*fractal and fractional*

*Article*

# New Results on Controllability for a Class of Fractional Integrodifferential Dynamical Systems with Delay in Banach Spaces †

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**Abstract:** The present work addresses some new controllability results for a class of fractional integrodifferential dynamical systems with a delay in Banach spaces. Under the new definition of controllability, first introduced by us, we obtain some sufficient conditions of controllability for the considered dynamic systems. To conquer the difficulties arising from time delay, we also introduce a suitable delay item in a special complete space. In this work, a nonlinear item is not assumed to have Lipschitz continuity or other growth hypotheses compared with most existing articles. Our main tools are resolvent operator theory and fixed point theory. At last, an example is presented to explain our abstract conclusions.

**Keywords:** fractional integrodifferential dynamical systems; delay; controllability; resolvent operator; fixed point theory



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## 1. Introduction

The purpose of this work is to investigate the controllability of the following fractional integrodifferential dynamical systems with a delay in Banach spaces:

$$\begin{cases} {}^{C}D^{r}z(t) = \mathcal{A}z(t) + g(t, z_t, H_z(t)) + \mathcal{B}x(t), \ \ t \in V := [0, T], \\ z(t) = \psi(t), \ \ t \in [-c, 0], \end{cases} \tag{1}$$

where the state variable $z(\cdot)$ takes values in Banach space $E$. ${}^{C}D^{r}$ denotes the Caputo derivative with order $r \in (0,1)$. $\mathcal{A} : D(\mathcal{A}) \subset E \to E$ is a closed linear unbounded operator on $E$. $x$ is a control function defined in $L^2(V; U)$, where $U$ is a Banach space. $\mathcal{B} : U \to E$ is a bounded linear operator. $\psi(t) \in C([-c, 0]; E)$. $g$ is a given nonlinear item satisfying some appropriate hypotheses and

$$H_z(t) = \int_0^t h(t, s, z_s) ds$$

with $h : \Delta \times L([-c, 0]; E) \to E$ where $\Delta = \{(t, s) \in V \times V : s \le t\}$. $L([-c, 0]; E)$ is the space of $E$-valued Bochner integrable functions on $[-c, 0]$ with the norm $\|z\|_{L[-c,0]} = \int_{-c}^{0} \|z(t)\| dt$.

The theory of fractional calculus has a long-standing history, and has received considerable attention due mainly to its potential and wide applications in various fields, such as viscoelasticity, signal processing, pure mathematics, control, electromagnetics, etc. (see [1–7]). In the modeling of many phenomena in various science and technology fields, fractional differential equations, including both ordinary and partial ones, are considered to be more powerful tools than their corresponding integer-order counterparts. Many phenomena, such as electronics, fluid dynamics, biological models and chemical kinetics, cannot be described through classical differential equations; in these cases, integrodifferential equations play an important role in describing most of these natural phenomena. For more details of fractional calculus theory, and the results of integrodifferential equations, one can see [8–14], and the references therein.

Time delay occurs frequently and is inevitable in various practical systems of the real world [15–19]. This is especially true for dynamical evolution processes which are closely related to time. Hence, if we intend to accurately describe the evolution systems, we must consider the effect of time delay. With the development of the applications for fractional calculus, research into the controllability of fractional dynamical systems with delay is increasingly extensive [20–25].

It is well known that control theory is an interdisciplinary subject involving economics, engineering and mathematics, which investigates and analyses some dynamical behaviors of various systems [26–32]. It is worth noting that controllability is of importance in some research fields of networks such as logical control networks, and steady-state design of large-dimensional Boolean networks. Logical control networks are widely used in controllability, evolutionary games, stability and optimal control, and many fundamental results have been established for them [33–35]. With the rapid development of control theory, the problem of controllability for a special kind of logical control networks, Boolean control networks, was also investigated by researchers. For more details of the recent works in this regard, we refer readers to [36–38]. Controllability is one of the fundamental concepts in mathematical control theory. On the one hand, in the study of controllability for fractional dynamical systems, the hypothesis of noncompact semigroups is especially important, as the compactness of the semigroups is only applicable in finite-dimensional spaces, since the inverse of control operator cannot be ensured if the state space is infinite-dimensional. Some technical errors caused by the compactness of semigroups have been pointed out by Hernández et al. [39]. On the other hand, how to introduce the mild solutions in infinite dimensional spaces is another particularly important step. For example, Hernández et al. [40] also pointed out that the definition of mild solutions in some articles, such as [23,41], was inappropriate because it was only a simple extension of the integer-order mild solutions. We know that a fractional evolution dynamical system is usually transformed into a form of Volterra integral equation to obtain its mild solutions. Therefore, the theory of resolvent operators is a powerful tool to study such systems. Compared with the mild solutions constructed by some probability density functions (El-Borai [42]), it is found that in the investigation of evolution dynamical systems with unbounded operators in infinite dimensional spaces, resolvent operators seem to be more appropriate since they are direct generalizations of $C_0$-semigroups and cosine families. This is why we adopted the resolvent operator theory to define mild solutions and investigate the controllability of the considered fractional dynamical systems in this paper.

Some excellent results concerning the controllability of various nonlinear fractional dynamical systems were obtained. However, most of these controllability problems were investigated under the hypothesis that the nonlinear item $f$ is Lipschitz continuous, compact or satisfies some other growth conditions, see [20–23,43,44] for example. We point out that, as a more stronger smooth condition than continuity, Lipschitz continuity, is only regarded as an idealized supposition in many cases, which is difficult to apply to practical problems. Furthermore, there are scarcely any results on the controllability of fractional integrodifferential dynamical systems with delay, except for [20,22–24]. However, in [20], the authors still supposed the nonlocal item to be Lipschitz continuous, and that the nonlinear function satisfied certain growth conditions. Notice that in [22–24], authors hobtained controllability results for fractional delay differential and integrodifferential dynamical systems with the nonlinear functions also being Lipschitz continuous. Therefore, a very natural question is whether the considered fractional integrodifferential dynamical systems with delay are controllable when the nonlinear item is only continuous, rather than Lipschitz continuous. This is also the main motivation for the present work.

Compared with the above-mentioned research, the main contributions of this work are as below: (i) Under the new definition of controllability, we suppose that the nonlinear function here only has continuity rather than Lipschitz continuity and other certain growth assumptions. (ii) In order to overcome the obstacles caused by time delay, we utilize a special complete space $L([-c, 0]; E)$ in which to define the suitable time delay item $z_t$.

The organization of the rest of this work is as follows. Some necessary preparations are given in Section 2. In Section 3, sufficient conditions of the controllability for system (1) are obtained. An example is provided in Section 4 to illustrate the effectiveness of the abstract results.

## 2. Preliminaries

**Notation 1.** *Let $R$ denote the set of real numbers, $R^+$ the set of positive numbers. $\Gamma$ is the gamma function. $I$ represents the identity operator. Suppose $E$ to be a Banach space along with the norm $\|\cdot\|$. $\mathcal{B} : U \to E$ is a bounded linear operator where $U$ is also a Banach space. $\mathcal{A} : D(\mathcal{A}) \subset E \to E$ is a closed linear unbounded operator on $E$. Denoted by $\mathcal{D}$, the dense domain of closed linear unbounded operator $\mathcal{A}$ equipped with the graph norm $\|z\|_{\mathcal{D}} = \|z\| + \|\mathcal{A}z\|$; $C(V; E)$ stands for the space of E-valued continuous functions on $V$ with the norm $\|\cdot\|_{C(V;E)}$. $C([-c, T]; E)$ denotes the Banach space of continuous functions from $[-c, T]$ to $E$ with the usual supreme norm. Denote the norm of the space $C([-c, 0]; E)$ by $\|\cdot\|_c$ for brevity. $C^r(V; E)$, $r \in (0, 1)$, represents the space formed of all the r-Hölder E-valued continuous functions from $V$ into $E$ equipped with the norm $\|z\|_{C^r(V;E)} = \|z\|_{C(V;E)} + [|z|]_{C^r(V;E)}$, where $[|z|]_{C^r(V;E)} = \sup\limits_{t,s\in V, t\neq s} \dfrac{\|z(t) - z(s)\|}{(t - s)^r}$. Assume that $J \subset R$, and $1 \le p \le \infty$. For measurable function $z : J \to R$, define the norm*

$$\|z\|_{L^p(J)} = \begin{cases} \left(\int_J |z(t)|^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf\limits_{\mu(\overline{J}=0)} \{ \sup\limits_{t\in J-\overline{J}} |z(t)|\}, & p = \infty, \end{cases}$$

*where $\mu(\overline{J})$ is the Lebesgue measure on $\overline{J}$. Let $L^p(J, R)$ be the Banach space of all Lebesgue functions $z : J \to R$ with $\|\cdot\|_{L^p(J)} < \infty$. The space of bounded linear operators from $E$ into Banach space $F$ is defined as $\mathcal{L}(E, F)$ provided with the operator norm $\|\cdot\|_{\mathcal{L}(E,F)}$, and $\mathcal{L}(E, E)$ is written as $\mathcal{L}(E)$ with norm $\|\cdot\|_{\mathcal{L}(E)}$.*

To deal with the inconveniences caused by delay during the investigation of controllability in the sequel, we utilize a special complete space $L([-c, 0]; E)$. For $z \in C(V; E)$ and $t \in V$, define a function $z_t$:

$$z_t(\theta) = \begin{cases} z(t+\theta), & t+\theta \ge 0, \\ \psi(t+\theta), & t+\theta \le 0, \end{cases} \tag{2}$$

for any $\theta \in [-c, 0]$. It is not difficult to deduce that $z_t \in L([-c, 0]; E)$.

The basic definitions of fractional calculus are presented as follows. For further details, please see [11] and the references therein.

The Riemann–Liouville fractional integral of order $r > 0$ and the lower limit zero for a continuous function $u$ is given by

$$I^r_{0^+} u(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} u(s) ds, \ t > 0,$$

if the right side integral is pointwise defined on $(0, +\infty)$.

The Riemann–Liouville derivative and the lower limit zero for a continuous function $u : (0, \infty) \to R$ is defined as

$$^{L}D^{r}_{0+}u(t) = \frac{1}{\Gamma(n-r)}\frac{d^n}{dt^n}\int_0^t \frac{u(s)}{(t-s)^{r-n+1}}ds,\ t > 0,\ n-1 < r < n,\ r > 0,$$

and the corresponding Caputo fractional derivative of order $r > 0$ with the lower limit zero for a continuous function $u : (0, \infty) \to R$ is given by

$$^{C}D^{r}_{0+}u(t) = ^{L}D^{r}_{0+}\left(u(t) - \sum_{k=0}^{n-1}\frac{u^k(0)}{k!}t^k\right),\ t > 0,\ n-1 < r < n,\ r > 0.$$

It is noted that if $u(t) \in C^n[0, \infty)$, then one can obtain

$$^{C}D^{r}_{0+}u(t) = \frac{1}{\Gamma(n-r)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{r-n+1}}ds = I^{n-r}u^{(n)}(t),\ t > 0,\ n-1 < r < n,\ r > 0.$$

Throughout this paper, we suppose that the following integral equation

$$z(t) = \frac{1}{\Gamma(r)}\int_0^t \frac{\mathcal{A}z(s)}{(t-s)^{1-r}}ds,\ t \geq 0, \tag{3}$$

has an associated resolvent operator $\{\mathcal{N}(t)\}_{t\geq 0}$ on $E$.

**Definition 1** ([45]). *Bounded linear operator $\{\mathcal{N}(t)\}_{t\geq 0} \subset \mathcal{L}(E)$ is defined as a resolvent operator for (3) if the following assumptions are satisfied:*
*(I) $\mathcal{N}(t)$ is strongly continuous on $R^+$ and $\mathcal{N}(0) = I$;*
*(II) $\mathcal{N}(t)\mathcal{D} \subset \mathcal{D}$, $\mathcal{A}\mathcal{N}(t)z = \mathcal{N}(t)\mathcal{A}z$ for all $z \in \mathcal{D}$ and every $t \geq 0$;*
*(III) $\mathcal{N}(t)z = z + \frac{1}{\Gamma(r)}\int_0^t \frac{\mathcal{A}z(s)}{(t-s)^{1-r}}ds,\ z \in \mathcal{D}, t \geq 0$.*

**Definition 2** ([45]). *A resolvent operator $\mathcal{N}(t)$ for (3) is called analytic, if the function $\mathcal{N}(\cdot) : R^+ \to \mathcal{L}(E)$ admits an analytic extension to a sector $\sum(0, \theta_0) = \{\lambda \in C : |arg(\lambda)| < \theta_0\}$ for some $0 < \theta_0 \leq \frac{\pi}{2}$.*

**Definition 3** ([45]). *$z \in C(V; E)$ is defined as a mild solution to the Volterra integral equation*

$$z(t) = k(t) + \frac{1}{\Gamma(r)}\int_0^t \frac{\mathcal{A}z(s)}{(t-s)^{1-r}}ds,\ \ t \in V, \tag{4}$$

*where $k \in L^1(V; E)$, if $\int_0^t \frac{z(s)}{(t-s)^{1-r}}ds \in \mathcal{D}$ for all $t \in V$ and*

$$z(t) = k(t) + \frac{1}{\Gamma(r)}\mathcal{A}\int_0^t \frac{z(s)}{(t-s)^{1-r}}ds$$

*holds for $V$.*

**Lemma 1** ([45]). *Suppose $\mathcal{N}(t)$ is an analytic resolvent operator of (4) and $k \in C^r(V; E), k(0) = 0$. Then*

$$z(t) = \mathcal{N}(t)k(t) + \int_0^t \dot{\mathcal{N}}(t-s)[k(s) - k(t)]ds,\ \ t \in V,$$

*is a mild solution of (4) and $z \in C^r(V; E)$.*

**Lemma 2** (Mönch). *Assume that $D$ is a closed and convex subset of a Banach space $E$ and $z_0 \in D$. Suppose that the continuous operator $\mathcal{A} : D \to D$ satisfies: $C \subset D$ countable, $C \subset \overline{co}(\{z_0\} \bigcup \mathcal{A}(C)) \to C$ is relatively compact. Then, $\mathcal{A}$ has a fixed point in $D$.*

**Lemma 3** (Hölder Inequality)**.** *Assume that $p_1, p_2 \geq 1$, and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. If $z_1 \in L^{p_1}(J, R)$, $z_2 \in L^{p_2}(J, R)$, then, $z_1 z_2 \in L^1(J, R)$ and $\|z_1 z_2\|_{L^1 J} \leq \|z_1\|_{L^{p_1} J} \|z_2\|_{L^{p_2} J}$.*

The theory of Kuratowski's measures of noncompactness is crucial to the following proof work. For more details, see [46].

**Lemma 4.** *Suppose $E$ to be a Banach space and $\zeta(\cdot)$ to be the Kuratowski's measures of noncompactness.*
*(1) Let $D_1, D_2$ be bounded sets of $E$ and $\lambda \in R$. Then*

*(i) $\zeta(D_1) = 0$ if, and only if, $D_1$ is relatively compact;*
*(ii) $\zeta(D_1) \leq \zeta(D_2)$ if $D_1 \subset D_2$;*
*(iii) $\zeta(\lambda D_1) = |\lambda| \zeta(D_1)$;*
*(iv) $\zeta(D_1 + D_2) \leq \zeta(D_1) + \zeta(D_2)$;*
*(v) $\zeta(x_0 \cup D_1) = \zeta(D_1)$, $x_0 \in E$;*

*(2) Let $D \subset C(V; E)$ be bounded. Then, $D(t)$ is bounded in $E$ and $\zeta(D(t)) \leq \zeta(D)$.*
*(3) Let $D \subset C(V; E)$ be bounded and equicontinuous. Then, $\zeta(D(t))$ is continuous on $V$, and $\zeta(D) = \max_{t \in V} \zeta(D(t))$.*
*(4) Let $D = \{u_n\} \subset C(V; E)$ be countable. If there exists $\Phi \in L^1(W)$ such that $\|u_n(t)\| \leq \Phi(t)$ a.e.$t \in W$, $n = 1, 2, \cdots$, then $\zeta(D(t))$ is integrable on $V$, and*

$$\zeta\left(\left\{\int_V u_n(t)dt : n \in N\right\}\right) \leq 2 \int_V \zeta(D(t))dt.$$

On the premise of no confusion, Kuratowski's measures of noncompactness of a bounded subset in spaces $E, C(V; E)$ and $L([-c, 0]; E)$ are all denoted by $\zeta(\cdot)$.

Finally, we introduce some important results:

**Lemma 5.** *If $z_n$ converges to $z_0$ in $C(V; E)$ as $n \to +\infty$, then one has that $(z_n)_t$ converges to $(z_0)_t$ in $L([-c, 0]; E)$ for each $t \in V$ as $n \to +\infty$.*

**Proof.** By means of (2), we have

$$\|(z_n)_t - (z_0)_t\|_{L[-c,0]} = \begin{cases} \int_0^t \|z_n(s) - z_0(s)\| ds, & t \leq c, \\ \int_{t-c}^t \|z_n(s) - z_0(s)\| ds, & t \geq c. \end{cases}$$

This indicates that $\|(z_n)_t - (z_0)_t\|_{L[-c,0]} \leq c\|z_n - z_0\|_{C(V;E)}$. This completes the proof. □

On the basis of the definition of Kuratowski's measures of noncompactness and Lemma 5, it is not difficult to obtain:

**Lemma 6.** *Assume that $\{z_n\}_{n=1}^{\infty}$ is a bounded countable sequence in $C(V; E)$. Then one can obtain*

$$\zeta(\{(z_n)_t\}) \leq c\zeta(\{z_n\}), \ \forall t \in V.$$

## 3. Main Results

On the basis of the Riemann–Liouville fractional integral, together with Definition 3, the mild solution to system (1) can be obtained as follows:

**Definition 4.** *For each $x \in L^2(V;U)$, a function $z \in C(W;E)$ is said to be a mild solution of fractional dynamical system (1) on $W$, if $\int_0^t \frac{z(s)}{(t-s)^{1-r}}ds \in \mathcal{D}$ for all $t \in [0,T_0]$ and $z$ satisfies the following integral equation*

$$z(t) = \begin{cases} \psi(0) + \frac{1}{\Gamma(r)}\mathcal{A}\int_0^t \frac{z(s)}{(t-s)^{1-r}}ds + \frac{1}{\Gamma(r)}\int_0^t \frac{g(s,z_s,H_z(s))}{(t-s)^{1-r}}ds + \frac{1}{\Gamma(r)}\int_0^t \frac{\mathcal{B}x(s)}{(t-s)^{1-r}}ds, \quad t \in [0,T_0], \\ \psi(t), \quad t \in [-c,0], \end{cases}$$

*where $W = [-c,T_0]$, $T_0 \in (0,T]$.*

**Definition 5.** *The system (1) is said to be controllable on $V = [0,T]$, if, for every $\psi(t) \in C([-c,0];E)$ and $z_1 \in E$, there exists a control $x \in L^2(V,U)$ and a constant $T_0 \in (0,T]$, such that a mild solution $z$ of system (1) on $W = [-c,T_0]$ satisfies $z(T_0) = z_1$.*

**Remark 1.** *Compared with the existing definitions in [20,21,43,44,47], etc., in which $z_1$ is obtained at the right endpoint $T$, the present definition, which we introduced with $z_1$ arriving at $T_0 \in (0,T]$, is weaker.*

Next, we impose the main hypotheses on the components of the systems:

**Hypothesis 1 (H1).** *$g : V \times L([-c,0];E) \times E \to E$ is continuous and takes bounded sets in $V \times L([-c,0];E) \times E$ into bounded sets in $E$.*

**Hypothesis 2 (H2).** *$h : \Delta \times L([-c,0];E) \to E$ is continuous where $\Delta = \{(t,s) \in V \times V : s \leq t\}$.*

**Hypothesis 3 (H3).** *(i) The linear operator $\mathcal{B} : L^2(V;U) \to L^1(V;E)$ is bounded, and there exists a constant $Q_1 > 0$ such that $\|\mathcal{B}\|_{\mathcal{L}(U,E)} \leq Q_1$;*
*(ii) The linear operator $\Lambda(t)$ defined by*

$$\Lambda(t)x = \mathcal{N}(t)\mathcal{B}_x(t) + \int_0^t \dot{\mathcal{N}}(t-s)(\mathcal{B}_x(s) - \mathcal{B}_x(t))ds$$

*where $\mathcal{B}_x(t) = \frac{1}{\Gamma(r)}\int_0^t \frac{\mathcal{B}x(s)}{(t-s)^{1-r}}ds$, has an invertible operator $\Lambda^{-1}(t)$ takeing values in $L^2(V;U)/ker\Lambda(t)$ for each $t \in V$, and there exists a constant $Q_2 > 0$ such that $\sup \|\Lambda^{-1}(\cdot)\|_{\mathcal{L}(E;L^2(V;U)/ker\Lambda(\cdot))} \leq Q_2$.*

**Hypothesis 4 (H4).** *(i) There exist constants $r_i \in (0,r)$ and real-valued functions $k_i \in L^{\frac{1}{r_i}}(V;R^+)$, $i = 1,2$, such that for any bounded subsets $D_1 \subset L([-c,0];E)$, $D_2 \subset E$,*

$$\zeta(g(t,D_1,D_2)) \leq k_1(t)\zeta(D_1) + k_2(t)\zeta(D_2),\ t \in V.$$

*(ii) There exists a function $l \in L^1(V;R^+)$ such that, for any bounded subset $D \subset L([-c,0];E)$,*

$$\zeta(h(t,s,D)) \leq l(s)\zeta(D),\ (t,s) \in \Delta.$$

*(iii) There exists a constant $l_0 > 0$ such that*

$$\zeta(\Lambda^{-1}(\cdot)(D)(t)) \leq l_0\zeta(D),\ t \in V,$$

*for any bounded set $D \subset E$.*

We point out that resolvent operator $\{\mathcal{N}(t)\}_{t\geq 0}$ is supposed to be analytic in the rest of this work. In light of [45], we can assume that $N_1, N_2$ are positive numbers, such that

$\|\dot{\mathcal{N}}(t)\|_{\mathcal{L}(E)} \leq N_1 t^{-1}$, $\|\ddot{\mathcal{N}}(t)\|_{\mathcal{L}(E)} \leq N_2 t^{-2}$ for all $t \in (0, T]$; $N_0$ is a positive constant, such that $\|\mathcal{N}(t)\|_{\mathcal{L}(E)} \leq N_0$ for all $t \in V$.

For simplicity, take

$$\mathcal{G}_z(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{g(s, z_s, H_z(s))}{(t-s)^{1-r}} ds,$$

and let

$$\theta_i = \frac{1-r_i}{r-r_i}, \quad \vartheta_i = \frac{r-1}{1-r_i}, \quad i = 1, 2;$$

$$K_1 = 2c\theta_1^{1-r_1} \|k_1\|_{L^{\frac{1}{r_1}}} + 4c\theta_2^{1-r_2} \|k_2\|_{L^{\frac{1}{r_2}}} \|l\|_{L^1};$$

$$K_2 = 2c\theta_1 \|k_1\|_{L^{\frac{1}{r_1}}} + 4c\theta_2 \|k_2\|_{L^{\frac{1}{r_2}}} \|l\|_{L^1};$$

and

$$Q = N_0 K_1 + 4N_1(K_1 + K_2).$$

In addition, for the purpose of simplifying our next work, we provide the next two necessary conclusions:

**Lemma 7.** *(i) Assume that* $g(\cdot, z_\cdot, H_z(\cdot)) : V \times L([-c, 0]; E) \times E \to E$ *is continuous. Then,* $\mathcal{G}_z \in C^r(V; E)$ *and* $[|\mathcal{G}_z|]_{C^r} \leq \dfrac{2}{\Gamma(r+1)} \|g(\cdot, z_\cdot, H_z(\cdot))\|_{C(V;E)}$.
*(ii) Assume that* $x(\cdot) \in L^2(V; U)$*. Then,* $\mathcal{B}_x \in C^r(V; E)$ *and* $[|\mathcal{B}_x|]_{C^r} \leq \dfrac{2}{\Gamma(r+1)} \|\mathcal{B}\|_{\mathcal{L}(U,E)}$ $\|x(\cdot)\|_{L^2}$.

**Proof.** For $t \in [0, T)$ and $\iota > 0$ such that $t + \iota \in V$, we have

$$\begin{aligned}
\|\mathcal{G}_z(t+\iota) - \mathcal{G}_z(t)\| &\leq \frac{1}{\Gamma(r)} \int_0^t \left[ \frac{1}{(t-s)^{1-r}} - \frac{1}{(t+\iota-s)^{1-r}} \right] \|g(s, z_s, H_z(s))\| ds \\
&\quad + \frac{1}{\Gamma(r)} \int_t^{t+\iota} \frac{\|g(s, z_s, H_z(s))\|}{(t+\iota-s)^{1-r}} ds \\
&\leq \frac{1}{\Gamma(r)} \|g(\cdot, z_\cdot, H_z(\cdot))\|_{C(V;E)} \left( \frac{(t+\iota)^r - t^r + \iota^r}{r} + \frac{\iota^r}{r} \right) \\
&\leq \frac{2}{\Gamma(r+1)} \|g(\cdot, z_\cdot, H_z(\cdot))\|_{C(V;E)} \iota^r,
\end{aligned}$$

which implies that $[|\mathcal{G}_z|]_{C^r} \leq \dfrac{2}{\Gamma(r+1)} \|g(\cdot, z_\cdot, H_z(\cdot))\|_{C(V;E)}$ and $\mathcal{G}_z \in C^r(V; E)$.

Repeating a similar process, we can obtain $[|\mathcal{B}_x|]_{C^r} \leq \dfrac{2}{\Gamma(r+1)} \|\mathcal{B}\|_{\mathcal{L}(U,E)} \|x(\cdot)\|_{L^2}$ and $\mathcal{B}_x \in C^r(V; E)$. This completes the proof. □

**Lemma 8.** *Assume that (H1), (H2) and (H4) (i), (ii) hold. Then, operators* $\mathcal{K}_i : C(V; E) \to C(V; E)$ $(i = 1, 2)$*, defined by*

$$(\mathcal{K}_1 z)(t) = \int_0^t \left[ (t-s)^{r-1} - (t_1 - s)^{r-1} \right] g(s, z_s, H_z(s)) ds, \ t \in V, \ t < t_1,$$

$$(\mathcal{K}_2 z)(t) = \int_t^{t_1} (t_1 - s)^{r-1} g(s, z_s, H_z(s)) ds, \ t, \ t_1 \in V, \ t < t_1,$$

*satisfy* $\zeta((\mathcal{K}_i D)(t)) \leq K_i (t_1 - t)^r \zeta(D)$ $(i = 1, 2)$ *for any countable bounded set* $D \subset C(V; E)$.

**Proof.** Obviously, we can check that $\mathcal{K}_i$ $(i = 1, 2)$ takes bounded sets in $C(V; E)$ into bounded sets in $C(V; E)$. Generally, a bounded countable set is chosen $D = \{z_n\}_{n=1}^{\infty}$. From Lemma 6, we have

$$\zeta(\{(z_n)_s\}) \leq c\zeta(\{z_n\}).$$

By means of Lemma 3, Lemma 4 (4) and the well-known inequality

$$t^{\varrho} - s^{\varrho} \geq (t-s)^{\varrho}, \ \varrho \in [1, +\infty), \ 0 < s \leq t,$$

one has

$$\begin{aligned}
\zeta((\mathcal{K}_1 D)(t)) &= \zeta(\{(\mathcal{K}_1 z_n)(t)\}) \\
&= \zeta\left(\left\{\int_0^t \left[(t-s)^{r-1} - (t_1-s)^{r-1}\right] g(s, (z_n)_s, H_{z_n}(s)) ds\right\}\right) \\
&\leq 2\int_0^t \left[(t-s)^{r-1} - (t_1-s)^{r-1}\right] \zeta(\{g(s, (z_n)_s, H_{z_n}(s))\}) ds \\
&\leq 2\int_0^t \left[(t-s)^{r-1} - (t_1-s)^{r-1}\right] k_1(s)\zeta(\{(z_n)_s\}) ds \\
&\quad +4\|l\|_{L^1}\int_0^t \left[(t-s)^{r-1} - (t_1-s)^{r-1}\right] k_2(s)\zeta(\{(z_n)_s\}) ds \\
&\leq 2\left(\int_0^t \left[(t-s)^{r-1} - (t_1-s)^{r-1}\right]^{\frac{1}{1-r_1}} ds\right)^{1-r_1} \|k_1\|_{L^{\frac{1}{r_1}}} c\zeta(\{z_n\}) \\
&\quad +4\|l\|_{L^1}\left(\int_0^t \left[(t-s)^{r-1} - (t_1-s)^{r-1}\right]^{\frac{1}{1-r_2}} ds\right)^{1-r_2} \|k_2\|_{L^{\frac{1}{r_2}}} c\zeta(\{z_n\}) \\
&\leq 2\left(\int_0^t \left[(t-s)^{\vartheta_1} - (t_1-s)^{\vartheta_1}\right] ds\right)^{1-r_1} \|k_1\|_{L^{\frac{1}{r_1}}} c\zeta(\{z_n\}) \\
&\quad +4\|l\|_{L^1}\left(\int_0^t \left[(t-s)^{\vartheta_2} - (t_1-s)^{\vartheta_2}\right] ds\right)^{1-r_2} \|k_2\|_{L^{\frac{1}{r_2}}} c\zeta(\{z_n\}) \\
&\leq \frac{2\|k_1\|_{L^{\frac{1}{r_1}}}}{(1+\vartheta_1)^{1-r_1}} \left[t^{1+\vartheta_1} - t_1^{1+\vartheta_1} + (t_1-t)^{1+\vartheta_1}\right]^{1-r_1} c\zeta(\{z_n\}) \\
&\quad + \frac{4\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}}{(1+\vartheta_2)^{1-r_2}} \left[t^{1+\vartheta_2} - t_1^{1+\vartheta_2} + (t_1-t)^{1+\vartheta_2}\right]^{1-r_2} c\zeta(\{z_n\}) \\
&\leq \left(2\theta_1^{1-r_1}\|k_1\|_{L^{\frac{1}{r_1}}}(t_1-t)^{(1+\vartheta_1)(1-r_1)} + 4\theta_2^{1-r_2}\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}(t_1-t)^{(1+\vartheta_2)(1-r_2)}\right) c\zeta(\{z_n\}) \\
&\leq \left(2\theta_1^{1-r_1}\|k_1\|_{L^{\frac{1}{r_1}}}(t_1-t)^{r-r_1} + 4\theta_2^{1-r_2}\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}(t_1-t)^{r-r_2}\right) c\zeta(\{z_n\}) \\
&\leq \left(2c\theta_1^{1-r_1}\|k_1\|_{L^{\frac{1}{r_1}}} + 4c\theta_2^{1-r_2}\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}\right)(t_1-t)^r \zeta(\{z_n\}) \\
&= K_1(t_1-t)^r \zeta(D).
\end{aligned}$$

In the same way, one can obtain

$$\begin{aligned}
\zeta((\mathcal{K}_2 D)(t)) &= \zeta(\{(\mathcal{K}_2 z_n)(t)\}) \\
&= \zeta\bigg(\bigg\{\int_t^{t_1}(t_1-s)^{r-1}g(s,(z_n)_s,H_{z_n}(s))ds\bigg\}\bigg) \\
&\leq 2\int_t^{t_1}(t_1-s)^{r-1}\zeta(\{g(s,(z_n)_s,H_{z_n}(s))\})ds \\
&\leq 2\int_t^{t_1}(t_1-s)^{r-1}k_1(s)\zeta(\{(z_n)_s\})ds \\
&\quad +4\|l\|_{L^1}\int_t^{t_1}(t_1-s)^{r-1}k_2(s)\zeta(\{(z_n)_s\})ds \\
&\leq 2\bigg(\int_t^{t_1}(t_1-s)^{\frac{r-1}{1-r_1}}ds\bigg)^{1-r_1}\|k_1\|_{L^{\frac{1}{r_1}}}c\zeta(\{z_n\}) \\
&\quad +4\|l\|_{L^1}\bigg(\int_t^{t_1}(t_1-s)^{\frac{r-1}{1-r_2}}ds\bigg)^{1-r_2}\|k_2\|_{L^{\frac{1}{r_2}}}c\zeta(\{z_n\}) \\
&\leq \bigg(2\theta_1(t_1-t)^{\frac{1}{\theta_1}}\|k_1\|_{L^{\frac{1}{r_1}}}+4\theta_2(t_1-t)^{\frac{1}{\theta_2}}\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}\bigg)c\zeta(\{z_n\}) \\
&\leq \bigg(2c\theta_1\|k_1\|_{L^{\frac{1}{r_1}}}+4c\theta_2\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}\bigg)(t_1-t)^r\zeta(\{z_n\}) \\
&= K_2(t_1-t)^r\zeta(D).
\end{aligned}$$

The conclusion follows. □

**Theorem 1.** *If assumptions (H1)–(H4) hold, then the dynamical system (1) is controllable on $V$.*

**Proof.** We let constant

$$K=\sup\Big\{\|g(t,z_t,H_z(t))\|:\|z_t\|_{L[-c,0]}\leq c(\|\psi\|_c+R_0),\ \|H_z(t)\|\leq K_0,\ t\in V\Big\},$$

where $R_0=N_0\|\psi(0)\|+\dfrac{N_0r+2N_1}{r^2}$, and

$$K_0=\sup\bigg\{\|\int_0^t h(t,s,z_s)ds\|:(t,s)\in\Delta,\ \|z_s\|_{L[-c,0]}\leq c(\|\psi\|_c+R_0)\bigg\}.$$

From (H3), for an arbitrary function $z(\cdot)\in C(V;E)$ and any $z_1\in E$, define a feedback control

$$x_z(t):=\Lambda^{-1}(T_0)\bigg(z_1-\mathcal{N}(T_0)(\psi(0)+\mathcal{G}_z(T_0))-\int_0^{T_0}\dot{\mathcal{N}}(T_0-s)(\mathcal{G}_z(s)-\mathcal{G}_z(T_0))ds\bigg)(t),\ \ t\in V,$$

where

$$T_0=\min\Bigg\{T,\ \bigg(\frac{1}{K+Q_1N_3}\bigg)^{\frac{1}{r}},\ \bigg(\frac{r\Gamma^3(r+1)}{1+Q+2l_0Q_1Q_2QT^r(N_0+8N_1T^r)}\bigg)^{\frac{1}{r}}\Bigg\}, \tag{5}$$

and $N_3=Q_2\bigg(\|z_1\|+N_0(\|\psi(0)\|+\dfrac{KT^r}{r})+\dfrac{2KN_1T^r}{r^2}\bigg)$. Take $W=[-c,T_0]$. By considering Lemmas 1 and 7, in what follows, it suffices to show that, when using this control, the operator $\mathcal{P}:C(W;E)\to C(W;E)$ defined by

$$(\mathcal{P}z)(t)=\begin{cases}\mathcal{N}(t)(\psi(0)+\mathcal{G}_z(t)+\mathcal{B}_{x_z}(t)) \\ +\displaystyle\int_0^t\dot{\mathcal{N}}(t-s)(\mathcal{G}_z(s)-\mathcal{G}_z(t))ds+\int_0^t\dot{\mathcal{N}}(t-s)(\mathcal{B}_{x_z}(s)-\mathcal{B}_{x_z}(t))ds,\ \ t\in[0,T_0], \\ \psi(t),\ \ t\in[-c,0],\end{cases} \tag{6}$$

has a fixed point, from which it follows that this fixed point is a mild solution to the system (1) on $W$. Clearly, $(\mathcal{P}z)(T_0)=z_1$, which means that the control $x_z$ steers the system (1) from the initial function $\psi$ to $z_1$ in finite time $T_0$. Denote

$$\Omega=\bigg\{z\in C(W;E)\mid\sup_{t\in[0,T_0]}\|z_t\|_{L[-c,0]}\leq c(\|\psi\|_c+R_0);\ \ z(t)=\psi(t),\ t\in[-c,0]\bigg\},$$

then $\Omega$ is obviously a closed convex set in $C(W;E)$. Subsequently, we will use Mönch fixed point theorem. To this end, we proceed the following four steps.

**Step I.** $\mathcal{P}(\Omega) \subseteq \Omega$. From (H3), we have

$$\begin{aligned}\|x_z(t)\| &\leq Q_2\Big(\|z_1\| + N_0\|\psi(0)\| + N_0\tfrac{KT^r}{r} + \int_0^{T_0}\frac{N_1}{T_0-s}\|\mathcal{G}_z(T_0) - \mathcal{G}_z(s)\|ds\Big) \\ &\leq Q_2\Big(\|z_1\| + N_0\|\psi(0)\| + N_0\tfrac{KT^r}{r} + \tfrac{2N_1K}{r}\int_0^{T_0}\frac{1}{T_0-s}(T_0-s)^r ds\Big) \\ &\leq Q_2\Big(\|z_1\| + N_0(\|\psi(0)\| + \tfrac{KT^r}{r}) + \tfrac{2KN_1T^r}{r^2}\Big) = N_3,\ \ t \in V.\end{aligned}$$

From (H3) and (6), for any $z \in \Omega$ and $t \in [0, T_0]$, it follows that

$$\begin{aligned}\|(\mathcal{P}z)(t)\| &\leq N_0\Big(\|\psi(0)\| + \frac{KT_0^r}{r} + \frac{Q_1N_3T_0^r}{r}\Big) \\ &\quad + \int_0^t \frac{N_1}{t-s}\frac{2K(t-s)^r}{r}ds + \int_0^t \frac{N_1}{t-s}\frac{2Q_1N_3}{r}(t-s)^r ds \\ &\leq N_0\Big(\|\psi(0)\| + \frac{KT_0^r}{r} + \frac{Q_1N_3T_0^r}{r}\Big) + \frac{2N_1KT_0^r}{r^2} + \frac{2Q_1N_1N_3T_0^r}{r^2} \\ &\leq N_0\|\psi(0)\| + \frac{(N_0r + 2N_1)(K + Q_1N_3)}{r^2}T_0^r \\ &\leq N_0\|\psi(0)\| + \frac{N_0r + 2N_1}{r^2} = R_0.\end{aligned}$$

On the other hand,

$$\|(\mathcal{P}z)_t\|_{L[-c,0]} = \int_{-c}^0 (\mathcal{P}z)_t(\theta)d\theta = \begin{cases} \displaystyle\int_{t-c}^0 \psi(s)ds + \int_0^t (\mathcal{P}z)(s)ds, & t \leq c, \\ \displaystyle\int_{t-c}^t (\mathcal{P}z)(s)ds, & t \geq c, \end{cases}$$

which implies $\|(\mathcal{P}z)_t\|_{L[-c,0]} \leq c\|\psi\|_c + c\|\mathcal{P}z\|_{C([0,t];E)}$. Then, one can obtain

$$\sup_{t\in[0,T_0]} \|(\mathcal{P}z)_t\|_{L[-c,0]} \leq c(\|\psi\|_c + R_0).$$

It is clear that $(\mathcal{P}z)(t) = \psi(t)$ for any $z \in \Omega,\ t \in [-c, 0]$. Then, we conclude that $\mathcal{P}(\Omega) \subseteq \Omega$.

**Step II.** $\mathcal{P} : \Omega \to \Omega$ is equicontinuous.

For any $z \in \Omega$ and $t_1, t_2 \in W = [-c, T_0]$ with $t_1 < t_2$, we have the following discussion.
(i) $0 \leq t_1 < t_2 \leq T_0$. Note that

$$\begin{aligned} &(\mathcal{P}z)(t_2) - (\mathcal{P}z)(t_1) \\ =\ &(\mathcal{N}(t_2) - \mathcal{N}(t_1))\psi(0) \\ &+\mathcal{N}(t_2)\mathcal{G}_z(t_2) - \mathcal{N}(t_1)\mathcal{G}_z(t_1) + \mathcal{N}(t_2)\mathcal{B}_{x_z}(t_2) - \mathcal{N}(t_1)\mathcal{B}_{x_z}(t_1) \\ &+ \int_0^{t_2}\dot{\mathcal{N}}(t_2-s)(\mathcal{G}_z(s) - \mathcal{G}_z(t_2))ds - \int_0^{t_1}\dot{\mathcal{N}}(t_1-s)(\mathcal{G}_z(s) - \mathcal{G}_z(t_1))ds \\ &+ \int_0^{t_2}\dot{\mathcal{N}}(t_2-s)(\mathcal{B}_{x_z}(s) - \mathcal{B}_{x_z}(t_2))ds - \int_0^{t_1}\dot{\mathcal{N}}(t_1-s)(\mathcal{B}_{x_z}(s) - \mathcal{B}_{x_z}(t_1))ds.\end{aligned}$$

Clearly,

$$\|(\mathcal{P}z)(t_2) - (\mathcal{P}z)(t_1)\| \leq \|\mathfrak{O}_1\| + \|\mathfrak{O}_2\| + \|\mathfrak{O}_3\| + \|\mathfrak{O}_4\|,$$

where

$$\begin{aligned}\mathfrak{O}_1 &= (\mathcal{N}(t_2) - \mathcal{N}(t_1))\psi(0), \\ \mathfrak{O}_2 &= (\mathcal{N}(t_2)\mathcal{G}_z(t_2) - \mathcal{N}(t_1)\mathcal{G}_z(t_1)) + (\mathcal{N}(t_2)\mathcal{B}_{x_z}(t_2) - \mathcal{N}(t_1)\mathcal{B}_{x_z}(t_1)), \\ \mathfrak{O}_3 &= \int_0^{t_2}\dot{\mathcal{N}}(t_2-s)(\mathcal{G}_z(s) - \mathcal{G}_z(t_2))ds - \int_0^{t_1}\dot{\mathcal{N}}(t_1-s)(\mathcal{G}_z(s) - \mathcal{G}_z(t_1))ds, \\ \mathfrak{O}_4 &= \int_0^{t_2}\dot{\mathcal{N}}(t_2-s)(\mathcal{B}_{x_z}(s) - \mathcal{B}_{x_z}(t_2))ds - \int_0^{t_1}\dot{\mathcal{N}}(t_1-s)(\mathcal{B}_{x_z}(s) - \mathcal{B}_{x_z}(t_1))ds.\end{aligned}$$

The strong continuity of $\mathcal{N}(\cdot)$ indicates that $\|\mathfrak{O}_1\| \to 0$, as $|t_1 - t_2| \to 0$. By Lemma 7, one can obtain

$$
\begin{aligned}
\|\mathfrak{O}_2\| &\leq \|\mathcal{N}(t_2)\|_{\mathcal{L}(E)}\|\mathcal{G}_z(t_2) - \mathcal{G}_z(t_1)\| + \|(\mathcal{N}(t_2) - \mathcal{N}(t_1))\mathcal{G}_z(t_1)\| \\
&\quad + \|\mathcal{N}(t_2)\|_{\mathcal{L}(E)}\|\mathcal{B}_{x_z}(t_2) - \mathcal{B}_{x_z}(t_1)\| + \|(\mathcal{N}(t_2) - \mathcal{N}(t_1))\mathcal{B}_{x_z}(t_1)\| \\
&\leq N_0[|\mathcal{G}_z|]_{C^r}(t_2 - t_1)^r + \int_{t_1}^{t_2} \|\dot{\mathcal{N}}(s)\mathcal{G}_z(t_1)\|ds \\
&\quad + N_0 Q_1 N_3 \frac{{t_2}^r - {t_1}^r + (t_2 - t_1)^r}{r} + N_0 \int_{t_1}^{t_2} \frac{\|\mathcal{B}x_z(s)\|}{(t_2 - s)^{1-r}} ds \\
&\quad + \|(\mathcal{N}(t_2) - \mathcal{N}(t_1))\|_{\mathcal{L}(E)} \frac{Q_1 N_3 T_0^r}{r} \\
&\leq N_0[|\mathcal{G}_z|]_{C^r}(t_2 - t_1)^r + N_1[|\mathcal{G}_z|]_{C^r} \int_{t_1}^{t_2} \frac{s^r}{s} ds \\
&\quad + N_0 Q_1 N_3 \frac{{t_2}^r - {t_1}^r + (t_2 - t_1)^r}{r} + N_0 Q_1 N_3 \frac{(t_2 - t_1)^r}{r} \\
&\quad + \|(\mathcal{N}(t_2) - \mathcal{N}(t_1))\|_{\mathcal{L}(E)} \frac{Q_1 N_3 T_0^r}{r} \\
&\leq N_0 \frac{2K}{\Gamma(r+1)}(t_2 - t_1)^r + N_1 \frac{2K}{\Gamma(r+1)} \frac{(t_2 - t_1)^r}{r} + N_0 Q_1 N_3 \frac{{t_2}^r - {t_1}^r + (t_2 - t_1)^r}{r} \\
&\quad + N_0 Q_1 N_3 \frac{(t_2 - t_1)^r}{r} + \|(\mathcal{N}(t_2) - \mathcal{N}(t_1))\|_{\mathcal{L}(E)} \frac{Q_1 N_3 T_0^r}{r} \to 0, \ as \ |t_1 - t_2| \to 0.
\end{aligned}
$$

By means of Lemma 7, we can obtain

$$
\begin{aligned}
\|\mathfrak{O}_3\| &\leq \int_0^{t_1} \|\dot{\mathcal{N}}((t_2 - t_1) + s) - \dot{\mathcal{N}}(s)\|_{\mathcal{L}(E)}\|\mathcal{G}_z(t_1 - s) - \mathcal{G}_z(t_1)\|ds \\
&\quad + \left\| \int_0^{t_1} \dot{\mathcal{N}}((t_2 - t_1) + s)(\mathcal{G}_z(t_1) - \mathcal{G}_z(t_2))ds \right\| \\
&\quad + \int_{t_1}^{t_2} \|\dot{\mathcal{N}}(t_2 - s)\|_{\mathcal{L}(E)}\|\mathcal{G}_z(s) - \mathcal{G}_z(t_2)\|ds \\
&\leq \int_0^{t_1} \int_s^{s+(t_2-t_1)} \|\ddot{\mathcal{N}}(\xi)\|_{\mathcal{L}(E)}[|\mathcal{G}_z|]_{C^r} s^r d\xi ds \\
&\quad + \|\mathcal{N}(t_2) - \mathcal{N}((t_2 - t_1))\|_{\mathcal{L}(E)}\|\mathcal{G}_z(t_1) - \mathcal{G}_z(t_2)\| + N_1[|\mathcal{G}_z|]_{C^r} \int_{t_1}^{t_2} (t_2 - s)^{r-1} ds \\
&\leq [|\mathcal{G}_z|]_{C^r} N_2 \int_0^{t_1} \int_s^{s+(t_2-t_1)} \xi^{r-2} d\xi ds + 2N_0[|\mathcal{G}_z|]_{C^r}(t_2 - t_1)^r + N_1[|\mathcal{G}_z|]_{C^r} \frac{(t_2 - t_1)^r}{r} \\
&\leq \frac{2(t_2 - t_1)^r}{r(1 - r)}[|\mathcal{G}_z|]_{C^r} N_2 + 2N_0[|\mathcal{G}_z|]_{C^r}(t_2 - t_1)^r + \frac{N_1[|\mathcal{G}_z|]_{C^r}(t_2 - t_1)^r}{r} \\
&\leq \frac{4N_2 K(t_2 - t_1)^r}{r(1 - r)\Gamma(r+1)} + \frac{4N_0 K(t_2 - t_1)^r}{\Gamma(r+1)} + \frac{2N_1 K(t_2 - t_1)^r}{r\Gamma(r+1)} \to 0, \ as \ |t_1 - t_2| \to 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|\mathfrak{O}_4\| &\leq \frac{2(t_2 - t_1)^r}{r(1 - r)}[|\mathcal{B}_{x_z}|]_{C^r} N_2 + 2N_0[|\mathcal{B}_{x_z}|]_{C^r}(t_2 - t_1)^r + N_1[|\mathcal{B}_{x_z}|]_{C^r} \frac{(t_2 - t_1)^r}{r} \\
&\leq \frac{4Q_1 N_2 N_3 (t_2 - t_1)^r}{r(1 - r)\Gamma(r+1)} + \frac{4Q_1 N_0 N_3 (t_2 - t_1)^r}{\Gamma(r+1)} + \frac{2Q_1 N_1 N_3 (t_2 - t_1)^r}{r\Gamma(r+1)} \\
&\quad \to 0, \ as \ |t_1 - t_2| \to 0.
\end{aligned}
$$

(ii) $-c \leq t_1 < t_2 \leq 0$. From the continuity of $\psi(\cdot)$, we have

$$\|(\mathcal{P}z)(t_2) - (\mathcal{P}z)(t_1)\| = \|\psi(t_2) - \psi(t_1)\| \to 0, \ as \ \ |t_1 - t_2| \to 0.$$

(iii) $-c \leq t_1 < 0 < t_2 \leq T_0$. Then,

$$\begin{aligned}
\|(\mathcal{P}z)(t_2)-(\mathcal{P}z)(t_1)\| &\le \|(\mathcal{P}z)(t_2)-(\mathcal{P}z)(0)\|+\|(\mathcal{P}z)(0)-(\mathcal{P}z)(t_1)\| \\
&\le \|(\mathcal{N}(t_2)-I)\psi(0)\|+\|\mathcal{N}(t_2)\mathcal{G}_z(t_2)\|+\|\mathcal{N}(t_2)\mathcal{B}_{x_z}(t_2)\| \\
&\quad +\|\int_0^{t_2}\dot{\mathcal{N}}(t_2-s)(\mathcal{G}_z(s)-\mathcal{G}_z(t_2))ds\| \\
&\quad +\|\int_0^{t_2}\dot{\mathcal{N}}(t_2-s)(\mathcal{B}_{x_z}(s)-\mathcal{B}_{x_z}(t_2))ds\|+\|\psi(0)-\psi(t_1)\| \\
&\le \|(\mathcal{N}(t_2)-I)\psi(0)\|+\frac{N_0(K+Q_1N_3)}{r}t_2^r \\
&\quad +\frac{2N_1(K+Q_1N_3)}{r}\int_0^{t_2}\frac{1}{t_2-s}(t_2-s)^r ds+\|\psi(0)-\psi(t_1)\| \\
&\le \|(\mathcal{N}(t_2)-I)\psi(0)\|+\frac{N_0(K+Q_1N_3)}{r}t_2^r \\
&\quad +\frac{2N_1(K+Q_1N_3)}{r^2}t_2^r+\|\psi(0)-\psi(t_1)\|\to 0,\ as\ |t_1-t_2|\to 0.
\end{aligned}$$

Thus, $\|(\mathcal{P}z)(t_2)-(\mathcal{P}z)(t_1)\|\to 0$, as $|t_1-t_2|\to 0$, for every $z\in\Omega$. This deduces that $\mathcal{P}:\Omega\to\Omega$ is equicontinuous.

**Step III.** $\mathcal{P}$ is continuous on $\Omega$.

Let $y_n$ be a sequence, such that $y_n\to y$ in $\Omega$ as $n\to\infty$. We only consider the case $t\in[0,T_0]$ since the continuity of operator $\mathcal{P}$ is obvious under the case $t\in[-c,0]$.

For each $t\in[0,T_0]$, one has

$$\begin{aligned}
\|(\mathcal{P}y_n)(t)-(\mathcal{P}y)(t)\| &\le N_0\|\mathcal{G}_{y_n}(t)-\mathcal{G}_y(t)\|+N_0\|\mathcal{B}_{x_{y_n}}(t)-\mathcal{B}_{x_y}(t)\| \\
&\quad +\int_0^t\|\dot{\mathcal{N}}(t-s)[(\mathcal{G}_{y_n}(s)-\mathcal{G}_{y_n}(t))-(\mathcal{G}_y(s)-\mathcal{G}_y(t))]\|ds \\
&\quad +\int_0^t\|\dot{\mathcal{N}}(t-s)[(\mathcal{B}_{x_{y_n}}(s)-\mathcal{B}_{x_{y_n}}(t))-(\mathcal{B}_{x_y}(s)-\mathcal{B}_{x_y}(t))]\|ds \\
&\le N_0\int_0^t(t-s)^{r-1}\|g(s,(y_n)_s,H_{y_n}(s))-g(s,y_s,H_y(s))\|ds \\
&\quad +\frac{N_0Q_1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\|x_{y_n}(s)-x_y(s)\|ds \\
&\quad +\int_0^t\|\dot{\mathcal{N}}(t-s)[(\mathcal{G}_{y_n}(s)-\mathcal{G}_{y_n}(t))-(\mathcal{G}_y(s)-\mathcal{G}_y(t))]\|ds \\
&\quad +\int_0^t\|\dot{\mathcal{N}}(t-s)[(\mathcal{B}_{x_{y_n}}(s)-\mathcal{B}_{x_{y_n}}(t))-(\mathcal{B}_{x_y}(s)-\mathcal{B}_{x_y}(t))]\|ds
\end{aligned}$$

It is easy to check that the following inequalities hold:

$$(\cdot-s)^{r-1}\|g(s,(y_n)_s,H_{y_n}(s))-g(s,y_s,H_y(s))\|\le 2K(\cdot-s)^{r-1}\in L^1([0,T_0],R^+),$$

$$(\cdot-s)^{r-1}\|x_{y_n}(s)-x_y(s)\|\le 2N_3(\cdot-s)^{r-1}\in L^1([0,T_0],R^+),$$

$$\begin{aligned}
\|\dot{\mathcal{N}}(\cdot-s)[(\mathcal{G}_{y_n}(s)-\mathcal{G}_{y_n}(\cdot))-(\mathcal{G}_y(s)-\mathcal{G}_y(\cdot))]\| &\le \frac{N_1}{(\cdot-s)}\frac{2K(\cdot-s)^r}{r} \\
&\le \frac{2N_1K}{r}(\cdot-s)^{r-1}\in L^1([0,T_0],R^+), \\
\|\dot{\mathcal{N}}(\cdot-s)[(\mathcal{B}_{x_{y_n}}(s)-\mathcal{B}_{x_{y_n}}(\cdot))-(\mathcal{B}_{x_y}(s)-\mathcal{B}_{x_y}(\cdot))]\| &\le \frac{N_1}{(\cdot-s)}\frac{2Q_1N_3(\cdot-s)^r}{r} \\
&\le \frac{2Q_1N_1N_3}{r}(\cdot-s)^{r-1}\in L^1([0,T_0],R^+).
\end{aligned}$$

Moreover, one has

$$\begin{aligned}
&\|x_{y_n}(s)-x_y(s)\| \\
\le\ & Q_2\left(N_0\|\mathcal{G}_{y_n}(T_0)-\mathcal{G}_y(T_0)\|+\int_0^{T_0}\|\dot{\mathcal{N}}(T_0-s)[(\mathcal{G}_{y_n}(s)-\mathcal{G}_{y_n}(t))-(\mathcal{G}_y(s)-\mathcal{G}_y(t))]\|ds\right),
\end{aligned}$$

and

$$\begin{aligned}
\|\dot{\mathcal{N}}(T_0-s)[(\mathcal{G}_{y_n}(s)-\mathcal{G}_{y_n}(\cdot))-(\mathcal{G}_y(s)-\mathcal{G}_y(\cdot))]\| &\le \frac{N_1}{(T_0-s)}\frac{2K(\cdot-s)^r}{r} \\
&\le \frac{2N_1K}{r}(\cdot-s)^{r-1}\in L^1([0,T_0],R^+).
\end{aligned}$$

Then, Lebesgue's domination convergence theorem implies that $\|(\mathcal{P}y_n)(t)-(\mathcal{P}y)(t)\|\to 0$, as $n\to+\infty$. From Ascoli-Arzelà theorem, it follows that $\|\mathcal{P}y_n-\mathcal{P}y\|_{C(W;E)}\to 0$, as $n\to+\infty$. The proof is completed.

**Step IV.** Mönch's condition holds.

Suppose $B = \overline{co}\mathcal{P}(\Omega)$. From Step I and II, it is not difficult to check that $\mathcal{P}(B) \subseteq B$ and $B$ is equicontinuous.

For any $z \in B$, we take

$$(\mathcal{P}z)(t) = \begin{cases} (\mathcal{P}_1 z)(t) + (\mathcal{P}_2 z)(t) + (\mathcal{P}_3 z)(t) + (\mathcal{P}_4 z)(t), & t \in [0, T_0], \\ \psi(t), & t \in [-c, 0], \end{cases}$$

where

$$(\mathcal{P}_1 z)(t) = \mathcal{N}(t)\psi(0) + \mathcal{N}(t)\mathcal{G}_z(t),$$
$$(\mathcal{P}_2 z)(t) = \int_0^t \dot{\mathcal{N}}(t-s)(\mathcal{G}_z(s) - \mathcal{G}_z(t))ds,$$
$$(\mathcal{P}_3 z)(t) = \mathcal{N}(t)\mathcal{B}_{x_z}(t),$$
$$(\mathcal{P}_4 z)(t) = \int_0^t \dot{\mathcal{N}}(t-s)(\mathcal{B}_{x_z}(s) - \mathcal{B}_{x_z}(t))ds.$$

Suppose bounded set $D_0 \subset B$ is countable and $D_0 \subset \overline{co}(\{z_0\} \bigcup \mathcal{P}(D_0))$, we shall show that $\zeta(D_0) = 0$. Without loss of generality, we may suppose that $D_0 = \{z_n\}_{n=1}^{\infty}$.

From Lemmas 4 (4) and 6 and Hypothesis (H4) (i), (ii), for any $s \in V$, we have

$$\begin{aligned} \zeta(\{g(s,(z_n)_s, H_{z_n}(s))\}) &\leq k_1(s)\zeta(\{(z_n)_s\}) + k_2(s)\zeta\left(\left\{\int_0^s h(s,\eta,(z_n)_\eta)d\eta\right\}\right) \\ &\leq k_1(s)\zeta(\{(z_n)_s\}) + 2k_2(s)\int_0^s \zeta(\{h(s,\eta,(z_n)_\eta)\})d\eta \\ &\leq k_1(s)\zeta(\{(z_n)_s\}) + 2k_2(s)\int_0^s l(\eta)\zeta(\{(z_n)_\eta\})d\eta \\ &\leq (k_1(s) + 2k_2(s)\|l\|_{L^1})c\zeta(\{z_n\}). \end{aligned}$$

Then, for any $t \in [0, T_0]$, by Lemmas 3 and 6, one has

$$\begin{aligned} & 2\int_0^t (t-s)^{r-1}\zeta(\{g(s,(z_n)_s, H_{z_n}(s))\})ds \\ \leq\ & 2\int_0^t (t-s)^{r-1}k_1(s)ds \cdot c\zeta(\{z_n\}) + 4\|l\|_{L^1}\int_0^t (t-s)^{r-1}k_2(s)ds \cdot c\zeta(\{z_n\}) \\ \leq\ & 2\left(\int_0^t [(t-s)^{r-1}]^{\frac{1}{1-r_1}}ds\right)^{1-r_1}\|k_1\|_{L^{\frac{1}{r_1}}} \cdot c\zeta(\{z_n\}) \\ & +4\|l\|_{L^1}\left(\int_0^t [(t-s)^{r-1}]^{\frac{1}{1-r_2}}ds\right)^{1-r_2}\|k_2\|_{L^{\frac{1}{r_2}}} \cdot c\zeta(\{z_n\}) \\ \leq\ & \left(2\left(\frac{1-r_1}{r-r_1}\right)^{1-r_1} t^{r-r_1}\|k_1\|_{L^{\frac{1}{r_1}}} + 4\|l\|_{L^1}\left(\frac{1-r_2}{r-r_2}\right)^{1-r_2} t^{r-r_2}\|k_2\|_{L^{\frac{1}{r_2}}}\right)c\zeta(\{z_n\}) \\ \leq\ & \left(2c\theta_1^{1-r_1}\|k_1\|_{L^{\frac{1}{r_1}}} + 4c\theta_2^{1-r_2}\|k_2\|_{L^{\frac{1}{r_2}}}\|l\|_{L^1}\right)t^r\zeta(\{z_n\}) \\ \leq\ & K_1 T_0^r \zeta(\{z_n\}). \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta((\mathcal{P}_1 D_0)(t)) &= \zeta\left(\left\{\mathcal{N}(t)\psi(0) + \mathcal{N}(t)\frac{1}{\Gamma(r)}\int_0^t \frac{g(s,(z_n)_s, H_{z_n}(s))}{(t-s)^{1-r}}ds\right\}\right) \\ &\leq \frac{2N_0}{\Gamma(r)}\int_0^t (t-s)^{r-1}\zeta(\{g(s,(z_n)_s, H_{z_n}(s))\})ds \\ &\leq \frac{N_0 K_1}{\Gamma(r)} T_0^r \zeta(\{z_n\}) \\ &= \frac{N_0 K_1}{\Gamma(r)} T_0^r \zeta(D_0),\ t \in [0, T_0]. \end{aligned} \tag{7}$$

For $t \in [0, T_0]$, in view of Lemmas 4 (4) and 8, we get

$$
\begin{aligned}
\zeta((\mathcal{P}_2 D_0)(t)) &= \zeta\left(\left\{\int_0^t \dot{\mathcal{N}}(t-s)\frac{1}{\Gamma(r)}\left(\int_0^s \frac{g(y,(z_n)_y,H_{z_n}(y))}{(s-y)^{1-r}}dy - \int_0^t \frac{g(y,(z_n)_y,H_{z_n}(y))}{(t-y)^{1-r}}dy\right)ds\right\}\right) \\
&\leq \frac{4N_1}{\Gamma(r)}\int_0^t \frac{1}{t-s}\left(\int_0^s ((s-y)^{r-1}-(t-y)^{r-1})\zeta(\{g(y,(z_n)_y,H_{z_n}(y))dy\right)ds \\
&\quad + \frac{4N_1}{\Gamma(r)}\int_0^t \frac{1}{t-s}\left(\int_s^t (t-y)^{r-1}\zeta(\{g(y,(z_n)_y,H_{z_n}(y))\})dy\right)ds \\
&\leq \frac{4N_1}{\Gamma(r)}\int_0^t \frac{1}{t-s}[K_1(t-s)^r\zeta(\{z_n\})]ds \\
&\quad + \frac{4N_1}{\Gamma(r)}\int_0^t \frac{1}{t-s}[K_2(t-s)^r\zeta(\{z_n\})]ds \\
&\leq \frac{4N_1K_1}{\Gamma(r+1)}t^r\zeta(\{z_n\}) + \frac{4N_1K_2}{\Gamma(r+1)}t^r\zeta(\{z_n\}) \\
&\leq \frac{4N_1(K_1+K_2)}{\Gamma(r+1)}t^r\zeta(\{z_n\}) \\
&\leq \frac{4N_1(K_1+K_2)}{\Gamma(r+1)}T_0^r\zeta(D_0).
\end{aligned} \tag{8}
$$

From (7) and (8), it follows that

$$
\begin{aligned}
\zeta(\{\mathcal{B}_{x_{z_n}}(t)\}) &\leq \frac{2l_0Q_1Q_2}{\Gamma(r)}\int_0^t (t-s)^{r-1}\zeta(\{z_1 - \mathcal{N}(T_0)\psi_0 - \mathcal{N}(T_0)\mathcal{G}_{z_n}(T_0)\})ds \\
&\quad + \frac{2l_0Q_1Q_2}{\Gamma(r)}\int_0^t (t-s)^{r-1}\zeta\left(\left\{\int_0^{T_0} \dot{\mathcal{N}}(T_0-s)(\mathcal{G}_{z_n}(s)-\mathcal{G}_{z_n}(T_0))ds\right\}\right)ds \\
&\leq \frac{2l_0Q_1Q_2}{\Gamma(r)}\int_0^t (t-s)^{r-1}[\zeta(\{(\mathcal{P}_1D_0)(T_0)\}) + \zeta(\{(\mathcal{P}_2D_0)(T_0)\})]ds \\
&\leq \frac{2l_0Q_1Q_2T^r}{\Gamma(r+1)}\left(\frac{N_0K_1}{\Gamma(r)} + \frac{4N_1(K_1+K_2)}{\Gamma(r+1)}\right)T_0^r\zeta(D_0) \\
&\leq \frac{2l_0Q_1Q_2T^r(N_0K_1 + 4N_1(K_1+K_2))}{\Gamma^2(r+1)}T_0^r\zeta(D_0) \\
&= \frac{2l_0Q_1Q_2QT^r}{\Gamma^2(r+1)}T_0^r\zeta(D_0),\ t\in[0,T_0],
\end{aligned} \tag{9}
$$

and this indicates

$$
\zeta((\mathcal{P}_3D_0)(t)) \leq \frac{2l_0N_0Q_1Q_2QT^r}{\Gamma^2(r+1)}T_0^r\zeta(D_0),\ t\in[0,T_0]. \tag{10}
$$

By means of Lemma 4 (4), Hypothesis (H3) and (9), for $t \in [0, T_0]$, one derives

$$
\begin{aligned}
\zeta((\mathcal{P}_4D_0)(t)) &= \zeta\left(\left\{\int_0^t \dot{\mathcal{N}}(t-s)\frac{1}{\Gamma(r)}\left(\int_0^s \frac{\mathcal{B}x_{z_n}(y)}{(s-y)^{1-r}}dy - \int_0^t \frac{\mathcal{B}x_{z_n}(y)}{(t-y)^{1-r}}dy\right)ds\right\}\right) \\
&\leq \frac{4N_1}{\Gamma(r)}\int_0^t \frac{1}{t-s}\left(\int_0^s ((s-y)^{r-1}-(t-y)^{r-1})\zeta(\{\mathcal{B}x_{z_n}(y)\})dy\right)ds \\
&\quad + \frac{4N_1}{\Gamma(r)}\int_0^t \frac{1}{t-s}\left(\int_s^t (t-y)^{r-1}\zeta(\{\mathcal{B}x_{z_n}(t)\})dy\right)ds \\
&\leq \frac{4N_1\zeta(\{\mathcal{B}x_{z_n}(t)\})}{\Gamma(r)}\int_0^t \frac{1}{t-s}\left(\frac{s^r}{r} - \frac{t^r-(t-s)^r}{r} + \frac{(t-s)^r}{r}\right)ds \\
&\leq \frac{4N_1\zeta(\{\mathcal{B}x_{z_n}(t)\})}{\Gamma(r+1)}\int_0^t \frac{1}{t-s}(s^r - t^r + 2(t-s)^r)ds \\
&\leq \frac{4N_1\zeta(\{\mathcal{B}x_{z_n}(t)\})}{\Gamma(r+1)}\int_0^t \frac{1}{t-s}\cdot 2(t-s)^r ds \\
&\leq \frac{8N_1\zeta(\{\mathcal{B}x_{z_n}(t)\})}{\Gamma(r+1)}\int_0^t (t-s)^{r-1}ds \\
&\leq \frac{16l_0Q_1Q_2QN_1T^{2r}}{r\Gamma^3(r+1)}T_0^r\zeta(D_0).
\end{aligned} \tag{11}
$$

Thus, (7), (8), (10) and (11) imply that

$$\begin{aligned}
\zeta((\mathcal{P}D_0)(t)) &\leq \zeta((\mathcal{P}_1D_0)(t)) + \zeta((\mathcal{P}_2D_0)(t)) + \zeta((\mathcal{P}_3D_0)(t)) + \zeta((\mathcal{P}_4D_0)(t)) \\
&\leq \frac{N_0K_1}{\Gamma(r)}T_0^r\zeta(D_0) + \frac{4N_1(K_1+K_2)}{\Gamma(r+1)}T_0^r\zeta(D_0) \\
&\quad + \frac{2l_0N_0Q_1Q_2QT^r}{\Gamma^2(r+1)}T_0^r\zeta(D_0) + \frac{16l_0Q_1Q_2QN_1T^{2r}}{r\Gamma^3(r+1)}T_0^r\zeta(D_0) \\
&\leq \frac{Q + 2l_0Q_1Q_2QT^r(N_0 + 8N_1T^r)}{r\Gamma^3(r+1)}T_0^r\zeta(D_0).
\end{aligned} \tag{12}$$

On the other hand, we have from the equicontinuity and boundedness of $\mathcal{P}(D_0)$

$$\zeta(\mathcal{P}(D_0)) = \max_{t\in W}\zeta((\mathcal{P}D_0)(t)). \tag{13}$$

Then, by the definition of $T_0$ and (12), (13), one can derive

$$\zeta(D_0) \quad \leq \quad \zeta(\overline{co}(\{z_0\} \cup \mathcal{P}(D_0))) \leq \zeta(\mathcal{P}(D_0)) < \zeta(D_0),$$

that is, $\zeta(D_0) = 0$, which shows that $D_0$ is relatively compact. By Lemma 2, we know that $\mathcal{P}$ has at least one fixed point $z \in B$, which is a mild solution to system (1) on $W$, satisfying $(\mathcal{P}z)(T_0) = z(T_0) = z_1$. The proof is now completed. □

**Remark 2.** *(I) Compactness of the resolvent operators associated with the system (1) is unnecessary. (II) By introducing the complete space $L([-c,0];E)$ and function $z_t$, the difficulties in the estimate of noncompactness measures caused by delay are effectively solved (Lemmas 5 and 6). Therefore, we generalize some related control results such as [20,21,43,44], etc.*

## 4. An Example

To illustrate our theory, we consider the fractional integrodifferential dynamical system with delay of the form

$$\begin{cases}
\dfrac{\partial^{\frac{3}{5}}}{\partial t^{\frac{3}{5}}}z(t,y) = \dfrac{\partial}{\partial y}z(t,y) \\
\quad +\dfrac{1}{3}\left(z(t+\theta,y) + \displaystyle\int_0^t \frac{(t-s)^4|z(s+\theta,y)|}{2+|z(s+\theta,y)|}ds\right) + \varpi\mu(t,y), \ \ t\in V, \ y\in(0,1), \\
z(t,0) = z(t,1) = 0, \ \ t \in V, \\
z(t,y) = \psi(t,y), \ \ t\in[-c,0], \ y \in (0,1),
\end{cases} \tag{14}$$

where $\psi$ is continuous and satisfies certain smoothness conditions, $\varpi > 0$, $\mu : V\times[0,1] \to [0,1]$ is continuous, and $V = [0,T]$, $\theta \in [-c,0]$.

Let $E = U := C([0,1])$ and let $\mathcal{A} : \mathcal{D} \subset E \to E$ given by $\mathcal{A}\nu = \nu^{'}$ with domain $\mathcal{D} = \{\nu \in E : \ \nu^{'} \in E, \ \nu(0) = \nu(1) = 0\}$. So, $\mathcal{A}$ generates a semigroup $\{\mathcal{T}(t) : t \geq 0\}$ on $X$, which is defined as $\mathcal{T}(t)\nu(s) = \nu(t+s)$ for $\nu \in E$, and $\mathcal{T}(t)$ is not a compact semigroup on $E$.

Furthermore, from the Corollary 2.4 in [45], it follows that the integral equation

$$z(t) = \frac{1}{\Gamma(r)}\int_0^t \frac{\mathcal{A}z(s)}{(t-s)^{1-r}}ds, \ t\geq 0,$$

admits an analytic resolvent operator $\mathcal{N}(t)$ on $E$.

Define

$$\begin{aligned}
&z(t)(y) = z(t,y), \\
&{}^CD^rz(t)(y) = \frac{\partial^{\frac{3}{5}}}{\partial t^{\frac{3}{5}}}z(t,y), \\
&g(t,z_t,H_z(t))(y) = \frac{1}{3}\left(z(t+\theta,y) + \int_0^t \frac{(t-s)^4|z(s+\theta,y)|}{2+|z(s+\theta,y)|}ds\right),
\end{aligned}$$

$h(t,s,z_s) = \dfrac{(t-s)^4|z(s+\theta,y)|}{2+|z(s+\theta,y)|}$, $(t,s) \in \Delta = \{(t,s) \in V \times V : s \leq t\}$,
$x(t)(y) = \mu(t,y)$.

It is easy to see that (H1) and (H2) hold. For $y \in (0,1)$, suppose that the linear operator $\Lambda(t)$ defined as

$$\begin{aligned}(\Lambda(t)u)(y) &= \frac{\varpi\mathcal{N}(t)}{\Gamma(\frac{3}{5})}\int_0^t (t-s)^{-\frac{2}{5}}\mu(s,y)ds \\ &\quad + \frac{\varpi}{\Gamma(\frac{3}{5})}\int_0^t \dot{\mathcal{N}}(t-s)\left(\int_0^s (s-\eta)^{-\frac{2}{5}}\mu(\eta,y)d\eta - \int_0^t (t-\eta)^{-\frac{2}{5}}\mu(\eta,y)d\eta\right)ds,\end{aligned}$$

satisfies the assumption (H3). In addition, simple verification can imply that (H4) holds with $k_1(t) = k_2(t) = \dfrac{1}{3}$, $t \in V$ and $l(s) = T^4$, $s \in V$. Consequently, if all the requirements of Theorem 1 are satisfied, then system (14) is controllable on $V$.

## 5. Conclusions and Future Work

Some new controllability results for a class of fractional integrodifferential dynamical systems with a delay in Banach spaces are derived in this paper by using resolvent operator theory and fixed-point theory. A new definition of controllability is introduced, and the nonlinearity is not supposed to be Lipschitz continuous compared, with most of the existing literature. A suitable delay item in a special complete space is also introduced to solve the difficulties caused by time delay. An explicit example is given to demonstrate the effectiveness of our results.

Drawing on the ideas of this paper, the controllability for a class of fractional integrodifferential dynamical inclusions with time delay and nonlocal conditions will be further studied in the future:

$$\begin{cases} D^r z(t) \in \mathcal{A}z(t) + g(t, z_t, H_z(t)) + \mathcal{B}x(t), \;\; a.e.\ t \in V := [0,T], \\ z(t) + h(z) = \psi(t), \;\; t \in [-c, 0], \end{cases}$$

where $h : C([-c,T],E) \to E$ is a given function. In common applications, the nonlocal conditions are usually described as $h(z) = \sum_{i=1}^{m} k_i z(\tau_i)$, where $k_i$ $(i = 1,2,\cdots,m)$ are given constants and $0 < \tau_1 < \tau_2 < \cdots < \tau_n \leq T$.

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