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# A Study of Coupled Systems of $\psi$-Hilfer Type Sequential Fractional Differential Equations with Integro-Multipoint Boundary Conditions

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**Abstract:** In this paper, the existence and uniqueness of solutions for a coupled system of $\psi$-Hilfer type sequential fractional differential equations supplemented with nonlocal integro-multi-point boundary conditions is investigated. The presented results are obtained via the classical Banach and Krasnosel'skiĭ's fixed point theorems and the Leray–Schauder alternative. Examples are included to illustrate the effectiveness of the obtained results.

**Keywords:** $\psi$-Hilfer fractional derivative; Riemann–Liouville fractional derivative; Caputo fractional derivative; system of fractional differential equations



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## 1. Introduction

Fractional calculus, as an extension of usual integer calculus, is a forceful tool to express real-world problems rather than integer-order differentiations, so that this idea has wide applications in various fields such as, mathematics, physics, engineering, biology, finance, economy and other sciences (see [1–3] and related references therein). Accordingly, many researchers have studied initial and boundary value problems for fractional differential equations (see [4–11]). Additionally, fractional differential equations involving coupled systems have nonlocal natures and applications in many real = world process. The investigation of types of integral and differential operators and the relationship between these operators plays a key role in studying fractional differential equations. Fractional operators of a function concerning another function were introduced by Kilbas et al. [5]. Later, Almeida [12] introduced the notion of the $\psi$-Caputo fractional operator. For some applications of $\psi$ operator, we refer to the papers [13–15]. Hilfer [16] extended both Riemann–Liouville and Caputo fractional derivatives by presenting a family of derivative operators. Different models based on Hilfer fractional derivative have been considered in [17–21], and references cited therein. Many applications of Hilfer fractional differential equations can be found in many fields of mathematics, physics, etc (see [22–24]). The study of boundary value problems for Hilfer-fractional differential equations of order in $(1,2]$, and nonlocal boundary conditions was initiated in [25] by studying the boundary value problem of the form:

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}u(z) = h(z,u(z)), \;\; z \in [c,d], \;\; 1 < \alpha_1 \leq 2, \;\; 0 \leq \beta_1 \leq 1, \\ u(c) = 0, \;\; u(d) = \displaystyle\sum_{i=1}^{m} \varepsilon_i I^{\phi_i} u(\xi_i), \;\; \phi_i > 0, \varepsilon_i \in \mathbb{R}, \xi_i \in [c,d], \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha_1,\beta_1}$ is the Hilfer fractional derivative of order $\alpha_1$, $1 < \alpha_1 \leq 2$, and parameter $\beta_1$, $0 \leq \beta_1 \leq 1$, $f : [c,d] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $c \geq 0$ and $I^{\phi_i}$ is the Riemann–Liouville fractional integral of order $\phi_i, i = 1,2,\ldots,m$. Several existence and uniqueness results were proved by using a variety of fixed point theorems.

Wongcharoen et al. [26] studied a system of Hilfer-type fractional differential equations of the form

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}u(z) = f_1(z,u(z),v(z)), \;\; z \in [c,d], \\ {}^{H}D^{\overline{\alpha}_1,\beta_1}v(z) = g_1(z,u(z),v(z)), \;\; z \in [c,d], \\ u(c) = 0, \; u(d) = \displaystyle\sum_{i=1}^{m} \overline{\theta}_i I^{\phi_i} v(\xi_i), \\ v(c) = 0, \; v(d) = \displaystyle\sum_{j=1}^{n} \overline{\zeta}_j I^{\overline{\phi}_j} u(z_i), \end{cases} \tag{2}$$

where ${}^{H}D^{\alpha_1,\beta_1;\psi}$ and ${}^{H}D^{\overline{\alpha_1},\beta_1;\psi}$ are the Hilfer fractional derivatives of orders $\alpha_1$ and $\overline{\alpha}_1$, $1 < \alpha_1, \overline{\alpha}_1 < 2$ and parameter $\beta_1$, $0 \leq \beta_1 \leq 1$, $f_1, g_1 : [c,d] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, $c \geq 0$, $\overline{\theta}_i, \overline{\zeta}_i \in \mathbb{R}$, and $I^{\phi_i}, I^{\overline{\phi}_j}$ are the Riemann–Liouville fractional integrals of order $\phi_i > 0, \overline{\phi}_j > 0$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

Sitho et al. [27] proved the existence and uniqueness of solutions for the following class of boundary value problems consisting of fractional-order $\psi$-Hilfer-type differential equations supplemented with nonlocal integro-multipoint boundary conditions of the form:

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1;\psi}u(z) = h(z,u(z)), \;\; z \in [c,d], \\ u(c) = 0, \;\; u(d) = \displaystyle\sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s)ds + \sum_{j=1}^{m} \lambda_j u(\xi_j), \end{cases} \tag{3}$$

where ${}^{H}D^{\alpha_1,\beta_1;\psi}$ is the $\psi$-Hilfer fractional derivative operator of order $\alpha_1$, $1 < \alpha_1 < 2$ and parameter $\beta_1$, $0 \leq \beta_1 \leq 1$, $f : [c,d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $c \geq 0$, $\mu_i, \lambda_j \in \mathbb{R}$, $\eta_i, \xi_j \in (c,d)$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots m$ and $\psi$ is a positive increasing function on $(c,d]$, which has a continuous derivative $\psi'(t)$ on $(c,d)$.

Recently, in [28], the boundary value problem (3) was extended to sequential $\psi$-Hilfer-type fractional differential equations involving integral multi-point boundary conditions of the form

$$\begin{cases} \left({}^{H}D^{\alpha_1,\beta_1;\psi} + k\,{}^{H}D^{\alpha_1-1,\beta_1;\psi}\right)u(z) = f(z,u(z)), \;\; k \in \mathbb{R}, \;\; z \in [c,d], \\ u(c) = 0, \;\; u(d) = \displaystyle\sum_{i=1}^{n} \mu_i \int_a^{\eta_i} \psi'(s)u(s)ds + \sum_{j=1}^{m} \theta_j u(\xi_j), \end{cases} \tag{4}$$

where the notations are the same as those of problem (3).

In the present research, inspired by the published articles in this direction, we study the existence and uniqueness of solutions for the following coupled system of sequential $\psi$-Hilfer-type fractional differential equations with integro-multi-point boundary conditions of the form

$$\begin{cases} ({}^{H}D^{\alpha_1,\beta_1;\psi} + k\,{}^{H}D^{\alpha_1-1,\beta_1;\psi})u(z) = f(z,u(z),v(z)), \;\; z \in [c,d], \\ ({}^{H}D^{\overline{\alpha}_1,\beta_1;\psi} + k\,{}^{H}D^{\overline{\alpha}_1-1,\beta_1;\psi})v(z) = g(z,u(z),v(z)), \;\; z \in [c,d], \\ u(c) = 0, \; u(d) = \displaystyle\sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s)v(s)ds + \sum_{j=1}^{m} \theta_j v(\xi_j), \\ v(c) = 0, \; v(d) = \displaystyle\sum_{r=1}^{p} \upsilon_r \int_c^{\varsigma_r} \psi'(s)u(s)ds + \sum_{s=1}^{q} \tau_s u(\sigma_s), \end{cases} \tag{5}$$

where ${}^{H}D^{\alpha_1,\beta_1;\psi}$ and ${}^{H}D^{\overline{\alpha}_1,\beta_1;\psi}$ are the $\psi$-Hilfer fractional derivatives of orders $\alpha_1$ and $\overline{\alpha}_1$, $1 < \alpha_1, \overline{\alpha}_1 < 2$ and parameter $\beta_1$, $0 \leq \beta_1 \leq 1$, $f, g : [c,d] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, $c \geq 0$, $\mu_i, \theta_j, \upsilon_r, \tau_s \in \mathbb{R}^+$, $\eta_i, \xi_j, \varsigma_r, \sigma_s \in (c,d)$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $r = 1,2,\ldots,p$, $s = 1,2,\ldots,q$ and $\psi$ is an increasing and positive monotone function on $(c,d]$ having a continuous derivative $\psi'$ on $(c,d)$.

The classical fixed point theorems are applied in order to obtain our main existence and uniqueness results. Thus, the Banach fixed point theorem is applied to obtain the uniqueness result, while Leray–Schuader alternative and Krasnosel'skiĭ's fixed point theorems are the basic tools used to present the existence results.

The rest of the paper is organized as follows: we recall some primitive concepts in Section 2. In Section 3, an auxiliary lemma is proved, which is a basic tool in proving the main results of the paper, which are presented in Section 4. The main results are supported by numerical examples.

## 2. Preliminaries

In this section, some basic concepts in connection to fractional calculus and fixed point theory are assigned. Throughout the paper, by $\mathcal{X} = C([c,d],\mathbb{R})$, we denote the Banach space of all continuous mappings from $[c,d]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x(t)|; t \in [c,d]\}$. It is clear that the space $\mathcal{X} \times \mathcal{X}$, endowed with the norm $\|(x,y)\| = \|x\| + \|y\|$, is a Banach space.

**Definition 1** ([2]). *Let $(c,d)$, $(-\infty \leq c < d \leq \infty)$ $\alpha > 0$ and $\psi(z)$ be a positive increasing function on $(c,d]$, with continuous derivative $\psi'(z)$ on $(c,d)$. The $\psi$-Riemann–Liouville fractional integral of a function $h$ with respect to a another function $\psi$ on $[c,d]$ is defined by*

$$I^{\alpha,\psi}h(z) = \frac{1}{\Gamma(\alpha)}\int_c^z \psi'(s)(\psi(z)-\psi(s))^{\alpha-1}h(s)ds,\ z > c > 0,$$

*where $\Gamma(.)$ is the Euler Gamma function.*

**Definition 2** ([29]). *Let $\psi \in C^n([c,d],\mathbb{R})$ with $\psi'(z) \neq 0$ and $\eta > 0$, $n \in \mathbb{N}$. The Riemann–Liouville derivatives of a function $h$ with connection to another function $\psi$ of order $\eta$ is represented as*

$$\begin{aligned} D^{\eta;\psi}h(z) &= \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n I_{c^+}^{n-\eta;\psi}, \\ &= \frac{1}{\Gamma(n-\eta)}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \int_c^z \psi'(s)(\psi(z)-\psi(s))^{n-\eta-1}h(s)ds, \end{aligned}$$

*where $n = [\eta] + 1$, $[\eta]$ represent the integer part of real number $\eta$.*

**Definition 3** ([29]). *Assume that $n - 1 < \eta < n$ with $n \in \mathbb{N}$ and $[c,d]$ is the interval so that $-\infty \leq c < d \leq \infty$ and $h, \psi \in C^n([c,d],\mathbb{R})$ are two functions, such that $\psi$ is increasing and $\psi'(z) \neq 0$ for all $z \in [c,d]$. The $\psi$-Hilfer fractional derivative of a function $h$ of order $\eta$ and type $0 \leq \overline{\eta} \leq 1$ is defined by*

$${}^{H}D_{c^+}^{\eta,\overline{\eta};\psi}h(z) = I_{c^+}^{\overline{\eta}(n-\eta);\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n I_{c^+}^{(1-\overline{\eta})(n-\eta);\psi}h(z) = I_{c^+}^{\gamma-\eta;\psi}D_{c^+}^{\gamma;\psi}h(z),$$

*where $n = [\eta] + 1$, $[\eta]$ represents the integer part of the real number $\eta$ with $\gamma = \eta + \overline{\eta}(n-\eta)$.*

**Lemma 1** ([29]). *If $h \in C^n(J,\mathbb{R})$, $n - 1 < \eta < n$, $0 \leq \overline{\eta} \leq 1$ and $\gamma = \eta + \overline{\eta}(n-\eta)$, then*

$$I_{c^+}^{\eta;\psi}\left({}^{H}D_{c^+}^{\eta,\overline{\eta};\psi}h\right)(z) = h(z) - \sum_{k=1}^{n}\frac{(\psi(z)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}\nabla_{\psi}^{[n-k]}I_{c^+}^{(1-\overline{\eta})(n-\eta);\psi}h(c),$$

*for all* $z \in J$, *where* $\nabla_{\psi}^{[n]} h(z) = \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n h(z)$.

Finally, we summarize the fixed point theorems used to prove the main results in this paper. $X$ is a Banach space in each theorem.

**Lemma 2.** *(Banach fixed point theorem [30]). Let $D$ be a closed set in $X$ and $T : D \to D$ satisfies*

$$|Tu - Tv| \le \lambda |u - v|, \ \text{for some } \lambda \in (0,1), \text{ and for all } u, v \in D.$$

*Then $T$ admits one fixed point in $D$.*

**Lemma 3.** *(Leray–Schauder alternative [31]). Let the set $\Omega$ be closed bounded convex in $X$ and $O$ an open set contained in $\Omega$ with $0 \in O$. Then, for the continuous and compact $T : \bar{U} \to \Omega$, either:*

$(\mathfrak{a})$ *$T$ admits a fixed point in $\bar{U}$, or*
$(\mathfrak{aa})$ *There exists $\mathfrak{u} \in \partial U$ and $\mu \in (0,1)$ with $\mathfrak{u} = \mu T(\mathfrak{u})$.*

**Lemma 4.** *(Krasnosel'skiĭ fixed point theorem [32]). Let $M$ be a closed, bounded, convex and nonempty subset of a Banach space $X$. Let $A, B$ be operators such that (i) $Ax + By \in M$ where $x, y \in M$, (ii) $A$ is compact and continuous and (iii) $B$ is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

## 3. An Auxiliary Result

We prove the following auxiliary lemma, concerning a linear variant of the coupled system (5), which is useful to present the coupled system (5) as a fixed point problem.

**Lemma 5.** *Let $c \ge 0, 1 < \alpha_1, \bar{\alpha}_1 < 2, 0 \le \beta_1 \le 1, \gamma = \alpha_1 + 2\beta_1 - \alpha_1\beta_1, \gamma_1 = \bar{\alpha}_1 + 2\beta_1 - \bar{\alpha}_1\beta_1$ and $\Lambda \neq 0$.*

*Then, for $h_1, h_2 \in C([c,d], \mathbb{R})$, the unique solution of the coupled system*

$$\begin{cases} ({}^H D^{\alpha_1,\beta_1;\psi} + k\, {}^H D^{\alpha_1-1,\beta_1;\psi}) u(z) = h_1(z), \ \ z \in [c,d], \\ ({}^H D^{\bar{\alpha}_1,\beta_1;\psi} + k\, {}^H D^{\bar{\alpha}_1-1,\beta_1;\psi}) v(z) = h_2(z), \ \ z \in [c,d], \\ u(c) = 0, \ u(d) = \sum\limits_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) v(s) ds + \sum\limits_{j=1}^{m} \theta_j v(\xi_j), \\ v(c) = 0, \ v(d) = \sum\limits_{r=1}^{p} v_r \int_c^{\varsigma_r} \psi'(s) u(s) ds + \sum\limits_{s=1}^{q} \tau_s u(\sigma_s), \end{cases} \tag{6}$$

*is given as*

$$\begin{aligned} u(z) \ = \ & -k \int_c^z \psi'(s) u(s) ds + I_{c^+}^{\alpha_1;\psi} h_1(z) \\ & + \frac{(\psi(z) - \psi(c))^{\gamma-1}}{\Lambda \Gamma(\gamma)} \Big[ \Delta \Big( \sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) I^{\bar{\alpha}_1;\psi} h_2(s) ds + \sum_{j=1}^{m} \theta_j I^{\bar{\alpha}_1;\psi} h_2(\xi_j) \\ & - k \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} \psi'(s) v(s) ds - k \sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) \int_c^s \psi'(t) v(t) dt ds \\ & + k \int_c^d \psi'(s) u(s) ds - I_{c^+}^{\alpha_1;\psi} h_1(d) \Big) + B \Big( \sum_{r=1}^{p} v_r \int_c^{\varsigma_r} \psi'(s) I^{\alpha_1;\psi} h_1(s) ds \\ & + \sum_{s=1}^{q} \tau_s I^{\alpha_1;\psi} h_1(\sigma_s) - k \sum_{s=1}^{q} \tau_s \int_c^{\sigma_s} \psi'(s) u(s) ds \\ & - k \sum_{r=1}^{p} v_r \int_a^{\varsigma_r} \psi'(s) \int_c^s \psi'(t) u(t) dt ds \end{aligned}$$

$$+k\int_{c}^{d}\psi'(s)v(s)ds-I_{c^+}^{\overline{\alpha}_1;\psi}h_2(d)\Big)\Big], \tag{7}$$

*and*

$$\begin{aligned} v(z) &= -k\int_{c}^{z}\psi'(s)v(s)ds+I_{c^+}^{\overline{\alpha}_1;\psi}h_2(z) \\ &+\frac{(\psi(z)-\psi(c))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\Big[A\Big(\sum_{r=1}^{p}v_r\int_{c}^{\varsigma_r}\psi'(s)I^{\alpha_1;\psi}h_1(s)ds+\sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi}h_1(\sigma_s) \\ &-k\sum_{s=1}^{q}\tau_s\int_{c}^{\sigma_s}\psi'(s)u(s)ds-k\sum_{r=1}^{p}v_r\int_{c}^{\varsigma_r}\psi'(s)\int_{c}^{s}\psi'(t)u(t)dtds \\ &+k\int_{c}^{c}\psi'(s)v(s)ds-I_{c^+}^{\overline{\alpha}_1\psi}h_2(d)\Big)+\Gamma\Big(\sum_{i=1}^{n}\mu_i\int_{c}^{\eta_i}\psi'(s)I^{\overline{\alpha}_1;\psi}h_2(s)ds \\ &+\sum_{j=1}^{m}\theta_j I^{\overline{\alpha}_1;\psi}h_2(\xi_j)-k\sum_{j=1}^{m}\theta_j\int_{c}^{\xi_j}\psi'(s)v(s)ds \\ &-k\sum_{i=1}^{n}\mu_i\int_{c}^{\eta_i}\psi'(s)\int_{c}^{s}\psi'(t)v(t)dtds \\ &+k\int_{c}^{d}\psi'(s)u(s)ds-I_{c^+}^{\alpha_1;\psi}h_1(d)\Big)\Big], \end{aligned} \tag{8}$$

*where*

$$\begin{aligned} A &= \frac{(\psi(d)-\psi(c))^{\gamma-1}}{\Gamma(\gamma)}, \\ B &= \sum_{i=1}^{n}\mu_i\int_{c}^{\eta_i}\frac{\psi'(s)(\psi(s)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}ds+\sum_{j=1}^{m}\theta_j\frac{(\psi(\xi_j)-\psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)}, \\ \Gamma &= \sum_{r=1}^{p}v_r\int_{c}^{\varsigma_r}\frac{\psi'(s)(\psi(s)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}ds+\sum_{s=1}^{q}\tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\gamma-1}}{\Gamma(\gamma)}, \\ \Delta &= \frac{(\psi(d)-\psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)}, \end{aligned} \tag{9}$$

*and*

$$\Lambda = A\Delta - B\Gamma.$$

**Proof.** Taking the operator $I^{\alpha}$ on both sides of equations in (6) and using Lemma 1, we conclude that

$$\begin{aligned} u(z) &= c_0\frac{(\psi(z)-\psi(c))^{-(2-\alpha_1)(1-\beta_1)}}{\Gamma(1-(2-\alpha_1)(1-\beta_1))}+c_1\frac{(\psi(z)-\psi(c))^{1-(2-\alpha_1)(1-\beta_1)}}{\Gamma(2-(2-\alpha_1)(1-\beta_1)} \\ &-k\int_{c}^{z}\psi'(s)u(s)ds+I_{c^+}^{\alpha_1;\psi}h_1(z) \\ &= c_0\frac{(\psi(z)-\psi(c))^{\gamma-2}}{\Gamma(\gamma-1)}+c_1\frac{(\psi(z)-\psi(c))^{\gamma-1}}{\Gamma(\gamma)} \\ &-k\int_{c}^{z}\psi'(s)u(s)ds+I_{c^+}^{\alpha_1;\psi}h_1(z), \\ v(z) &= d_0\frac{(\psi(z)-\psi(c))^{\gamma_1-2}}{\Gamma(\gamma_1-1)}+d_1\frac{(\psi(z)-\psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)} \\ &-k\int_{c}^{z}\psi'(s)v(s)ds+I_{c^+}^{\overline{\alpha}_1;\psi}h_2(z). \end{aligned}$$

Hence, due to $u(c), v(c) = 0$, we obtain $c_0, d_0 = 0$. Consequently,

$$u(z) = c_1 \frac{(\psi(z) - \psi(c))^{\gamma-1}}{\Gamma(\gamma)} - k \int_c^z \psi'(s)u(s)ds + I_{c^+}^{\alpha_1;\psi} h_1(z), \tag{10}$$

$$v(z) = d_1 \frac{(\psi(z) - \psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)} - k \int_c^z \psi'(s)v(s)ds + I_{c^+}^{\bar{\alpha}_1;\psi} h_2(z). \tag{11}$$

From $u(d) = \sum_{i=1}^n \mu_i \int_c^{\eta_i} \psi'(s)v(s)ds + \sum_{j=1}^m \theta_j v(\xi_j)$ and $v(d) = \sum_{r=1}^p v_r \int_c^{\varsigma_r} \psi'(s)u(s)ds$ $+ \sum_{s=1}^q \tau_s u(\sigma_s)$, we have

$$\begin{aligned} & c_1 \frac{(\psi(d) - \psi(c))^{\gamma-1}}{\Gamma(\gamma)} - k \int_c^d \psi'(s)u(s)ds + I_{c^+}^{\alpha_1;\psi} h_1(d) \\ = \;& d_1 \sum_{i=1}^n \mu_i \int_c^{\eta_i} \frac{\psi'(s)(\psi(s) - \psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)} ds + \sum_{i=1}^n \mu_i \int_c^{\eta_i} \psi'(s) I^{\bar{\alpha}_1;\psi} h_2(s)ds \\ & + d_1 \sum_{j=1}^m \theta_j \frac{(\psi(\xi_j) - \psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)} - k \sum_{j=1}^m \theta_j \int_c^{\xi_j} \psi'(s)v(s)ds + \sum_{j=1}^m \theta_j I^{\bar{\alpha}_1;\psi} h_2(\xi_j) \\ & - k \sum_{i=1}^n \mu_i \int_c^{\eta_i} \psi'(s) \int_c^s \psi'(t)v(t)dtds, \end{aligned}$$

and

$$\begin{aligned} & d_1 \frac{(\psi(d) - \psi(c))^{\gamma_1-1}}{\Gamma(\gamma_1)} - k \int_c^d \psi'(s)v(s)ds + I_{c^+}^{\bar{\alpha}_1\psi} h_2(d) \\ = \;& c_1 \sum_{r=1}^p v_r \int_a^{\varsigma_r} \frac{\psi'(s)(\psi(s) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} ds + \sum_{r=1}^p v_r \int_c^{\varsigma_r} \psi'(s) I^{\alpha_1;\psi} h_1(s)ds \\ & + c_1 \sum_{s=1}^q \tau_s \frac{(\psi(\sigma_s) - \psi(c))^{\gamma-1}}{\Gamma(\gamma)} - k \sum_{s=1}^q \tau_s \int_a^{\sigma_s} \psi'(s)u(s)ds + \sum_{s=1}^q \tau_s I^{\alpha_1;\psi} h_1(\sigma_s) \\ & - k \sum_{r=1}^p v_r \int_a^{\varsigma_r} \psi'(s) \int_c^s \psi'(t)u(t)dtds, \end{aligned}$$

or

$$\begin{aligned} Ac_1 - Bd_1 &= P, \\ -\Gamma c_1 + \Delta d_1 &= Q, \end{aligned}$$

where $A, B, \Gamma, \Delta$ are defined by (9) and

$$\begin{aligned} P \;=\; & \sum_{i=1}^n \mu_i \int_c^{\eta_i} \psi'(s) I^{\bar{\alpha}_1;\psi} h_2(s)ds + \sum_{j=1}^m \theta_j I^{\bar{\alpha}_1;\psi} h_2(\xi_j) - k \sum_{j=1}^m \theta_j \int_c^{\xi_j} \psi'(s)v(s)ds \\ & - k \sum_{i=1}^n \mu_i \int_c^{\eta_i} \psi'(s) \int_c^s \psi'(t)v(t)dtds + k \int_c^d \psi'(s)u(s)ds - I_{c^+}^{\alpha_1;\psi} h_1(d), \end{aligned}$$

and

$$\begin{aligned} Q \;=\; & \sum_{r=1}^p v_r \int_c^{\varsigma_r} \psi'(s) I^{\alpha_1;\psi} h_1(s)ds + \sum_{s=1}^q \tau_s I^{\alpha_1;\psi} h_1(\sigma_s) - k \sum_{s=1}^q \tau_s \int_c^{\sigma_s} \psi'(s)u(s)ds \\ & - k \sum_{r=1}^p v_r \int_c^{\varsigma_r} \psi'(s) \int_c^s \psi'(t)u(t)dtds + k \int_c^d \psi'(s)v(s)ds - I_{c^+}^{\bar{\alpha}_1\psi} h_2(d). \end{aligned}$$

By solving the above system, we find

$$c_1 = \frac{\Delta P + BQ}{\Lambda}, \quad d_1 = \frac{AQ + \Gamma P}{\Lambda}.$$

Substituting the values of $c_1$ and $d_1$ into Equations (10) and (11), respectively, we obtain the solutions (7) and (8). The converse is obtained by direct computation. The proof is finished. □

## 4. Existence and Uniqueness Results

Keeping in mind Lemma 5, we define an operator $\mathcal{P} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by

$$\mathcal{P}(u,v)(z) := \big(\mathcal{P}_1(u,v)(z), \mathcal{P}_2(u,v)(z)\big), \tag{12}$$

where

$$\begin{aligned}
\mathcal{P}_1(u,v)(z) \;=\; & -k\int_c^z \psi'(s)u(s)ds + I_{c^+}^{\alpha_1;\psi} f_{uv}(z) \\
& + \frac{(\psi(z)-\psi(c))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big[\Delta\Big(\sum_{i=1}^{n}\mu_i \int_c^{\eta_i}\psi'(s) I^{\bar{\alpha}_1;\psi} g_{uv}(s)ds + \sum_{j=1}^{m}\theta_j I^{\bar{\alpha}_1;\psi} g_{uv}(\xi_j) \\
& - k\sum_{j=1}^{m}\theta_j \int_c^{\xi_j}\psi'(s)v(s)ds - k\sum_{i=1}^{n}\mu_i \int_c^{\eta_i}\psi'(s)\int_c^s \psi'(t)v(t)dtds \\
& + k\int_c^d \psi'(s)u(s)ds - I_{c^+}^{\alpha_1;\psi} f_{uv}(d)\Big) + B\Big(\sum_{r=1}^{p} v_r \int_c^{\varsigma_r}\psi'(s) I^{\alpha_1;\psi} f_{uv}(s)ds \\
& + \sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi} f_{uv}(\sigma_s) - k\sum_{s=1}^{q}\tau_s \int_c^{\sigma_s}\psi'(s)u(s)ds \\
& - k\sum_{r=1}^{p} v_r \int_c^{\varsigma_r}\psi'(s)\int_c^s \psi'(t)u(t)dtds \\
& + k\int_c^d \psi'(s)v(s)ds - I_{c^+}^{\bar{\alpha}_1\psi} g_{uv}(d)\Big)\Big],
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\mathcal{P}_2(u,v)(z) \;=\; & -k\int_c^z \psi'(s)v(s)ds + I_{c^+}^{\bar{\alpha}_1;\psi} g_{uv}(z) \\
& + \frac{(\psi(z)-\psi(c))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\Big[A\Big(\sum_{r=1}^{p} v_r \int_c^{\varsigma_r}\psi'(s) I^{\alpha_1;\psi} f_{uv}(s)ds + \sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi} h_1(\sigma_s) \\
& - k\sum_{s=1}^{q}\tau_s \int_c^{\sigma_s}\psi'(s)u(s)ds - k\sum_{r=1}^{p} v_r \int_c^{\varsigma_r}\psi'(s)\int_c^s \psi'(t)u(t)dtds \\
& + k\int_c^d \psi'(s)v(s)ds - I_{c^+}^{\bar{\alpha}_1\psi} g_{uv}(d)\Big) + \Gamma\Big(\sum_{i=1}^{n}\mu_i \int_c^{\eta_i}\psi'(s) I^{\bar{\alpha}_1;\psi} g_{uv}(s)ds \\
& + \sum_{j=1}^{m}\theta_j I^{\bar{\alpha}_1;\psi} g_{uv}(\xi_j) - k\sum_{j=1}^{m}\theta_j \int_c^{\xi_j}\psi'(s)v(s)ds \\
& - k\sum_{i=1}^{n}\mu_i \int_c^{\eta_i}\psi'(s)\int_c^s \psi'(t)v(t)dtds \\
& + k\int_c^d \psi'(s)u(s)ds - I_{c^+}^{\alpha_1;\psi} f_{uv}(d)\Big)\Big],
\end{aligned} \tag{14}$$

where

$$f_{uv}(z) = f(z,u(z),v(z)), \quad g_{uv}(z) = g(z,u(z),v(z)).$$

For the sake of convenience, we use the following notations:

$$
\begin{aligned}
\mathcal{A}_1 &= \frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
&\quad +|B|\Big(\sum_{r=1}^{p} v_r \frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)} + \sum_{s=1}^{q} \tau_s \frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\Big], && (15)\\
\mathcal{A}_2 &= \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n} \mu_i \frac{(\psi(\eta_i)-\psi(c))^{\bar{\alpha}_1+1}}{\Gamma(\bar{\alpha}_1+2)} \\
&\quad +\sum_{j=1}^{m} \theta_j \frac{(\psi(\xi_j)-\psi(c))^{\bar{\alpha}_1}}{\Gamma(\bar{\alpha}_1+1)}\Big) + |B|\frac{(\psi(d)-\psi(c))^{\bar{\alpha}_1}}{\Gamma(\bar{\alpha}_1+1)}\Big], && (16)\\
\mathcal{A}_3 &= |k|(\psi(d)-\psi(c)) + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta||k|(\psi(d)-\psi(c)) \\
&\quad +|B|\Big(|k|\sum_{s=1}^{q} \tau_s(\psi(\sigma_s)-\psi(c)) + \frac{1}{2}|k|\sum_{r=1}^{p} v_r(\psi(\varsigma_r)-\psi(c))^2\Big)\Big], && (17)\\
\mathcal{A}_4 &= \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(|k|\sum_{j=1}^{m} \theta_j(\psi(\xi_j)-\psi(c)) \\
&\quad +\frac{1}{2}|k|\sum_{i=1}^{n} \mu_i(\psi(\eta_i)-\psi(c))^2\Big) + |B||k|(\psi(d)-\psi(c))\Big], && (18)\\
\mathcal{B}_1 &= \frac{(\psi(d)-\psi(c))^{\gamma_1-1}}{|\Lambda|\Gamma(\gamma_1)}\Big[|A|\Big(v_r\sum_{i=1}^{n} \frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} + \sum_{s=1}^{q} \tau_s \frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big) \\
&\quad +|\Gamma|\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big], && (19)\\
\mathcal{B}_2 &= \frac{(\psi(d)-\psi(c))^{\bar{\alpha}_1}}{\Gamma(\bar{\alpha}_1+1)} + \frac{(\psi(d)-\psi(c))^{\gamma_1-1}}{|\Lambda|\Gamma(\gamma_1)}\Big[|A|\frac{(\psi(d)-\psi(c))^{\bar{\alpha}_1}}{\Gamma(\bar{\alpha}_1+1)} \\
&\quad +|\Gamma|\Big(\sum_{i=1}^{n} \mu_i \frac{(\psi(\eta_i)-\psi(c))^{\bar{\alpha}_1+1}}{\Gamma(\bar{\alpha}_1+2)} + \sum_{j=1}^{m} \theta_j \frac{(\psi(\xi_j)-\psi(c))^{\bar{\alpha}_1}}{\Gamma(\bar{\alpha}_1+1)}\Big)\Big], && (20)\\
\mathcal{B}_3 &= \frac{(\psi(d)-\psi(c))^{\gamma_1-1}}{|\Lambda|\Gamma(\gamma_1)}\Big[|A|\Big(|k|\sum_{j=1}^{m} \tau_s(\psi(\sigma_s)-\psi(c)) \\
&\quad +\frac{1}{2}|k|\sum_{r=1}^{p} v_r(\psi(\varsigma_r)-\psi(c))^2\Big) + |\Gamma||k|(\psi(d)-\psi(c))\Big], && (21)\\
\mathcal{B}_4 &= |k|(\psi(d)-\psi(c)) + \frac{(\psi(d)-\psi(c))^{\gamma_1-1}}{|\Lambda|\Gamma(\gamma_1)}\Big[|A||k|(\psi(d)-\psi(c)) \\
&\quad +|\Gamma|\Big(|k|\sum_{j=1}^{m} \theta_j)(\psi(\xi_j)-\psi(c)) + \frac{1}{2}|k|\sum_{i=1}^{n} \mu_i(\psi(\eta_i)-\psi(c))^2\Big)\Big]. && (22)
\end{aligned}
$$

### *4.1. Existence and Uniqueness Result via Banach Fixed Point Theorem*

Here, by using the Banach contraction mapping principle, we prove an existence and uniqueness result.

**Theorem 1.** *Assume that $\Lambda \neq 0$ and $f, g : [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ two functions satisfying the condition:*
*$(H_1)$ there exist positive real constants $\ell_1, \ell_2$ such that, for all $z \in [c,d]$ and $u_i, v_i \in \mathbb{R}, i = 1,2$ we have*

$$
\begin{aligned}
|f(z,u_1,v_1) - f(z,u_2,v_2)| &\leq \ell_1\big(|u_1-u_2| + |v_1-v_2|\big),\\
|g(z,u_1,v_1) - g(z,u_2,v_2)| &\leq \ell_2\big(|u_1-u_2| + |v_1-v_2|\big).
\end{aligned}
$$

*Then, system (5) admits a unique solution on $[c,d]$ provided that*

$$\ell_1(\mathcal{A}_1+\mathcal{B}_1) + \ell_2(\mathcal{A}_2+\mathcal{B}_2) + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}_3 + \mathcal{B}_4 < 1, \tag{23}$$

*where $\mathcal{A}_i, \mathcal{B}_i, i = 1,2,3,4$ are given by (15)–(18) and (19)–(22) respectively.*

**Proof.** We transform system (5) into a fixed point problem, $(u,v)(z) = \mathcal{P}(u,v)(z)$, where the operator $\mathcal{P}$ is defined as in (12). Applying the Banach contraction mapping principle, we show that the operator $\mathcal{P}$ has a unique fixed point, which is the unique solution of system (5).

Let $\sup_{z\in[c,d]}|f(z,0,0)| := \overline{M} < \infty$ and $\sup_{z\in[c,d]}|g(z,0,0)| := \overline{N} < \infty$. Next, we set $B_r := \{(u,v) \in \mathcal{X}\times\mathcal{X} : \|(u,v)\| \le r\}$ with

$$r \ge \frac{\overline{M}(\mathcal{A}_1+\mathcal{B}_1)+\overline{N}(\mathcal{A}_2+\mathcal{B}_2)}{1-[\ell_1(\mathcal{A}_1+\mathcal{B}_1)+\ell_2(\mathcal{A}_2+\mathcal{B}_2)+\mathcal{A}_3+\mathcal{A}_4+\mathcal{B}_3+\mathcal{B}_4]}. \tag{24}$$

Observe that $B_r$ is a bounded, closed, and convex subset of $\mathcal{X}$.

First, we show that $\mathcal{P}B_r \subset B_r$.

For any $(u,v) \in B_r$, $z \in [c,d]$, using the condition $(H_1)$, we have

$$\begin{aligned}|f_{uv}(z)| = |f(z,u(z),v(z))| &\le |f(z,u(z),v(z)) - f(z,0,0)| + |f(z,0,0)| \\ &\le \ell_1(|u(z)|+|v(z)|)+\overline{M} \\ &\le \ell_1(\|u\|+\|v\|)+\overline{M} \le \ell_1 r+\overline{M},\end{aligned}$$

and

$$|g_{uv}(z)| = |g(z,u(z),v(z))| \le \ell_2(\|u\|+\|v\|)+\overline{N} \le \ell_2 r+\overline{N}.$$

Then, we obtain

$$\begin{aligned}
& |\mathcal{P}_1(u,v)(z)| \\
\le\ & |k|\int_c^d \psi'(s)|u(s)|ds + I_{c^+}^{\alpha_1;\psi}|f_{uv}|(z) \\
& + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^n \mu_i \int_c^{\eta_i}\psi'(s)I^{\overline{\alpha}_1;\psi}|g_{uv}|(s)ds + \sum_{j=1}^m \theta_j I^{\overline{\alpha}_1;\psi}|g_{uv}|(\xi_j) \\
& + |k|\sum_{j=1}^m \theta_j \int_c^{\xi_j}\psi'(s)|v(s)|ds + |k|\sum_{i=1}^n \mu_i\int_c^{\eta_i}\psi'(s)\int_c^s \psi'(t)|v(t)|dtds \\
& + |k|\int_c^d \psi'(s)|u(s)|ds + I_{c^+}^{\alpha_1;\psi}|f_{uv}|(d)\Big) + |B|\Big(\sum_{r=1}^p v_r\int_c^{\varsigma_r}\psi'(s)I^{\alpha_1;\psi}|f_{uv}|(s)ds \\
& + \sum_{s=1}^q \tau_s I^{\alpha_1;\psi}|f_{uv}|(\sigma_s) + |k|\sum_{s=1}^q \tau_s\int_c^{\sigma_s}\psi'(s)|u(s)|ds \\
& + |k|\sum_{r=1}^p v_r\int_a^{\varsigma_r}\psi'(s)\int_c^s\psi'(t)|u(t)|dtds + |k|\int_c^d\psi'(s)|v(s)|ds + I_{c^+}^{\overline{\alpha}_1\psi}|g_{uv}|(d)\Big)\Big] \\
\le\ & |k|(\psi(d)-\psi(c))\|u\| + \frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\ell_1 r+\overline{M}) \\
& + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^n \mu_i\frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)}(\ell_2 r+\overline{N}) \\
& + \sum_{j=1}^m \theta_j\frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}(\ell_2 r+\overline{N}) + |k|\sum_{j=1}^m\theta_j(\psi(\xi_j)-\psi(c))\|v\| \\
& + \frac{1}{2}|k|\sum_{i=1}^n \mu_i(\psi(\eta_i)-\psi(c))^2\|v\| + |k|(\psi(d)-\psi(c))\|u\| \\
& + \frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\ell_1 r+\overline{M})\Big) + |B|\Big(\sum_{r=1}^p v_r\frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}(\ell_1 r+\overline{M}) \\
& + \sum_{s=1}^q \tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}(\ell_1 r+\overline{M}) + |k|\sum_{s=1}^q\tau_s(\psi(\sigma_s)-\psi(c))\|u\|
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}|k|\sum_{r=1}^{p} v_r(\psi(\varsigma_r)-\psi(c))^2\|u\|+|k|(\psi(d)-\psi(c)\|v\| \\
& +\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}(\ell_2 r+\overline{N})\Big)\Big] \\
\leq\ & (\ell_1 r+\overline{M})\Big\{\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
& +|B|\Big(\sum_{r=1}^{p} v_r\frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_2+2)}+\sum_{s=1}^{q}\tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\Big]\Big\} \\
& +(\ell_2 r+\overline{N})\Big\{\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n}\mu_i\frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)} \\
& +\sum_{j=1}^{m}\theta_j\frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big)+|B|\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big]\Big\}+r\Big\{|k|(\psi(d)-\psi(c)) \\
& +\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta||k|(\psi(d)-\psi(c))+|B|\Big(|k|\sum_{s=1}^{q}\tau_s(\psi(\sigma_s)-\psi(c)) \\
& +\frac{1}{2}|k|\sum_{r=1}^{p} v_r(\psi(\varsigma_r)-\psi(c))^2\Big)\Big]\Big\}+r\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(|k|\sum_{j=1}^{m}\theta_j(\psi(\xi_j)-\psi(c)) \\
& +\frac{1}{2}|k|\sum_{i=1}^{n}\mu_i(\psi(\eta_i)-\psi(c))^2\Big)+|B||k|(\psi(d)-\psi(c))\Big] \\
=\ & \mathcal{A}_1(\ell_1 r+\overline{M})+\mathcal{A}_2(\ell_2 r+\overline{N})+r(\mathcal{A}_3+\mathcal{A}_4).
\end{aligned}
$$

Hence

$$\|\mathcal{P}_1(u,v)\|\leq\mathcal{A}_1(\ell_1 r+\overline{M})+\mathcal{A}_2(\ell_2 r+\overline{N})+r(\mathcal{A}_3+\mathcal{A}_4).$$

Similarly, we find that

$$\|\mathcal{P}_2(u,v)\|\leq\mathcal{B}_1(\ell_1 r+\overline{M})+\mathcal{B}_2(\ell_2 r+\overline{N})+r(\mathcal{B}_3+\mathcal{B}_4).$$

Consequently, we have

$$
\begin{aligned}
\|\mathcal{P}(x,y)\| \leq\ & \Big[\ell_1(\mathcal{A}_1+\mathcal{B}_1)+\ell_2(\mathcal{A}_2+\mathcal{B}_2)+\mathcal{A}_3+\mathcal{A}_4+\mathcal{B}_3+\mathcal{B}_4\Big]r \\
& +(\mathcal{A}_1+\mathcal{B}_1)\overline{M}+(\mathcal{A}_2+\mathcal{B}_2)\overline{N}\leq r,
\end{aligned}
$$

which implies that $\mathcal{P}B_r\subset B_r$.

Next we show that $\mathcal{P}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ is a contraction.

Using condition $(H_1)$, for any $(u_1,v_1)$, $(u_2,v_2)\in\mathcal{X}\times\mathcal{X}$ and for each $z\in[c,d]$, we have

$$
\begin{aligned}
& |\mathcal{P}_1(u_1,v_1)(z)-\mathcal{P}_1(u_2,v_2)(z)| \\
\leq\ & |k|\int_c^d\psi'(s)|u_1(s)-u_2(s)|ds+I_{c^+}^{\alpha_1;\psi}|f_{u_1v_1}-f_{u_2v_2}|(z) \\
& +\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n}\mu_i\int_c^{\eta_i}\psi'(s)I^{\overline{\alpha}_1;\psi}|g_{u_1v_1}-g_{u_2v_2}|(s)ds \\
& +\sum_{j=1}^{m}\theta_j I^{\overline{\alpha}_1;\psi}|g_{u_1v_1}-g_{u_2v_2}|(\xi_j)+|k|\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}\psi'(s)|v_1(s)-v_2(s)|ds \\
& +|k|\sum_{i=1}^{n}\mu_i\int_c^{\eta_i}\psi'(s)\int_c^{s}\psi'(s)|v(t)|dtds+|k|\int_c^d\psi'(t)|u_1(s)-u_2(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
&+I_{c^+}^{\alpha_1;\psi}|f_{u_1v_1}-f_{u_2v_2}|(b)\Big)+|B|\Big(\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)I^{\alpha_1;\psi}|f_{u_1v_1}-f_{u_2v_2}|(s)ds\\
&+\sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi}|f_{u_1v_1}-f_{u_2v_2}|(\sigma_s)+|k|\sum_{s=1}^{q}\tau_s\int_c^{\sigma_s}\psi'(s)|u_1(s)-u_2(s)|ds\\
&+|k|\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)\int_c^{s}\psi'(t)|u(t)|dtds+|k|\int_c^{d}\psi'(s)|v_1(s)-v_2(s)|ds\\
&+I_{c^+}^{\overline{\alpha}_1\psi}|g_{u_1v_1}-g_{u_2v_2}|(d)\Big)\Big]\\
\le\ & |k|(\psi(d)-\psi(c))\|u_1-u_2\|+\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1(\|u_1-u_2\|+\|v_1-v_2\|)\\
&+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n}\mu_i\frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)}\ell_2(\|u_1-u_2\|+\|v_1-v_2\|)\\
&+\sum_{j=1}^{m}\theta_j\frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}(\ell_2 r+N)+|k|\sum_{j=1}^{m}\theta_j(\psi(\xi_j)-\psi(c))\|v_1-v_2\|\\
&+\frac{1}{2}|k|\sum_{i=1}^{n}\mu_i(\psi(\eta_i)-\psi(c))^2\|v_1-v_2\|+|k|(\psi(d)-\psi(c))\|u_1-u_2\|\\
&+\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1(\|u_1-u_2\|+\|v_1-v_2\|)\Big)\\
&+|B|\Big(\sum_{r=1}^{p}v_r\frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\ell_1(\|u_1-u_2\|+\|v_1-v_2\|)\\
&+\sum_{s=1}^{q}\tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1(\|u_1-u_2\|+\|v_1-v_2\|)\\
&+|k|\sum_{s=1}^{q}\tau_s(\psi(\sigma_s)-\psi(c))\|u_1-u_2\|+\frac{1}{2}|k|\sum_{r=1}^{p}v_r(\psi(\varsigma_r)-\psi(c))^2\|u_1-u_2\|\\
&+|k|(\psi(d)-\psi(c)\|v_1-v_2\|+\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\ell_2(\|u_1-u_2\|+\|v_1-v_2\|)\Big)\Big]\\
\le\ & \ell_1(\|u_1-u_2\|+\|v_1-v_2\|)\Big\{\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
&+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}+|B|\Big(\sum_{r=1}^{p}v_r\frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\\
&+\sum_{s=1}^{q}\tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\Big]\Big\}\\
&+\ell_2(\|u_1-u_2\|+\|v_1-v_2\|)\Big\{\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n}\mu_i\frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)}\\
&+\sum_{j=1}^{m}\theta_j\frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big)+|B|\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big]\Big\}\\
&+\|u_1-u_2\|\Big\{|k|(\psi(d)-\psi(c))+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta||k|(\psi(d)-\psi(c))\\
&+|B|\Big(|k|\sum_{s=1}^{q}\tau_s(\psi(\sigma_s)-\psi(c))+\frac{1}{2}|k|\sum_{r=1}^{p}v_r(\psi(\varsigma_r)-\psi(c))^2\Big)\Big]\Big\}\\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(|k|\sum_{j=1}^{m}\theta_j(\psi(\xi_j)-\psi(a))\\
&+\frac{1}{2}|k|\sum_{i=1}^{n}\mu_i(\psi(\eta_i)-\psi(c))^2\Big)+|B||k|(\psi(d)-\psi(c))\Big]\|v_1-v_2\|
\end{aligned}
$$

$$
\begin{aligned}
&= (\ell_1\mathcal{A}_1 + \ell_2\mathcal{A}_2)(\|u_1 - u_2\| + \|v_1 - v_2\|) + \mathcal{A}_3\|u_1 - u_2\| + \mathcal{A}_4\|v_1 - v_2\|. \\
&\le (\ell_1\mathcal{A}_1 + \ell_2\mathcal{A}_2) + \mathcal{A}_3 + \mathcal{A}_4)(\|v_1 - v_2\| + \|v_1 - v_2\|),
\end{aligned}
$$

and therefore

$$
\|\mathcal{P}_1(u_1, v_1) - \mathcal{P}_1(u_2, v_2)\| \le (\ell_1\mathcal{A}_1 + \ell_2\mathcal{A}_2) + \mathcal{A}_3 + \mathcal{A}_4)(\|u_1 - u_2\| + \|v_1 - v_2\|). \tag{25}
$$

Similarly, we find that

$$
\|\mathcal{P}_2(u_1, v_1) - \mathcal{P}_2(u_2, v_2)\| \le (\ell_1\mathcal{B}_1 + \ell_2\mathcal{B}_2) + \mathcal{B}_3 + \mathcal{B}_4)(\|u_1 - u_2\| + \|v_1 - v_2\|). \tag{26}
$$

From (25) and (26), it yields

$$
\begin{aligned}
\|\mathcal{P}(u_1, u_1) - \mathcal{P}(u_2, u_2)\| \le & \Big[\ell_1(\mathcal{A}_1 + \mathcal{B}_1) + \ell_2(\mathcal{A}_2 + \mathcal{B}_2) + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}_3 + \mathcal{B}_4\Big] \\
& \times \Big(\|u_1 - u_2\| + \|v_1 - v_2\|\Big).
\end{aligned}
$$

Since $\ell_1(\mathcal{A}_1 + \mathcal{B}_1) + \ell_2(\mathcal{A}_2 + \mathcal{B}_2) + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}_3 + \mathcal{B}_4 < 1$, by (23), the operator $\mathcal{P}$ is a contraction. Therefore, using the Banach contraction mapping principle (Lemma 1), the operator $\mathcal{P}$ has a unique fixed point. Hence, system (5) has a unique solution on $[c, d]$. The proof is completed. □

*4.2. Existence Result via Leray-Schauder Alternative*

The Leray–Schauder alternative (Lemma 3) is used in the proof of our first existence result.

**Theorem 2.** *Let $\Lambda \neq 0$, and $f, g : [c, d] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. Assume that:*

$(H_2)$ *There exist real constants $u_i, v_i \ge 0$ for $i = 1, 2$ and $u_0, v_0 > 0$ such that for all $u, v \in \mathbb{R}$, we have*

$$
\begin{aligned}
|f(z, u(z), v(z))| &\le u_0 + u_1|u| + u_2|v|, \\
|g(z, u(z), v(z))| &\le v_0 + v_1|u| + v_2|v|.
\end{aligned}
$$

*If $(\mathcal{A}_1 + \mathcal{B}_1)u_1 + (\mathcal{A}_2 + \mathcal{B}_2)v_1 + \mathcal{A}_3 + \mathcal{B}_3 < 1$ and $(\mathcal{A}_1 + \mathcal{B}_1)u_2 + (\mathcal{A}_2 + \mathcal{B}_2)v_2 + \mathcal{A}_4 + \mathcal{B}_4 < 1$, where $\mathcal{A}_i$, $\mathcal{B}_i$ for $i = 1, 2$ are given by (15)–(18) and (19)–(22), respectively, then the system (5) admits at least one solution on $[c, d]$.*

**Proof.** Obviously, the operator $\mathcal{P}$ is continuous, due to the continuity of the functions $f, g$ on $[c, d] \times \mathbb{R}^2$. Now, show that the operator $\mathcal{P} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is completely continuous. Let $B_r \subset \mathcal{X} \times \mathcal{X}$ be a bounded set, where $B_r = \{(u, v) \in \mathcal{X} \times \mathcal{X} : \|(u, v)\| \le r\}$. Then, for any $(u, v) \in B_r$, there exist positive real numbers $W_1$ and $W_2$ such that $|f_{uv}(z)| = |f(z, u(t), v(z))| \le W_1$ and $|g_{uv}(z)| = |g(z, u(z), v(z))| \le W_2$.

Thus, for each $(u, v) \in B_r$ we have

$$
\begin{aligned}
& |\mathcal{P}_1(u, v)(z)| \\
\le\ & |k| \int_c^d \psi'(s)|u(s)|ds + I_{c^+}^{\alpha_1;\psi}|f_{uv}(z)| \\
& + \frac{(\psi(d) - \psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \Big[|\Delta|\Big(\sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) I^{\bar{\alpha}_1;\psi}|g_{uv}(s)|ds + \sum_{j=1}^{m} \theta_j I^{\bar{\alpha}_1;\psi}|g_{uv}(\xi_j)| \\
& + |k| \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} \psi'(s)|v(s)|ds + |k| \sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) \int_c^{s} \psi'(t)|v(t)|dtds \\
& + |k| \int_c^d \psi'(s)|u(s)|ds + I_{c^+}^{\alpha_1;\psi}|f_{uv}|(d)\Big) + |B|\Big(\sum_{r=1}^{p} v_r \int_c^{\varsigma_r} \psi'(s) I^{\alpha_1;\psi}|f_{uv}|(s)ds
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi}|f_{uv}|(\sigma_s)+|k|\sum_{s=1}^{q}\tau_s\int_c^{\sigma_s}\psi'(s)|u(s)|ds \\
& +|k|\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)\int_c^{s}\psi'(t)|u(t)|dtds \\
& +|k|\int_c^{d}\psi'(s)|v(s)|ds+I_{c^+}^{\overline{\alpha}_1\psi}|g_{uv}|(d)\Big)\Big] \\
\leq\ & |k|(\psi(d)-\psi(c))\|u\|+\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}W_1 \\
& +\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n}\mu_i\frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)}W_2+\sum_{j=1}^{m}\theta_j\frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}W_2 \\
& +|k|\sum_{j=1}^{m}\theta_j(\psi(\xi_j)-\psi(c))\|v\|+\frac{1}{2}|k|\sum_{i=1}^{n}\mu_i(\psi(\eta_i)-\psi(c))^2\|v\| \\
& +|k|(\psi(d)-\psi(c))\|u\|+\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}W_1\Big)+|B|\Big(\sum_{r=1}^{p}v_r\frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}W_1 \\
& +\sum_{s=1}^{q}\tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}W_1+|k|\sum_{s=1}^{q}\tau_s(\psi(\sigma_s)-\psi(c))\|u\| \\
& +\frac{1}{2}|k|\sum_{r=1}^{n}v_r(\psi(\varsigma_r)-\psi(c))^2\|u\|+|k|(\psi(d)-\psi(c)\|v\|+\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}W_2\Big)\Big] \\
\leq\ & W_1\Big\{\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
& +|B|\Big(\sum_{r=1}^{n}v_r\frac{(\psi(\varsigma_r)-\psi(a))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}+\sum_{s=1}^{q}\tau_s\frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\Big]\Big\} \\
& +W_2\Big\{\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(\sum_{i=1}^{n}\mu_i\frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)} \\
& +\sum_{j=1}^{m}\theta_j\frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big)+|B|\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big]\Big\}+r\Big\{|k|(\psi(d)-\psi(c)) \\
& +\frac{(\psi(d)-\psi(c))^{\gamma-1}}{\Gamma(\gamma)}\Big[|\Delta||k|(\psi(d)-\psi(c))+|B|\Big(|k|\sum_{s=1}^{q}\tau_s(\psi(\sigma_s)-\psi(c)) \\
& +\frac{1}{2}|k|\sum_{r=1}^{p}v_r(\psi(\varsigma_r)-\psi(c))^2\Big)\Big]\Big\}+r\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[|\Delta|\Big(|k|\sum_{j=1}^{m}\theta_j(\psi(\xi_j)-\psi(c)) \\
& +\frac{1}{2}|k|\sum_{i=1}^{n}\mu_i(\psi(\eta_i)-\psi(c))^2\Big)+|B||k|(\psi(d)-\psi(c))\Big] \\
=\ & \mathcal{A}_1W_1+\mathcal{A}_2W_2+r(\mathcal{A}_3+\mathcal{A}_4),
\end{aligned}
$$

which yields

$$\|\mathcal{P}_1(u,v)\|\leq\mathcal{A}_1W_1+\mathcal{A}_2W_2+r(\mathcal{A}_3+\mathcal{A}_4).$$

Similarly, we obtain that

$$\|\mathcal{P}_2(u,v)\|\quad\leq\quad\mathcal{B}_1W_1+\mathcal{B}_2W_2+r(\mathcal{B}_3+\mathcal{B}_4).$$

Hence, from the above inequalities, we find that the operator $\mathcal{P}$ is uniformly bounded, since

$$\|\mathcal{P}(u,v)\|\leq(\mathcal{A}_1+\mathcal{B}_1)W_1+(\mathcal{B}_1+\mathcal{B}_2)W_2+r(\mathcal{A}_3+\mathcal{A}_4+\mathcal{B}_3+\mathcal{B}_4).$$

Next, we prove that the operator $\mathcal{P}$ is equicontinuous. Let $\tau_1, \tau_2 \in [c,d]$ with $\tau_1 < \tau_2$. Then, we have

$$\begin{aligned}
& |\mathcal{P}_1(u,v)(\tau_2) - \mathcal{P}_1(u,v)(\tau_1)| \\
\leq\ & \Big| I_{c^+}^{\alpha_1;\psi} f_{uv}(\tau_2) - I_{c^+}^{\alpha_1;\psi} f_{uv} z(\tau_1) \Big| \\
& + \frac{(\psi(\tau_2)-\psi(c))^{\gamma-1} - (\psi(\tau_1)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \Big[ |\Delta| \Big( \sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) I^{\overline{\alpha}_1;\psi} |g_{uv}(s)| ds \\
& + \sum_{j=1}^{m} \theta_j I^{\overline{\alpha}_1;\psi} |g_{uv}(\xi_j)| + |k| \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} \psi'(s)|v(s)| ds \\
& + |k| \sum_{i=1}^{n} \mu_i \int_c^{\eta_i} \psi'(s) \int_c^{s} \psi'(t)|v(t)| dt ds + |k| \int_c^{d} \psi'(s)|u(s)| ds + I_{c^+}^{\alpha_1;\psi} |f_{uv}|(d) \Big) \\
& + |B| \Big( \sum_{r=1}^{p} v_r \int_c^{\varsigma_r} \psi'(s) I^{\alpha_1;\psi} |f_{uv}|(s) ds + \sum_{s=1}^{q} \tau_s I^{\alpha_1;\psi} |f_{uv}|(\sigma_s) \\
& + |k| \sum_{s=1}^{q} \tau_s \int_c^{\sigma_s} \psi'(s)|u(s)| ds + |k| \sum_{r=1}^{p} v_r \int_c^{\varsigma_r} \psi'(s) \int_c^{s} \psi'(t)|u(t)| dt ds \\
& + |k| \int_c^{d} \psi'(s)|v(s)| ds + I_{c^+}^{\overline{\alpha}_1 \psi} |g_{uv}|(d) \Big) \Big] \\
\leq\ & W_1 \Big| \int_c^{\tau_1} \psi'(s) \frac{(\psi(\tau_2)-\psi(s))^{\alpha_1-1} - (\psi(\tau_1)-\psi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} ds \\
& + \int_{\tau_1}^{\tau_2} \psi'(s) \frac{(\psi(\tau_2)-\psi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} ds \Big| \\
& + \frac{(\psi(\tau_2)-\psi(c))^{\gamma-1} - (\psi(\tau_1)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \Big[ |\Delta| \Big( \sum_{i=1}^{n} \mu_i \frac{(\psi(\eta_i)-\psi(c))^{\overline{\alpha}_1+1}}{\Gamma(\overline{\alpha}_1+2)} W_2 \\
& + \sum_{j=1}^{m} \theta_j \frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)} W_2 + |k| \sum_{j=1}^{m} \theta_j (\psi(\xi_j)-\psi(c)) \|v\| \\
& + \frac{1}{2}|k| \sum_{i=1}^{n} \mu_i (\psi(\eta_i)-\psi(c))^2 \|v\| + |k|(\psi(d)-\psi(c))\|u\| \\
& + \frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} W_1 \Big) + |B| \Big( \sum_{r=1}^{p} v_r \frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)} W_1 \\
& + \sum_{s=1}^{q} \tau_s \frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} W_1 + |k| \sum_{s=1}^{q} \tau_s (\psi(\sigma_s)-\psi(c)) \|u\| \\
& + \frac{1}{2}|k| \sum_{r=1}^{p} v_r (\psi(\varsigma_r)-\psi(c))^2 \|u\| + |k|(\psi(d)-\psi(c)\|v\| + \frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)} W_2 \Big) \Big] \\
\leq\ & \frac{W_1}{\Gamma(\alpha_1+1)} \Big[ 2(\psi(\tau_2)-\psi(\tau_1))^{\alpha_1} + (\psi(\tau_2)-\psi(c))^{\alpha_1} - (\psi(\tau_1)-\psi(c))^{\alpha_1} \Big] \\
& + \frac{[(\psi(\tau_2)-\psi(c))^{\gamma-1} - (\psi(\tau_1)-\psi(c))^{\gamma-1}]}{|\Lambda|\Gamma(\gamma)} \Big\{ W_1 |\Delta| \frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
& + |B| W_1 \Big( \sum_{r=1}^{p} v_r \frac{(\psi(\varsigma_r)-\psi(c))^{\alpha_1+1}}{\Gamma(\alpha_1+2)} + \sum_{s=1}^{q} \tau_s \frac{(\psi(\sigma_s)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)} \Big) \\
& + W_2 \Big[ |\Delta| \Big( \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} + \sum_{j=1}^{m} \theta_j \frac{(\psi(\xi_j)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)} \Big) + |B| \frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)} \Big] \\
& + r \Big[ |\Delta||k|(\psi(d)-\psi(c)) + |B| \Big( |k| \sum_{s=1}^{q} \tau_s (\psi(\sigma_s)-\psi(c))
\end{aligned}$$

$$+\frac{1}{2}|k|\sum_{r=1}^{p}v_r(\psi(\varsigma_r)-\psi(c))^2\Big)\Big]+r\Big[|\Delta|\Big(|k|\sum_{j=1}^{m}\theta_j(\psi(\xi_j)-\psi(c))$$
$$+\frac{1}{2}|k|\sum_{i=1}^{n}\mu_i(\psi(\eta_i)-\psi(c))^2\Big)+|B||k|(\psi(d)-\psi(c))\Big]\Big\}.$$

Therefore, we obtain

$$|\mathcal{P}_1(u,v)(\tau_2)-\mathcal{P}_1(u,v)(\tau_1)|\to 0, \quad \text{as } \tau_1\to\tau_2.$$

Analogously, we can obtain the following inequality:

$$|\mathcal{P}_2(u,v)(\tau_2)-\mathcal{P}_2(u,v)(\tau_1)|\to 0, \quad \text{as } \tau_1\to\tau_2.$$

Hence, the set $\mathcal{P}\Phi$ is equicontinuous. Accordingly, the Arzelá–Ascoli theorem implies that the operator $\mathcal{P}$ is completely continuous.

Finally, we show the boundedness of the set $\Xi=\{(u,v)\in\mathcal{X}\times\mathcal{X}:(u,v)=\mu\mathcal{P}(u,v), 0\le\mu\le1\}$. Let any $(u,v)\in\Xi$, then $(u,v)=\mu\mathcal{P}(u,v)$. We have, for all $z\in[c,d]$,

$$u(z)=\mu\mathcal{P}_1(u,v)(z), \quad v(z)=\mu\mathcal{P}_2(u,v)(z).$$

Then, we obtain

$$\begin{aligned}\|u\| &\le (u_0+u_1\|u\|+u_2\|v\|)\mathcal{A}_1+(v_0+v_1\|u\|+v_2\|v\|)\mathcal{A}_2+\|u\|(\mathcal{A}_3+\mathcal{B}_3),\\ \|v\| &\le (u_0+u_1\|u\|+u_2\|v\|)\mathcal{B}_1+(v_0+v_1\|u\|+v_2\|v\|)\mathcal{B}_2+\|v\|(\mathcal{A}_4+\mathcal{B}_4),\end{aligned}$$

which imply that

$$\begin{aligned}\|u\|+\|v\| \le\ & (\mathcal{A}_1+\mathcal{B}_1)u_0+(\mathcal{A}_2+\mathcal{B}_2)v_0+\Big[(\mathcal{A}_1+\mathcal{B}_1)u_1+(\mathcal{A}_2+\mathcal{B}_2)v_1\\ &+\mathcal{A}_3+\mathcal{B}_3\Big]\|u\|+\Big[(\mathcal{A}_1+\mathcal{B}_1)u_2+(\mathcal{A}_2+\mathcal{B}_2)v_2+\mathcal{A}_4+\mathcal{B}_4\Big]\|v\|.\end{aligned}$$

Thus, we obtain

$$\|(u,v)\|\le\frac{(\mathcal{A}_1+\mathcal{B}_1)u_0+(\mathcal{A}_2+\mathcal{B}_2)v_0}{M^*}, \tag{27}$$

where $M^*=\min\{1-(\mathcal{A}_1+\mathcal{B}_1)u_1-(\mathcal{A}_2+\mathcal{B}_2)v_1-(\mathcal{A}_3+\mathcal{B}_3),1-(\mathcal{A}_1+\mathcal{B}_1)u_2-(\mathcal{A}_2+\mathcal{B}_2)v_2-(\mathcal{A}_4+\mathcal{B}_4)\}$, which shows that the set $\Xi$ is bounded. Therefore, via the Leray–Schauder alternative (Lemma 3), the operator $\mathcal{P}$ has at least one fixed point. Hence, we deduce that problem (5) admits a solution on $[c,d]$, which completes the proof. □

### *4.3. Existence Result via Krasnosel'skiǐ's Fixed Point Theorem*

Now we apply Krasnosel'skiǐ's fixed point theorem (Lemma 4) to prove our second existence result.

**Theorem 3.** *Let $\Lambda\neq0$ and $f,g:[c,d]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous functions which satisfy the condition $(H_1)$ in Theorem 1. In addition, we assume that there exist two positive constants $Z_1,Z_2$ such that, for all $z\in[c,d]$ and $u_i,v_i\in\mathbb{R}, i=1,2$, we have*

$$\begin{aligned}&|\,f(z,u_1,v_1)\,|\le Z_1\\ &|\,f(z,u_1,v_1)\,|\le Z_2.\end{aligned} \tag{28}$$

*Moreover, assume that $\mathcal{A}_3+\mathcal{A}_4<1$, $\mathcal{B}_3+\mathcal{B}_4<1$ and $\left[\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1+\frac{(d-c)^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\ell_2\right]<1$. Then, problem (5) admits at least one solution on $[c,d]$.*

**Proof.** Let the operator $\mathcal{P}$, defined by (12), be decomposed into four operators as

$$
\begin{aligned}
\mathcal{M}(u,v)(z) &= k\int_c^t \psi'(s)u(s)ds \\
&+\frac{(\psi(z)-\psi(c))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big[\Delta\Big(\sum_{i=1}^{n}\mu_i\int_c^{\eta_i}\psi'(s)I^{\bar{\alpha}_1;\psi}g_{uv}(s)ds+\sum_{j=1}^{m}\theta_j I^{\bar{\alpha}_1;\psi}g_{uv}(\xi_j) \\
&-k\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}\psi'(s)v(s)ds-k\sum_{i=1}^{n}\mu_i\int_c^{\eta_i}\psi'(s)\int_c^{s}\psi'(t)v(t)dtds \\
&+k\int_c^{d}\psi'(s)u(s)ds-I_{c^+}^{\alpha_1;\psi}f_{uv}(d)\Big)+B\Big(\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)I^{\alpha_1;\psi}f_{uv}(s)ds \\
&+\sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi}f_{uv}(\sigma_s)-k\sum_{s=1}^{q}\tau_s\int_c^{\sigma_s}\psi'(s)u(s)ds \\
&-k\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)\int_c^{s}\psi'(t)u(t)dtds \\
&+k\int_c^{d}\psi'(s)v(s)ds-I_{c^+}^{\bar{\alpha}_1;\psi}g_{uv}(d)\Big)\Big], \\
\mathcal{N}(u,v)(z) &= I_{c^+}^{\alpha_1;\psi}f_{uv}(z), \\
\mathcal{T}(u,v)(z) &= -k\int_c^z \psi'(s)v(s)ds \\
&+\frac{(\psi(z)-\psi(c))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\Big[A\Big(\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)I^{\alpha_1;\psi}f_{uv}(s)ds+\sum_{s=1}^{q}\tau_s I^{\alpha_1;\psi}h_1(\sigma_s) \\
&-k\sum_{s=1}^{q}\tau_s\int_c^{\sigma_s}\psi'(s)u(s)ds-k\sum_{r=1}^{p}v_r\int_c^{\varsigma_r}\psi'(s)\int_c^{s}\psi'(t)u(t)dtds \\
&+k\int_c^{d}\psi'(s)v(s)ds-I_{c^+}^{\bar{\alpha}_1\psi}g_{uv}(d)\Big)+\Gamma\Big(\sum_{i=1}^{n}\mu_i\int_c^{\eta_i}\psi'(s)I^{\bar{\alpha}_1;\psi}g_{uv}(s)ds \\
&+\sum_{j=1}^{m}\theta_j I^{\bar{\alpha}_1;\psi}g_{uv}(\xi_j)-k\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}\psi'(s)v(s)ds \\
&-k\sum_{i=1}^{n}\mu_i\int_c^{\eta_i}\psi'(s)\int_c^{s}\psi'(t)v(t)dtds \\
&+k\int_c^{d}\psi'(s)u(s)ds-I_{c^+}^{\alpha_1;\psi}f_{uv}(d)\Big)\Big], \\
\mathcal{R}(u,v)(z) &= I_{c^+}^{\bar{\alpha}_1;\psi}g_{uv}(z).
\end{aligned}
\tag{29}
$$

Hence, $\mathcal{P}_1(u,v)(z) = \mathcal{M}(u,v)(z)+\mathcal{N}(u,v)(z)$ and $\mathcal{P}_2(u,v)(z) = \mathcal{T}(u,v)(z)+\mathcal{R}(u,v)(z)$. Let $B_\delta = \{(u,v)\in\mathcal{X}\times\mathcal{X};\|(u,v)\|\le\delta\}$, in which

$$\delta \ge \max\Big\{\frac{\mathcal{A}_1 Z_1+\mathcal{A}_2 Z_2}{1-(\mathcal{A}_3+\mathcal{A}_4)},\frac{\mathcal{B}_1 Z_1+\mathcal{B}_2 Z_2}{1-(\mathcal{B}_3+\mathcal{B}_4)}\Big\}.$$

First, we show that $\mathcal{P}_1(x,y)+\mathcal{P}_2(u,v)\in B_\delta$ for all $(x,y),(u,v)\in B_\delta$. As in the proof of Theorem 1, we have

$$
\begin{aligned}
&|\ \mathcal{M}(x,y)(z)+\mathcal{N}(u,v)(z)\ |\le \mathcal{A}_1 Z_1+\mathcal{A}_2 Z_2+(\mathcal{A}_3+\mathcal{A}_4)\delta\le\delta, \\
&|\ \mathcal{R}(x,y)(z)+\mathcal{S}(u,v)(z)\ |\le \mathcal{B}_1 Z_1+\mathcal{B}_2 Z_2+(\mathcal{B}_3+\mathcal{B}_4)\delta\le\delta.
\end{aligned}
\tag{30}
$$

Accordingly, $\mathcal{P}_1(x,y)+\mathcal{P}_2(u,v)\in B_\delta$ and the condition $(i)$ of Lemma 4 is satisfied. In the next step, we show that the operator $(\mathcal{N},\mathcal{R})$ is a contraction mapping. For $(x,y),(u,v)\in B_\delta$, we obtain

$$
\begin{aligned}
|\ \mathcal{N}(x,y)(z)-\mathcal{N}(u,v)(z)\ | \quad &\le I^{\alpha_1}\ |\ f_{x,y}-f_{u,v}\ |\ (z) \\
&\le \ell_1(\|x-u\|+\|y-v\|)I^{\alpha_1}(1)(d) \\
&\le \ell_1\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}(\|x-u\|+\|y-v\|),
\end{aligned}
\tag{31}
$$

and

$$\begin{aligned} \mid \mathcal{R}(x,y)(z)-\mathcal{R}(u,v)(z)\mid \quad &\leq I^{\overline{\alpha}_1}\mid g_{x,y}-g_{u,v}\mid (z) \\ &\leq \ell_2(\|x-u\|+\|y-v\|)I^{\alpha_1}(1)(d) \\ &\leq \ell_2\frac{(d-c)^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}(\|x-u\|+\|y-v\|). \end{aligned} \tag{32}$$

In view of (31) and (32), we obtain

$$\begin{aligned} &\|(\mathcal{N},\mathcal{R})(x,y)-(\mathcal{N},\mathcal{R})(u,v)\| \\ &\leq \left[\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1+\frac{(d-c)^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\ell_2\right](\|x-u\|+\|y-v\|). \end{aligned} \tag{33}$$

Since $\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1+\frac{(d-c)^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\ell_2<1$, the operator $(\mathcal{N},\mathcal{R})$ is a contraction and we conclude that the condition $(iii)$ of Lemma 4 is satisfied. In the next step, we verify the condition $(ii)$ of Lemma 4 for the operator $(\mathcal{M},\mathcal{T})$. By using the continuity of the functions $f,g$, one can see that the operator $(\mathcal{M},\mathcal{T})$ is continuous. On the other hand, for any $(u,v)\in B_\delta$, as in the proof of Theorem 1, we have

$$\begin{aligned} &\mid \mathcal{M}(u,v)(z)\mid \leq \Big(\mathcal{A}_1-\frac{(\psi(d)-\psi(c))^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)Z_1+\mathcal{A}_2Z_2+(\mathcal{A}_3+\mathcal{A}_4)\delta=P^*, \\ &\mid \mathcal{T}(u,v)(z)\mid \leq \mathcal{B}_1Z_1+\Big(\mathcal{B}_2-\frac{(\psi(d)-\psi(c))^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\Big)Z_2+(\mathcal{B}_3+\mathcal{B}_4)\delta=Q^*. \end{aligned} \tag{34}$$

Accordingly, we have $\|(\mathcal{M},\mathcal{T})(u,v)\|\leq P^*+Q^*$, which implies that $(\mathcal{M},\mathcal{T})B_\delta$ is uniformly bounded. Finally, it is shown that the set $(\mathcal{M},\mathcal{T})B_\delta$ is equicontinuous. For this aim, let $\tau_1,\tau_2\in[c,d]$ with $\tau_1<\tau_2$. For any $(u,v)\in B_\delta$, similar to the proofs of equicontinuous for the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ in the Theorem 2, we can show that $|\mathcal{M}(u,v)(\tau_2)-\mathcal{M}(u,v)(\tau_1)|,|\mathcal{T}(u,v)(\tau_2)-\mathcal{S}(u,v)(\tau_1)|\longrightarrow 0$ as $\tau_1\longrightarrow\tau_2$. Consequently, the set $(\mathcal{M},\mathcal{T})B_\delta$ is equicontinuous, and by applying the Arzelá–Ascoli theorem, the operator $(\mathcal{M},\mathcal{T})$ will be compact on $B_\delta$. Therefore, by applying Lemma 4, problem (5) has at least one solution on $[c,d]$. This completes the proof. □

**Example 1.** *Consider the coupled system of $\psi$-Hilfer-type sequential fractional differential equations with integro-multipoint boundary conditions:*

$$\begin{cases} \left({}^HD^{\frac{5}{4},\frac{1}{2};(1-e^{-2t})}+k\,{}^HD^{\frac{1}{4},\frac{1}{2};(1-e^{-2t})}\right)x(t)=f(t,x(t),y(t)), \quad t\in\left[\frac{1}{11},\frac{12}{11}\right], \\ \left({}^HD^{\frac{7}{4},\frac{1}{2};(1-e^{-2t})}+k\,{}^HD^{\frac{3}{4},\frac{1}{2};(1-e^{-2t})}\right)y(t)=g(t,x(t),y(t)), \quad t\in\left[\frac{1}{11},\frac{12}{11}\right], \\ x\left(\frac{1}{11}\right)=0, \quad x\left(\frac{12}{11}\right)=\frac{1}{7}\int_{\frac{1}{11}}^{\frac{5}{11}}e^{-2s}y(s)ds+\frac{2}{13}y\left(\frac{4}{11}\right)+\frac{3}{17}y\left(\frac{8}{11}\right) \\ +\frac{4}{19}y\left(\frac{10}{11}\right)+\frac{5}{23}y(1), \quad y\left(\frac{1}{11}\right)=0, \quad y\left(\frac{12}{11}\right)=\frac{6}{29}\int_{\frac{1}{11}}^{\frac{3}{11}}e^{-2s}x(s)ds \\ +\frac{7}{31}\int_{\frac{1}{11}}^{\frac{7}{11}}e^{-2s}x(s)ds+\frac{8}{37}x\left(\frac{2}{11}\right)+\frac{9}{41}x\left(\frac{6}{11}\right)+\frac{10}{43}x\left(\frac{9}{11}\right). \end{cases} \tag{35}$$

Here $\alpha_1=5/4$, $\overline{\alpha}_1=7/4$, $\beta_1=1/2$, $\psi(t)=(1-e^{-2t})$, $\psi'(t)=2e^{-2t}$, $c=1/11$, $d=12/11$, $\mu_1=1/14$, $\eta_1=5/11$, $\theta_1=2/13$, $\theta_2=3/17$, $\theta_3=4/19$, $\theta_4=5/23$, $\xi_1=4/11$, $\xi_2=8/11$, $\xi_3=10/11$, $\xi_4=1$, $\upsilon_1=3/29$, $\upsilon_2=7/62$, $\varsigma_1=3/11$, $\varsigma_2=7/11$, $\tau_1=8/37$, $\tau_2=9/41$, $\tau_3=10/43$, $\sigma_1=2/11$, $\sigma_2=6/11$, $\sigma_3=9/11$, $n=1$, $m=4$, $p=2$,

$q = 3$. We find that $\gamma = 13/8$, $\gamma_1 = 15/8$, $A \approx 0.9090586723$, $B \approx 0.5135134292$, $\Gamma \approx 0.4618499072$, $\Delta \approx 0.7876865883$, $\Lambda \approx 0.4788871945$, $\mathcal{A}_1 \approx 1.685246952$, $\mathcal{A}_2 \approx 0.6395331644$, $\mathcal{A}_3 \approx 0.1399736659$, $\mathcal{A}_4 \approx 0.09267043416$, $\mathcal{B}_1 \approx 0.9074990107$, $\mathcal{B}_2 \approx 1.026284484$, $\mathcal{B}_3 \approx 0.06726442842$, $\mathcal{B}_4 \approx 0.1432037148$.

$(i)$ If the nonlinear unbounded functions $f$ and $g$ are given by

$$f(z,u,v) = \frac{11e^{-\left(z-\frac{1}{11}\right)}}{2(11z+87)}\left(\frac{u^2+2|u|}{1+|u|}\right) + \frac{1}{17}\left(\cos^2 z + 1\right)\sin|v| + \frac{1}{7}, \tag{36}$$

$$g(z,u,v) = \frac{\tan^{-1}|u|}{9(1+\sin^4 z)} + \frac{11}{10(11z+43)}\left(\frac{3v^2+4|v|}{1+|v|}\right) + \frac{3}{5}, \tag{37}$$

then we can verify the Lipchitz conditions as

$$|f(z,u_1,v_1) - f(z,u_2,v_2)| \le \frac{1}{8}(|u_1-u_2| + |v_1-v_2|)$$
$$|g(z,u_1,v_1) - g(z,u_2,v_2)| \le \frac{1}{9}(|u_1-u_2| + |v_1-v_2|),$$

in which Lipchitz constants $\ell_1 = 1/8$ and $\ell_2 = 1/9$. In addition, we can compute that

$$\ell_1(\mathcal{A}_1+\mathcal{B}_1) + \ell_2(\mathcal{A}_2+\mathcal{B}_2) + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}_3 + \mathcal{B}_4 \approx 0.9522963385 < 1.$$

Therefore, all assumptions of Theorem 1 are fulfilled and the conclusion of Theorem 1 can be applied—that the coupled system of $\psi$-Hilfer-type sequential fractional differential equations with integro-multipoint boundary conditions (35) with (36)-(37) has a unique solution on $[1/11, 12/11]$.

$(ii)$ Consider the nonlinear functions $f$ and $g$ given by

$$f(z,u,v) = \frac{1}{z+3} + \frac{1}{6}\left(\frac{u^{16}}{1+|u|^{15}}\right) + \frac{1}{7}|v|e^{-u^4}, \tag{38}$$

$$g(z,u,v) = \frac{\cos^2 \pi z + 1}{3} + \frac{1}{14}\left(1+\sin^4 v^8\right)|u| + \frac{1}{5}\left(\frac{|v|^{23}}{2+v^{22}}\right). \tag{39}$$

Observe that the above two functions $f$ and $g$ are bounded by

$$|f(z,u,v)| \le \frac{11}{34} + \frac{1}{6}|u| + \frac{1}{7}|v|,$$
$$|g(z,u,v)| \le \frac{2}{3} + \frac{1}{7}|u| + \frac{1}{5}|v|.$$

Thus, we choose constants from Theorem 2 by $u_0 = 11/34$, $v_0 = 2/3$, $u_1 = 1/6$, $v_1 = 1/7$, $u_2 = 1/7$ and $v_2 = 1/5$. By direct computation, we have $(\mathcal{A}_1+\mathcal{B}_1)u_1 + (\mathcal{A}_2+\mathcal{B}_2)v_1 + \mathcal{A}_3 + \mathcal{B}_3 \approx 0.8773363712 < 1$ and $(\mathcal{A}_1+\mathcal{B}_1)u_2 + (\mathcal{A}_2+\mathcal{B}_2)v_2 + \mathcal{A}_4 + \mathcal{B}_4 \approx 0.9394299590 < 1$. Applying Theorem 2, we deduce that the boundary value problem (35) with (38) and (39) has at least one solution on $[1/11, 12/11]$.

$(iii)$ Let the nonlinear bounded functions $f$ and $g$ defined by

$$f(z,u,v) = \frac{11}{24}z + \frac{1}{2} + \frac{9}{16}\left(\frac{|u|}{1+|u|}\right) + \frac{1}{2}\sin|v|, \tag{40}$$

$$g(z,u,v) = 1 + \cos\pi z + \frac{7}{10}\tan^{-1}|u| + \frac{4}{5}\left(\frac{|v|}{1+|v|}\right). \tag{41}$$

It is obvious that these two functions are bounded since

$$|f(z,u,v)| \le \frac{33}{16}, \quad \text{and} \quad |g(z,u,v)| \le \frac{7}{2}.$$

In addition, the condition $(H_1)$ in Theorem 1 is satisfied with $\ell_1 = 9/16$ and $\ell_2 = 4/5$. Hence, we obtain $\mathcal{A}_3 + \mathcal{A}_4 \approx 0.2326441001 < 1$, $\mathcal{B}_3 + \mathcal{B}_4 \approx 0.2104681432 < 1$ and

$$\left[\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}\ell_1 + \frac{(d-c)^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\ell_2\right] \approx 0.9938694509 < 1.$$

Therefore, problem (35) with (40) and (41) has at least one solution on $[1/11, 12/11]$ by using the benefit of Theorem 3. Finally, we give a remark that Theorem 1 cannot be used for this problem because

$$\ell_1(\mathcal{A}_1 + \mathcal{B}_1) + \ell_2(\mathcal{A}_2 + \mathcal{B}_2) + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}_3 + \mathcal{B}_4 \approx 3.234185965 > 1.$$

## 5. Conclusions

In this paper, we investigated a coupled system of fractional differential equations involving $\psi$-Hilfer fractional derivatives, supplemented with integro-multi-point boundary conditions. Firstly, we proved the equivalence between a linear variant of the system (5) and the fractional integral Equations (7) and (8). After that, the existence of a unique solution for the system (5) was proved by using Banach's fixed point theorem. The Leray–Schauder alternative and Krasnosel'skiĭ's fixed point theorem were used to obtain the existence of solutions for the system (5). Moreover, examples were constructed to illustrate our main results. The obtained results are new and enrich the literature on coupled systems for nonlinear $\psi$-Hilfer fractional differential equations. The used methods are standard, but their configuration on the problem (5) is new.

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