f(b)}{2}.$$

Thus, Theorem 6 is proved. □

Remark 4. *It is worth noting that:*

1. *In Theorem 6, when $h=1$ and $c\in\mathbb{Z}$, the inequalities (46) are consistent with the inequalities (24) of Theorem 4:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\epsilon)}{\Omega(b-a)}\left[{}_{\frac{a+b}{2}}\nabla^{-\epsilon}f(b)+\nabla_{\frac{a+b}{2}}^{-\epsilon}f(a)\right]\leq\frac{f(a)+f(b)}{2},$$

where
$$\Omega = \int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\tau + 1\right)^{\overline{\epsilon-1}} \hat{\Delta}\tau.$$

In fact, using the time scale substitution rule (see Theorem 2), we have

$$\begin{aligned}\int_{\mathbb{T}_{[\frac{a+b}{2},b]}} \left(\frac{b-a}{2}\tau + 1\right)^{\overline{\epsilon-1}} \hat{\Delta}\tau &= \int_{w(b)}^{w\left(\frac{a+b}{2}\right)} \Big(F(\Phi^{-1})\Big)(\tau)\hat{\Delta}\tau \\ &= \int_{\frac{a+b}{2}}^{b} F(\tau)(-\Phi^{\nabla})(\tau)\nabla\tau \\ &= \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} (b-\rho(\tau))^{\overline{\epsilon-1}}\nabla\tau,\end{aligned}$$

where $\Phi(\tau) = \frac{\tau}{2}a + \frac{2-\tau}{2}b$ *and* $F(\tau) = (b-\tau+1)^{\overline{\epsilon-1}}$.

2. *If we take* $\epsilon = 1$*, then the inequalities (46) of Theorem 6 are consistent with the inequalities (36) of Theorem 5.*

4. Examples

In this section, we give two examples to illustrate our results.

Example 1. *Let* $f(t) : [a,b] \cap \mathbb{N}_{c,h} \to \mathbb{R}$ *be defined as* $f(t) = \frac{1}{t}$*, where* $a, b \in \mathbb{N}_{c,h}$ *and* $c \in \mathbb{Z}^+$*. Obviously,* f *is a convex function. Hence, we have*

$$\frac{2}{a+b} \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \frac{1}{t}\Delta_h t + \int_{\frac{a+b}{2}}^{b} \frac{1}{t}\nabla_h t\right] \le \frac{\frac{1}{a}+\frac{1}{b}}{2}.$$

If we take $h \to 0$*, then*

$$\frac{2}{a+b} \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \frac{1}{t}dt + \int_{\frac{a+b}{2}}^{b} \frac{1}{t}dt\right] \le \frac{\frac{1}{a}+\frac{1}{b}}{2}.$$

It follows that
$$\frac{2}{a+b} \le \frac{ln(b)-ln(a)}{b-a} \le \frac{\frac{1}{a}+\frac{1}{b}}{2}.$$

Example 2. *Let* $f(t) : [a,b] \cap \mathbb{N}_{c,h} \to \mathbb{R}$ *be defined as* $f(t) = (1+h)^{\frac{t}{h}}$*, where* $a, b \in \mathbb{N}_{c,h}$ *and* $c \in \mathbb{Z}^+$*. Due to the convexity of* f*, then the following H-H inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(t)\Delta_h t + \int_{\frac{a+b}{2}}^{b} f(t)\nabla_h t\right] \le \frac{f(a)+f(b)}{2},$$

hold. Hence, we have

$$\begin{aligned}&(1+h)^{\frac{a+b}{2h}} \\ \le\ &\frac{h}{b-a}\Big[(1+h)^{\frac{a}{h}} + (1+h)^{\frac{a}{h}+1} + \cdot + (1+h)^{\frac{a+b}{2h}-1} \\ &+(1+h)^{\frac{a+b}{2h}+1} + (1+h)^{\frac{a+b}{2h}+2} + \cdots + (1+h)^{\frac{b}{h}}\Big] \\ \le\ &\frac{(1+h)^{\frac{a}{h}} + (1+h)^{\frac{b}{h}}}{2}.\end{aligned}$$

It follows that

$$
\begin{aligned}
&(1+h)^{\frac{a+b}{2h}} \\
\leq\ & \frac{1}{b-a}\Big[(1+h)^{\frac{a+b}{2h}} - (1+h)^{\frac{a}{h}} + (1+h)^{\frac{b}{h}+1} - (1+h)^{\frac{a+b}{2h}+1}\Big] \\
\leq\ & \frac{(1+h)^{\frac{a}{h}} + (1+h)^{\frac{b}{h}}}{2}.
\end{aligned}
$$

Taking $u = f(a), v = f(b)$, then we obtain

$$(uv)^{\frac{1}{2}} \leq \frac{1}{f^{-1}(u) - f^{-1}(v)}\Big[(1+h)v - u - h(uv)^{\frac{1}{2}}\Big] \leq \frac{u+v}{2}.$$

Note that if we take $h \to 0$, then

$$(uv)^{\frac{1}{2}} \leq \frac{v-u}{\ln v - \ln u} \leq \frac{u+v}{2},$$

where we use the fact that $\lim_{h\to 0}(1+h)^{\frac{t}{h}} = e^t$.

5. Conclusions

The H-H inequalities play a meaningful role in mathematics, and they have been generalized to different forms. However, inequalities for the convex function defined on time scales are rarely studied. In this article, we established two types of discrete H-H inequalities on time scales: $\mathbb{Z}$ and $\mathbb{N}_{c,h}$, respectively. In addition, we proved two discrete fractional H-H inequalities for fractional sums. In the future, other generalized H-H inequalities, relying on different forms of convexity or sum operators, can be introduced by similar methods. Furthermore, the discrete H-H inequalities can be studied for the qualitative properties of difference equations.

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