fractal and fractional

Article

Optimal Control for $k \times k$ Cooperative Fractional Systems

Hassan M. Serag [1], Abd-Allah Hyder [2,3,*], Mahmoud El-Badawy [1] and Areej A. Almoneef [4]

1 Department of Mathematics, Faculty of Sciences, Al-Azhar University, Cairo 71524, Egypt
2 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia
3 Department of Engineering Mathematics and Physics, Faculty of Engineering, Al-Azhar University, Cairo 71524, Egypt
4 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
* Correspondence: abahahmed@kku.edu.sa

Abstract: This paper discusses the optimal control issue for elliptic $k \times k$ cooperative fractional systems. The fractional operators are proposed in the Laplace sense. Because of the nonlocality of the Laplace fractional operators, we reformulate the issue as an extended issue on a semi-infinite cylinder in $\mathbb{R}^{k+1}$. The weak solution for these fractional systems is then proven to exist and be unique. Moreover, the existence and optimality conditions can be inferred as a consequence.

Keywords: nonlocal operator; fractional laplace operator; weak solution; optimality conditions; cooperative systems

Citation: Serag, H.M.; Hyder, A.-A.; El-Badawy, M.; Almoneef, A.A. Optimal Control for $k \times k$ Cooperative Fractional Systems. *Fractal Fract.* **2022**, *6*, 559. https://doi.org/10.3390/fractalfract6100559

Academic Editors: Adriana Catas and Alina Alb Lupas

Received: 23 August 2022
Accepted: 30 September 2022
Published: 2 October 2022

1. Introduction

Different areas of mathematics, including harmonic and operator theories, have found nonlocal operators to be valuable for research. Additionally, they have attracted critical attentiveness due to their close relationship to practical issues, as they are essential to the modelling and simulation of complicated processes that take place over a variety of length scales. Furthermore, the nonlocal operators have numerous applications in various scientific scopes, including nonlocal continuum fields, porous media flow, fluids, science of material, turbulence, control problems, optimization and image processing [1–4].

This article discusses the elliptic $k \times k$ cooperative fractional system that contains the Laplace fractional operator $(-\Delta)^r$, which is one of the nonlocal operators.

Suppose $\Omega \subset \mathbb{R}^k$ is an open domain that is connected, bounded with the boundary $\partial\Omega$, where k is an integer such that $k \geq 2$, $0 < r < 1$ and, as a consequence, $k > 2r$. Then, we will investigate the following systems:

$$\begin{cases} (-\Delta)^r \psi_i = \sum_{j=1}^{k} a_{ij}\, \psi_j + f_i & \text{in } \Omega, \\ \psi_i = 0 & \text{in } \mathbb{R}^k \backslash \Omega, \\ \forall i = 1, 2, 3, \dots, k, \end{cases} \tag{1}$$

where $\{\psi_i\}_{i=1}^k$ are the states of the system and $\{f_i\}_{i=1}^k$ are the external sources. The systems in Equation (1) are said to be cooperative if $a_{ij} > 0$ for $i \neq j$; otherwise, the systems are said to be noncooperative.

Partial differential equations (PDEs) and their control have piqued the attention of researchers in a wide range of disciplines, including biology, ecology, economics, engineering and finance [5–9]. These findings have been used in both cooperative and noncooperative systems [10]. Compared with classical optimal control problems (OPCs), the study of fractional-order control problems (FOPCs) is very recent. However, it is becoming more well-known as a result of the significant roles that fractional differential equations (FDEs) play in physics, chemistry and engineering. Recent studies in various fields have shown that FDEs are capable of properly describing the dynamics of a wide range of systems. For example, viscoelasticity and heat transfer in memory materials, anomalous diffusion in fractal media, nonlocal electrostatics and image processing are just a few of the complex phenomena that may be studied by using FDEs to characterize their behavior [11–16]. There have been several studies that represented FOPCs. In [17,18], the distributed OCPs were covered for a system with a time fraction. These were followed by the derivation of the optimality requirements. In [19], the optimality requirements were derived for a fractional differential system with a Schrödinger operator. The harmonic extension approach was used in [20–22] to turn a nonlocal system into a local one. As a consequence, the optimality conditions were met.

It is commonly known that the fractional Laplace operator is nonlocal, which means that when ξ approaches infinity, the values of ψ have an effect on the values of $(-\Delta)^{\alpha}\psi(\xi)$, where $\xi \in \mathcal{O}$. Additionally, the support of the fractional Laplace operator $(-\Delta)^{\alpha}$ is non-compact even when the support of ψ is compact. This fundamental flaw might lead to various challenges. In fact, situations involving $(-\Delta)^{\alpha}$ cannot be investigated using the traditional local PDE techniques. For more details about nonlocal operators, see [23–25]. Caffarelli and Silvestre [26] showed that the fractional Laplace operator may be described as an operator that converts a Dirichlet boundary condition to a Neumann-type condition by employing an extension issue.

Let us assume $\mathcal{C}^+ \subset \mathbb{R}^{k+1}$ is a semi-infinite cylinder

$$\mathcal{C}^+ = \{(x, w) : x \in \mathbb{R}^k, w \in (0, \infty)\}, \tag{2}$$

where w is a newly defined extended variable. As a result, the nonlocal systems in Equation (1) is recast locally as follows:

$$\begin{cases} \nabla.(w^{\alpha}\nabla\Psi_i) = 0 & \text{in } \mathcal{C}^+, \\ \Psi_i(x,0) = 0 & \text{in } \mathbb{R}^k\backslash\Omega, \\ \frac{1}{s_r}\frac{\partial\Psi_i}{\partial\nu} = (f_i + \sum_{j=1}^{k} a_{ij} Tr_{\Omega}\Psi_j) & \text{on } \Omega\times\{0\}, \end{cases} \tag{3}$$

where $\frac{\partial\Psi}{\partial\nu} = -\lim_{w\to 0^+} w^{\alpha}\frac{\partial\Psi(x,w)}{\partial w}$, $\alpha = 1-2r$, ν is the outer unit normal to $\mathcal{C}^+$ at $\Omega\times\{0\}$, $\lim_{w\to\infty}\Psi(x,w) = 0$ and $s_r = 2^{\alpha}\frac{\Gamma(1-r)}{\Gamma(r)} > 0$.

In this paper, the control problem for cooperative fractional systems is investigated. There are a number of difficulties with the fractional Laplace operator since it is a nonlocal operator. To get around this, we employ an extension method to convert the nonlocal systems (Equation (1)) into local ones (Equation (3)). Then, the weak formulation is performed. Hence, we utilize the lemma of Lax–Milgram to show that the weak solution for the local systems exists and is unique. Furthermore, the optimality criteria for both local and nonlocal systems are derived using the Lions approach. If $r \to 1$ is employed, then the derived results are identical to the classical findings.

The following is the structure of this paper. Section 2 describes several functional spaces that can be used to model fractional cooperative systems and their expansions, as well as the existence results. We explore the weak solution and optimality criteria for a $k\times k$ cooperative fractional system in Section 3. A summary and discussion are provided in the Section 4. Section 5 contains some possible lines for future research in this topic.

2. Preliminaries

In our work, we employ the variational formulation to achieve optimal control of a cooperative system. Sobolev spaces are offered as the solution spaces to our problem as a consequence. This section is divided into three subsections. In Section 2.1, some definitions and the fractional Sobolev spaces are briefly explained. In Section 2.2, the weighted Sobolev spaces and their properties are reviewed. The primary eigenvalue problem's characterizations are described in Section 2.3.

2.1. Fractional Sobolev Spaces

We define the fractional order Sobolev space for $0 < r < 1$ [27] as

$$H^r(\Omega) = \left\{ \psi \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|\psi(x_1) - \psi(x_2)|^2}{|x_1 - x_2|^{k+2r}} dx_1 dx_2 < \infty \right\}, \tag{4}$$

$$H_0^r(\Omega) = \{\psi \in H^r(\Omega) : \psi = 0 \text{ on } \partial\Omega\}, \tag{5}$$

which are Hilbert spaces with the following norms:

$$\|\psi\|_{H^r(\Omega)} := \left(\int_\Omega |\psi|^2 dx + \int_\Omega \int_\Omega \frac{|\psi(x_1) - \psi(x_2)|^2}{|x_1 - x_2|^{k+2r}} dx_1 dx_2 \right)^{\frac{1}{2}}, \tag{6}$$

$$\|\psi\|_{H_0^r(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|\psi(x_1) - \psi(x_2)|^2}{|x_1 - x_2|^{k+2r}} dx_1 dx_2 \right)^{\frac{1}{2}}, \tag{7}$$

In addition, we define the Lion–Magenes space as follows [28]:

$$H_{00}^{\frac{1}{2}}(\Omega) = \left\{ \psi \in H^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{\psi^2(x)}{d(x, \partial\Omega)} dx < \infty \right\}, \tag{8}$$

where the distance between x and $\partial\Omega$ is $d(x, \partial\Omega)$. Combining Equations (4), (5) and (8) yields the following fractional Sobolev space for every $r \in (0, 1)$:

$$\mathsf{H}^r(\Omega) = \begin{cases} H^r(\Omega); & r \in \left(0, \frac{1}{2}\right), \\ H_{00}^{1/2}(\Omega); & r = \frac{1}{2}, \\ H_0^r(\Omega); & r \in \left(\frac{1}{2}, 1\right). \end{cases} \tag{9}$$

Additionally, we designate by $\mathsf{H}^{-r}(\Omega)$ the dual space of $\mathsf{H}^r(\Omega)$ in such a way that

$$(-\Delta)^r : \mathsf{H}^r(\Omega) \to \mathsf{H}^{-r}(\Omega), \tag{10}$$

where the fractional Laplace operator $(-\Delta)^r$ is given by [29]

$$(-\Delta)^r \psi(x) = c_{k,r} \text{ P.V. } \int_{\mathbb{R}^k} \frac{\psi(x) - \psi(z)}{|x - z|^{k+2r}} dz, \quad c_{k,r} > 0. \tag{11}$$

We use the embedding chain below:

$$\mathsf{H}^r(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathsf{H}^{-r}(\Omega). \tag{12}$$

After that, we obtain the subsequent chain of Sobolev spaces via the Cartesian product

$$(\mathsf{H}^r(\Omega))^k \hookrightarrow (L^2(\Omega))^k \hookrightarrow (\mathsf{H}^{-r}(\Omega))^k. \tag{13}$$

2.2. Extension Sobolev Spaces

Using the weighted space illustrated below, we may find the weak solution to Equation (3)

$$\mathbb{H}^r(\mathcal{C}^+) = \left\{ \Psi \in H^1_{\text{loc}}(\mathcal{C}^+) : \int_{C^+} w^\alpha |\nabla\Psi(x, w)|^2 dx dw < +\infty \right\}, \tag{14}$$

with the following norm:

$$\|\Psi\|_{\mathbb{H}^r(\mathcal{C}^+)} := \left(\int_{\mathcal{C}^+} w^\alpha |\nabla\Psi(x,w)|^2 dxdw\right)^{\frac{1}{2}}. \tag{15}$$

Additionally, we have the following spaces:

$$\mathbb{H}^r_\Omega(\mathcal{C}^+) = \left\{\Psi \in \mathbb{H}^r(\mathcal{C}^+) : \Psi|_{\mathbb{R}^k\times\{0\}} = 0 \text{ in } \mathbb{R}^k\backslash\Omega\right\}, \tag{16}$$

and

$$\mathsf{H}^r(\Omega) = \left\{\Psi|_{\Omega\times\{0\}} : \Psi \in \mathbb{H}^r_\Omega(\mathcal{C}^+)\right\}. \tag{17}$$

which provide a clear interpretation of the solutions to Equations (3) in a bounded domain Ω. The following embedding is also available:

Lemma 1. *Assume $1 \le p < \frac{2k}{k-2r}$, $k > 2r$. Then, $\mathrm{Tr}_\Omega(\mathbb{H}^r_\Omega(\mathcal{C}^+))$ is compactly embedded in $L^p(\Omega)$, and the trace operator $\mathrm{Tr}_\Omega : \mathbb{H}^r_\Omega(\mathcal{C}^+) \to \mathsf{H}^r(\Omega)$ satisfies the following inequality:*

$$\|\mathrm{Tr}_\Omega \Psi\|_{\mathsf{H}^r(\Omega)} \le \delta\|\Psi\|_{\mathbb{H}^r_\Omega(\mathcal{C}^+)},\ \delta > 0. \tag{18}$$

Furthermore, $\mathrm{Tr}_\Omega\Psi = \Psi(x,0) = \psi(x)$ is the trace of Ψ onto $\Omega \times \{0\}$.

2.3. Eigenvalue Problem

In this section, we will look at some new information from [29] about the eigenvalue of the fractional elliptic equation shown below:

$$\begin{cases} (-\Delta)^r\psi = \lambda\,\psi & \text{in } \Omega, \\ \psi = 0 & \text{in } \mathbb{R}^k\backslash\Omega. \end{cases} \tag{19}$$

Theorem 1 ([29])**.** *The first eigenvalue of Equation (19) is positive and can be characterized as follows:*

$$\lambda = \min_{\Psi\in\mathbb{H}^r_\Omega(\mathcal{C}^+)} \int_{\mathcal{C}^+} w^\alpha \nabla\Psi\cdot\nabla\Psi dxdw,\quad \|\Psi(x,0)\|_{L^2(\Omega)} = 1, \tag{20}$$

or its equivalent, shown below:

$$\lambda = \min_{\Psi\in\mathbb{H}^r_\Omega(\mathcal{C}^+)} \frac{\int_{\mathcal{C}^+} w^\alpha\nabla\Psi\cdot\nabla\Psi dxdw}{\int_\Omega |\Psi(x,0)|^2 dx},\quad \Psi(x,0) \ne 0. \tag{21}$$

3. $k \times k$ Cooperative Fractional Systems

Here, we show that weak solutions exist and create the optimality criteria for a $k \times k$ cooperative fractional system. This section contains two parts. Using the lemma of Lax–Milgram, we prove the existence and uniqueness of weak solutions in Section 3.1. The optimality requirements are derived using an adjoint problem in Section 3.2.

3.1. The Weak Solution

We first convert Equation (3) into a weak form. In fact, the first equation in Equation (3) is multiplied by a test function $\{\phi_i(x,w)\}_{i=1}^k \in \left(\mathbb{H}^r_\Omega(\mathcal{C}^+)\right)^k$, and by integrating over $\mathcal{C}^+$, we obtain

$$\int_{\mathcal{C}^+} \nabla.(w^\alpha\nabla\Psi_i)\phi_i(x,w)dx\,dt = 0. \tag{22}$$

Using Green's formula yields

$$\int_{\mathcal{C}^+} w^\alpha\nabla\Psi_i\nabla\phi_i(x,w)dx\,dw = -\int_{\Omega\times\{0\}} \mathrm{Tr}_\Omega\,\phi_i(x,w)\lim_{w\to0^+} w^\alpha\frac{\partial\Psi_i}{\partial w}dx. \tag{23}$$

Therefore, we have

$$\int_{\mathcal{C}^+} w^\alpha \nabla\Psi_i \nabla\phi_i(x,w)dx\,dw = \int_{\Omega\times\{0\}} \mathrm{Tr}_\Omega\,\phi_i(x,w)\frac{\partial\Psi_i}{\partial\nu}dx, \tag{24}$$

By using the systems in Equation (3), we obtain

$$\sum_{i=1}^{k}\int_{\mathcal{C}^+} w^\alpha \nabla\Psi_i \nabla\phi_i(x,w)dx\,dw = s_r\sum_{i,j=1}^{k}\int_{\Omega\times\{0\}} (f_i + a_{ij}\,\mathrm{Tr}_\Omega\,\Psi_{ij}(x,w))\,\mathrm{Tr}_\Omega\,\phi_i(x,w)dx. \tag{25}$$

To this end, a bilinear form on $\left(\mathbb{H}^r_\Omega(\mathcal{C}^+)\right)^k$ can be defined as

$$a(\Psi,\phi) = \sum_{i=1}^{k}\int_{\mathcal{C}^+} w^\alpha \nabla\Psi_i \nabla\phi_i(x,w)dxdw - s_r\sum_{i,j=1}^{k}\int_{\Omega\times\{0\}} a_{ij}\,\mathrm{Tr}_\Omega\,\Psi_j(x,w)\,\mathrm{Tr}_\Omega\,\phi_i(x,w)dx. \tag{26}$$

A linear form can also be defined as follows:

$$F(\phi) = s_r\sum_{i=1}^{k}\int_{\Omega\times\{0\}} f_i(x)\,\mathrm{Tr}_\Omega\,\phi_i(x,w)dx, \quad \forall\,\phi\in\left(\mathbb{H}^r_\Omega(\mathcal{C}^+)\right)^k. \tag{27}$$

Definition 1 ([10])**.** *We say that the square matrix $B = (b_{ij})$ is an M matrix if $b_{ij}\le 0$ for $i\neq j$ and $b_{ii} > 0$ and if all principal minors extracted from B are positive.*

Lemma 2. *The bilinear form in Equation (26) is coercive if the square matrix*

$$\lambda I - s_r A = \begin{pmatrix} \lambda - s_r a_{11} & -s_r a_{12} & \dots & -s_r a_{1k} \\ -s_r a_{21} & \lambda - s_r a_{22} & \dots & -s_r a_{2k} \\ . & . & \dots & . \\ . & . & \dots & . \\ -s_r a_{k1} & -s_r a_{k2} & \dots & \lambda - s_r a_{kk} \end{pmatrix} \tag{28}$$

is a non-singular M matrix.

Proof. Replacing $\phi = \{\phi_1\}_{i=1}^k$ by $Y = \{\Psi_i\}_{i=1}^k$ in Equation (26) yields

$$\begin{aligned} a(\Psi,\Psi) &= \sum_{i=1}^{k}\int_{\mathcal{C}^+} w^\alpha \nabla\Psi_i(x,w)\nabla\Psi_i(x,w)dxdw - s_r\sum_{i,j=1}^{k}\int_{\Omega\times\{0\}} a_{ij}\,\mathrm{Tr}_\Omega\,\Psi_j(x,w)\,\mathrm{Tr}_\Omega\,\Psi_i(x,w)dx \\ &= \sum_{i=1}^{k}\int_{\mathcal{C}^+} w^\alpha \nabla\Psi_i(x,w)\nabla\Psi_i(x,w)dxdw - s_r\sum_{i=1}^{k}\int_{\Omega\times\{0\}} a_{ii}\,\mathrm{Tr}_\Omega\,\Psi_i(x,t)\,\mathrm{Tr}_\Omega\,\Psi_i(x,w)dx \\ &\quad - s_r\sum_{i\neq j=1}^{n}\int_{\Omega\times\{0\}} a_{ij}\,\mathrm{Tr}_\Omega\,\Psi_j(x,w)\,\mathrm{Tr}_\Omega\,\Psi_i(x,w)dx. \end{aligned} \tag{29}$$

By applying Cauchy–Schwartz inequality, we have

$$\begin{aligned} a(\Psi,\Psi) \ge& \sum_{i=1}^{k}\int_{\mathcal{C}^+} w^\alpha|\nabla\Psi_i|^2dx\,dw - s_r\sum_{i=1}^{k}a_{ii}\int_{\Omega\times\{0\}}|\,\mathrm{Tr}_\Omega\,\Psi_i|^2dx \\ & - s_r\sum_{i\neq j=1}^{k}a_{ij}\left(\int_{\Omega\times\{0\}}|\,\mathrm{Tr}_\Omega\,\Psi_i|^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega\times\{0\}}|\,\mathrm{Tr}_\Omega\,\Psi_j|^2dx\right)^{\frac{1}{2}}. \end{aligned} \tag{30}$$

From Equation (20), we deduce

$$a(\Psi,\Psi) \geq \sum_{i=1}^{k}\Big(1-\frac{a_{ii}s_r}{\lambda}\Big)\|\Psi_i\|^2 - \frac{1}{\lambda}s_r\sum_{i\neq j}^{k} a_{ij}\|\Psi_j\|\|\Psi_i\|$$
$$= \sum_{i=1}^{k}\frac{1}{\lambda}(\lambda - a_{ii}s_r)\|\Psi_i\|^2 - \frac{s_r}{\lambda}\sum_{i\neq j=1}^{k} a_{ij}\|\Psi_j\|\|\Psi_i\|. \tag{31}$$

Hence from Equation (28), we obtain

$$a(\Psi,\Psi) \geq C\|\Psi\|^2_{(\mathbb{H}^r_\Omega(\mathcal{C}^+))^k}. \tag{32}$$

Thus, the bilinear form $a(\Psi,\varphi)$ is coercive. □

3.2. The Optimality Conditions

The fundamental goal of this subsection is to formulate the control problem. We create the adjoint state for the control problem. Furthermore, we created the optimality conditions using the Lions approach [6,7]. This subsection consists of two parts. The first part deduces the nonlocal optimality conditions. The second part reveals the local optimality conditions.

3.2.1. Fractional Optimal Control

Take into consideration the space of controls as $(L^2(\Omega))^k$. Then, for an element $u = \{u_i\}_{i=1}^k \in (L^2(\Omega))^k$, the state $\psi(u) = \{\psi_i(u)\}_{i=1}^k$ solves the following system:

$$\begin{cases} (-\Delta)^r\psi_i(u) = \sum_{j=1}^k a_{ij}\,\psi_j(u) + u_i & \text{in } \Omega, \\ \psi_i(u) = 0 & \text{in } \mathbb{R}^k\backslash\Omega, \\ \forall i = 1,2,3,\dots,k. \end{cases} \tag{33}$$

The equations for the observations are as follows:

$$\psi_i^*(u) = \psi_i(u), \quad i = 1,2,3,\dots,k. \tag{34}$$

For a given $\{z_{id}\}_{i=1}^k \in (L^2(\Omega))^k$ and $\{v_i\}_{i=1}^k \in (L^2(\Omega))^k$, when Equation (33) is applied, the cost functional is given by

$$\mathcal{Q}(v) = \frac{1}{2}\sum_{i=1}^{k}\|\psi_i(v) - z_{id}\|^2_{L^2(\Omega)} + (\mathsf{N}\,v,v)_{(L^2(\Omega))^k}, \tag{35}$$

where the positive definite Hermitian operator $\mathsf{N} \in \mathcal{L}((L^2(\Omega))^k)$ satisfies the following condition:

$$(\mathsf{N}\,v,v) \geq \gamma\|v\|^2_{(L^2(\Omega))^k}, \quad \gamma > 0. \tag{36}$$

Assume that $\mathcal{U}_{ad} \subset L^2(\Omega)$ is closed and convex. The problem of control is then presented by

$$\begin{cases} \text{Finding } u \in (\mathcal{U}_{ad})^k, \\ \text{such that } \mathcal{Q}(u) \leq \mathcal{Q}(v),\ \forall\, v \in (\mathcal{U}_{ad})^k. \end{cases} \tag{37}$$

Theorem 2. *There exists a unique optimal control $u \in (\mathcal{U}_{ad})^k$ if the cost functional is provided by Equations (35) and (36) is true. Additionally, the following equations explain this control:*

$$\begin{cases} (-\Delta)^r p_i(u) - \sum_{j=1}^k a_{ji}p_j(u) = \psi_i(u) - z_{id} & \textit{in } \Omega, \\ p_i(u) = 0 & \textit{on } \mathbb{R}^k\backslash\Omega. \end{cases} \tag{38}$$

together with

$$\sum_{i=1}^{k}(p_i, v_i - u_i) + (\mathsf{N}\,u, v-u)_{(L^2(\Omega)^k} \geq 0, \forall\, v \in (\mathcal{U}_{ad})^k, \tag{39}$$

where $\{p_i\}_{i=1}^{k} \in (\mathsf{H}^r(\Omega))^k$ *is the adjoint state.*

Proof. Due to the fact that $\mathsf{N} > 0$, the cost functional in Equation (35) is strictly convex. Additionally, $\mathcal{U}_{ad}$ is non-empty, closed, bounded and convex in $L^2(\Omega)$. Then, the optimal control exists and is unique.

Hence, the optimal control $\{u_i\}_{i=1}^{k} \in (\mathcal{U}_{ad})^k$ satisfies the following inequality:

$$\mathcal{Q}^{'}(u).(v-u) \geq 0,\ \forall\, v \in (\mathcal{U}_{ad})^k, \tag{40}$$

which is equivalent to [6]

$$\sum_{i=1}^{k}(\psi_i(u) - z_{id}, \psi_i(v) - \psi_i(u)) + (\mathsf{N}\,u, v-u) \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^k. \tag{41}$$

Now, since $(\mathbf{A}\,\psi, p) = (\psi, \mathbf{A}^*\,p)$, where

$$\mathbf{A}(\psi = \{\psi_i\}_{i=1}^{k}) = (-\Delta)^r\psi_i - \sum_{i=1}^{k} a_{ij}\psi_j, \tag{42}$$

and $\mathbf{A}^*$ its adjoint, then

$$\begin{aligned}(\mathbf{A}\,\psi, p) &= \left((-\Delta)^r\psi_i - \sum_{i=1}^{k} a_{ij}\,\psi_j, p_i\right)\\ &= ((-\Delta)^r\psi_i, p_i) - (\sum_{i=1}^{k} a_{ij}\psi_j, p_i)\\ &= (\psi_i, (-\Delta)^r p_i) - (\psi_i, \sum_{j=1}^{k} a_{ji}p_j)\\ &= (\psi, \mathbf{A}^*\,p),\end{aligned} \tag{43}$$

Consider the following adjoint system:

$$\begin{cases}(-\Delta)^r p_i(u) - \sum_{j=1}^{k} a_{ji}p_j(u) = \psi_i(u) - z_{id} & \text{in } \Omega,\\ p_i(u) = 0 & \text{on } \mathbb{R}^k \backslash \Omega.\end{cases} \tag{44}$$

By using Equations (33) and (44), we deduce that

$$\sum_{i=1}^{k}(p_i, v_i - u_i) + (\mathsf{N}\,u, v-u)_{(L^2(\Omega))^k} \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^k. \tag{45}$$

□

3.2.2. Extended Optimal Control

If $\psi(v) \in (\mathsf{H}^r(\Omega))^k$ is a solution to Equation (33) with $v = \{v_i\}_{i=1}^{k} \in (\mathsf{H}^{-r}(\Omega))^k$ and $\Psi(v) \in \left(\mathbb{H}^r_\Omega(\mathcal{C}^+)\right)^k$ solves the following systems:

$$\begin{cases}\nabla.(w^\alpha\nabla\Psi_i(u)) = 0 & \text{in } \mathcal{C}^+,\\ \Psi_i(x,0) = 0 & \text{in } \mathbb{R}^k\backslash\Omega,\\ \frac{1}{s_r}\frac{\partial\Psi_i(u)}{\partial\nu} = (u_i + \sum_{j=1}^{k} a_{ji}\ \mathrm{Tr}_\Omega\,\Psi_j(u)) & \text{on } \Omega\times\{0\}.\end{cases} \tag{46}$$

Then, we have

$$\operatorname{Tr}_{\Omega} \Psi(v) = \psi(v). \tag{47}$$

Consequently, the equivalent extended OCP is given by

$$\min \ \mathbb{Q}(v) = \frac{1}{2}\sum_{i=1}^{k} \| \operatorname{Tr}_{\Omega} \Psi_i(v) - z_{id} \|^2_{L^2(\Omega)} + (\mathsf{N}\, v, v)_{(L^2(\Omega))^k}, \quad \forall\, v \in (\mathcal{U}_{ad})^k, \tag{48}$$

Theorem 3. *If the cost functional is provided by Equation (48), and the condition in Equation (36) is met, then there exists a unique optimal control $u = \{u_i\}_{i=1}^k \in (\mathcal{U}ad)^k$. This control may also be explained using the following equations:*

$$\begin{cases} \nabla.(w^{\alpha}\nabla P_i) = 0 & in \ \mathcal{C}^+, \\ P_i(x,0) = 0 & in \ \mathbb{R}^k \backslash \Omega, \\ (\frac{\partial P_i}{\partial v}, \operatorname{Tr}_{\Omega} \Psi_i) = (\operatorname{Tr}_{\Omega} P_i, \frac{\partial \Psi_i}{\partial v}) & on \ \Omega \times \{0\}. \end{cases} \tag{49}$$

as well as

$$\sum_{i=1}^{k} (\operatorname{Tr}_{\Omega} P_i, v_i - u_i) + (\mathsf{N}\, u, v-u) \geq 0, \ \ \forall\, v \in (\mathcal{U}_{ad})^k, \tag{50}$$

where $P = \{P_i\}_{i=1}^k \in \left(\mathbb{H}^r_{\Omega}(\mathcal{C}^+)\right)^k$ is the adjoint state.

Proof. The optimal control $u \in (\mathcal{U}_{ad})^k$ is achieved only when

$$\mathbb{Q}^{'}(u).(v-u) \geq 0, \ \forall\, v \in (\mathcal{U}_{ad})^k, \tag{51}$$

which, again, is the same as [6]

$$\sum_{i=1}^{k} (\operatorname{Tr}_{\Omega} \Psi_i(v) - z_{id}, \operatorname{Tr}_{\Omega} \Psi_i(v) - \operatorname{Tr}_{\Omega} \Psi_i(u)) + (\mathsf{N}\, u, v-u) \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^k. \tag{52}$$

If $(\mathbf{A}\,\Psi, P) = (\Psi, \mathbf{A}^*\, P)$, then

$$\begin{aligned} (\mathbf{A}\,\Psi, P) &= \sum_{i=1}^{k} (\nabla.(t^{\alpha}\nabla\Psi_i), P_i) \\ &= \sum_{i=1}^{k} \int_{\mathcal{C}^+} \nabla.(w^{\alpha}\nabla\Psi_i) P_i dx\, dw \\ &= \sum_{i=1}^{k} \int_{\mathcal{C}^+} \nabla.(w^{\alpha}\nabla P_i) \Psi_i(x,w) dx\, dw \\ &\quad - \sum_{i=1}^{k} \int_{\Omega} \operatorname{Tr}_{\Omega} \Psi_i \frac{\partial P_i}{\partial v} dx + \sum_{i=1}^{k} \int_{\Omega} \operatorname{Tr}_{\Omega} P_i \frac{\partial \Psi_i}{\partial v} dx \\ &= (\Psi, \mathbf{A}^*\, P), \end{aligned} \tag{53}$$

Hence, the condition in Equation (49) has been met.
We consider the following adjoint systems:

$$\begin{cases} \nabla.(w^{\alpha}\nabla P_i) = 0 & \text{in } \mathcal{C}^+, \\ P_i(x,0) = 0 & \text{in } \mathbb{R}^k \backslash \Omega, \\ \frac{1}{s_r}\frac{\partial P_i}{\partial v} - \sum_{j=1}^{k} a_{ji} \operatorname{Tr}_{\Omega} P_j = \operatorname{Tr}_{\Omega} \Psi_i(v) - z_{id} & \text{on } \Omega \times \{0\}. \end{cases} \tag{54}$$

Then, Equation (52) is equivalent to

$$\sum_{i=1}^{k}\left(\frac{1}{s_r}\frac{\partial P_i}{\partial \nu}-\sum_{j=1}^{k}a_{ji}\,\mathrm{Tr}_{\Omega}\,P_j,\mathrm{Tr}_{\Omega}\,\Psi_i(v)-\mathrm{Tr}_{\Omega}\,\Psi_i(u)\right)+(\mathsf{N}\,u,v-u)\geq 0,\quad \forall\, v\in(\mathcal{U}_{ad})^k. \tag{55}$$

Consequently, by using the final condition from Equation (49), the optimality condition becomes

$$\sum_{i=1}^{k}(\mathrm{Tr}_{\Omega}\,P_i,v_i-u_i)+(\mathsf{N}\,u,v-u)\geq 0,\quad \forall\, v\in(\mathcal{U}_{ad})^k. \tag{56}$$

□

4. Summary and Conclusions

In this work, we examined the OCP for $k \times k$ fractional cooperative systems. Due to the difficulty created by the nonlocality of the fractional Laplace operator, we generalized our issue to local cooperative systems utilizing the harmonic extension technique. The Lax–Milgram lemma was used to illustrate the existence and uniqueness of the weak solution to the local system. Additionally, the conditions of optimality for both the local and nonlocal systems were proven using the Lions technique. The findings are equivalent to the standard results if $r \to 1$.

5. Open Problems

- Study the control problems for $k \times k$ cooperative fractional parabolic systems in the form

$$\begin{cases}\frac{\partial\psi_i}{\partial t}+(-\Delta)^r\psi_i=\sum_{j=1}^{k}a_{ij}\,\psi_j+f_i & \text{in }\Omega\times(0,T),\\ \psi_i=0 & \text{in }\mathbb{R}^k\backslash\Omega\times(0,T),\\ \psi_i(0)=o & \text{in }\Omega,\\ \forall i=1,2,3,\dots,k, & \end{cases} \tag{57}$$

- Study the control problems for $k \times k$ cooperative time and space fractional systems in the form

$$\begin{cases}D_t^s+(-\Delta)^r\psi_i=\sum_{j=1}^{k}a_{ij}\,\psi_j+f_i & \text{in }\Omega\times(0,T),\\ \psi_i=0 & \text{in }\mathbb{R}^k\backslash\Omega\times(0,T),\\ \psi_i(0)=o & \text{in }\Omega,\\ \forall i=1,2,3,\dots,k, & \end{cases} \tag{58}$$

where D_t^s is the Riemann–Liouville sense.

Author Contributions: Methodology and conceptualization, H.M.S., A.-A.H., M.E.-B. and A.A.A.; data curation and writing—original draft, H.M.S., A.-A.H., M.E.-B. and A.A.A.; investigation and visualization, H.M.S., A.-A.H., M.E.-B. and A.A.A.; validation, writing—reviewing and editing, H.M.S., A.-A.H., M.E.-B. and A.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Khalid University, Grant RGP.2/15/43.

Data Availability Statement: The corresponding author will provide the data used in this work upon reasonable request.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Groups Program under grant RGP.2/15/43.

Conflicts of Interest: The authors announce that they have no conflict of interest.

References

1. Muñoz, J. Local and Nonlocal Optimal Control in the Source. *Mediterr. J. Math.* **2022**, *19*, 27. [CrossRef]
2. Rabczuk, T.; Ren, H.; Zhuang, X. A Nonlocal Operator Method for Partial Differential Equations with Application to Electromagnetic Waveguide Problem. *Comput. Mater. Contin.* **2019**, *59*, 31–55. [CrossRef]
3. Ren, H.; Zhuang, X.; Trung, N.-T.; Rabczuk, T. A nonlocal operator method for finite deformation higher-order gradient elasticity. *Comput. Methods Appl. Mech. Eng.* **2021**, *384*, 113963. [CrossRef]
4. Seyedi, S.H.; Zayernouri, M. A data-driven dynamic nonlocal subgrid-scale model for turbulent flows. *Phys. Fluids* **2022**, *34*, 035104. [CrossRef]
5. Ceng, L.C.; Fu, Y.X.; Yin, J.; He, L.; He, L.; Hu, H.Y. The solvability of generalized systems of time-dependent hemivariational inequalities enjoying symmetric structure in reflexive Banach spaces. *Symmetry* **2021**, *13*, 1801. [CrossRef]
6. Lions, J.L. *Optimal Control of Systems Governed by Partial Differential Equations*; Springer: Berlin, Germany, 1971.
7. Lions, J.L. *Some Methods in the Mathematical Analysis of Systems and Their Control*; Science Press: Beijing, China, 1981.
8. Ceng, L.C.; Wen, C.F.; Yao, J.C.; Yao, Y. A system of evolution problems driven by a system of hemivariational inequalities. *J. Nonlinear Sci. Appl.* **2018**, *11*, 342–357. [CrossRef]
9. Gali, I.M.; Serag, H.M. Optimal control of cooperative elliptic systems defined on$\mathcal{R}^n$. *J. Egypt. Math. Soc.* **1995**, **3**, 33–39.
10. Fleckinger, J.; Serag, H.M. Semilinear cooperative elliptic systems on $\mathcal{R}^n$. *Rend. Mat. Appl.* **1995**, *15*, 98–108.
11. Guo, B.; Pu, X.; Huang, F. *Fractional Partial Differential Equations and Their Numerical Solutions*; World Scientific: Beijing, China, 2015.
12. Hyder, A.; El-Badawy, M. Fractional optimal control of Navier-Stokes equations. *Comput. Mater. Contin.* **2020**, *64*, 859–870. [CrossRef]
13. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Hackensack , NJ, USA, 2014.
14. Abdelhakem, M.; Mahmoud, D.; Baleanu, D. Shifted ultraspherical pseudo-Galerkin method for approximating the solutions of some types of ordinary fractional problems. *Adv. Differ. Equ.* **2021**, *2021*, 110. [CrossRef]
15. Abdelhakem, M.; Abdelhamied, D.; Alshehri, M.G.; El-kady, M. Shifted Legendre fractional pseudospectral differentiation matrices for solving fractional differential problems. *Fractals* 2022, *30*, 2240038. [CrossRef]
16. Abdelhakem, M.; Moussa, H.; Baleanu, D.; El-Kady, M. Shifted Chebyshev schemes for solving fractional optimal control problems. *J. Vib. Control* **2019**, *25*, 2143–2150. [CrossRef]
17. Mophou, G.M. Optimal control of fractional diffusion equation. *Comput. Math. Appl.* **2011**, *62*, 68–78. [CrossRef]
18. Mophou, G.M. Optimal control of fractional diffusion equation with state constraints. *Comput. Math. Appl.* **2011**, *62*, 1413–1426. [CrossRef]
19. Hyder, A.; El-Badawy, M. Distributed control for time fractional differential system involving Schrödinger operator. *J. Funct. Spaces* **2019**, *2019*, 1389787. [CrossRef]
20. Antil, H.; Otarola, E. A FEM for an Optimal Control Problem of Fractional Powers of Elliptic Operators. *SIAM J. Control Optim.* **2015**, *53*, 3432–3456. [CrossRef]
21. Antil, H.; Otarola, E.; Salgado, A.J. Optimization with respect to order in a fractional diffusion model: Analysis, approximation and algorithmic aspects. *J. Sci. Comput.* **2018**, *77*, 204–224. [CrossRef]
22. D'Elia, M.; Glusa, C.; Otarola, E. A priori error estimates for the optimal control of the integral fractional Laplacian. *SIAM J. Control Optim.* **2019**, *57*, 2775–2798. [CrossRef]
23. Wu, G.C.; Baleanu, D. Discrete fractional logistic map and its chaos. *Nonlinear Dyn.* **2014**, *75*, 283–287. [CrossRef]
24. Wu, G.C.; Baleanu, D.; Zeng, S.D. Discrete chaos in fractional sine and standard maps. *Phys. Lett.* **2014**, *A378*, 484–487. [CrossRef]
25. Lizama, C.; Murillo-Arcila, M.; Peris, A. Nonlocal operators are chaotic. *Chaos* **2020**, *30*, 103126. [CrossRef] [PubMed]
26. Caffarelli, L.; Silvestre, L. An extension problem related to the fractional Laplacian. *Commun. Partial. Differ. Equ.* **2017**, *32*, 1245–1260. [CrossRef]
27. Tartar, L. *An Introduction to Sobolev Spaces and Interpolation Spaces*; Volume 3 of Lecture Notes of the Unione Matematica Italiana; Springer: Berlin, Germany, 2007.
28. Lions, J.L.; Magenes, E. *Non-Homogeneous Boundary Value Problems and Applications Volume I*; Springer: New York, NY, USA, 1972.
29. Tian, Y. Some results on the eigenvalue problem for a fractional elliptic equation. *Bound. Value Probl.* **2019**, *2019*, 13. [CrossRef]