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Fractional Differential Equation Involving Mixed Nonlinearities with Nonlocal Multi-Point and Riemann-Stieltjes Integral-Multi-Strip Conditions

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Abstract: In this paper, we investigate a new class of boundary value problems involving fractional differential equations with mixed nonlinearities, and nonlocal multi-point and Riemann–Stieltjes integral-multi-strip boundary conditions. Based on the standard tools of the fixed point theory, we obtain some existence and uniqueness results for the problem at hand, which are well illustrated with the aid of examples. Our results are not only in the given configuration but also yield several new results as special cases. Some variants of the given problem are also discussed.

Keywords: fractional differential equation; mixed nonlinearities; multi-point; integral boundary conditions; existence; fixed point

1. Introduction

In this paper, we introduce and study a new boundary value problem of fractional differential equations involving mixed nonlinearities, and nonlocal multi-point and Riemann-Stieltjes integral-multi-strip boundary conditions. Precisely we consider the following problem:

$$^cD^p[^cD^q x(t) + f(t, x(t))] = g(t, x(t)), \;\; 0 < t < 1, \;\; 0 < p, q \leq 1, \tag{1}$$

$$x(0) = \sum_{j=1}^{m} \beta_j x(\sigma_j), \;\; bx(1) = a\int_0^1 x(s)dH(s) + \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} x(s)ds, \tag{2}$$

where $^cD^r$ denotes the Caputo fractional derivative of order r $(r = p, q)$, f and g are given continuous functions, $0 < \sigma_j < \xi_i < \eta_i < 1, a, b \in \mathbb{R}, \alpha_i, \beta_j \in \mathbb{R}, i = 1, 2, \ldots, n,\; j = 1, 2, \ldots, m$ and $H(.)$ is a function of bounded variation. One can note that the nonlinearities in (1) appear in the form:

$$g(t, x(t)) - {}^cD^p f(t, x(t)),$$

provided that it is possible to write (1) as $^cD^{p+q}x(t) + ^c D^p f(t, x(t)) = g(t, x(t))$. Notice that (1) is the neutral fractional differential equation.

Remark 1. *Letting $f(t, x(t)) = \lambda x(t)$, where λ is a constant, (1) becomes the Langevin equation with two fractional orders, which is a well known equation of mathematical physics and describes many interesting physical situations like fluctuating phenomena, anomalous diffusion, etc. [1]. In the limit $p, q \to 1^-$, the Equation (1)* *takes the form: $D^2x(t) + Df(t, x(t)) = g(t, x(t)), D = d/dt$, which is an equation of motion with nonlinear damping. Thus, (1) can be regarded as the fractional analogue of equation of motion. In case we fix $p = \alpha$, $q = \beta$, $f(t, x(t)) = (R_Z/L)x(t)$, $f(t, x(t)) = (1/LC)[-x(t) + e(t)]$, (1) takes the form of a fractional-order differential equation of the voltage function $x(t)$, see Equation (4) in [2]. The nonlocal conditions involved in the problem (1) appear in several applications of diffusion processes, computational fluid dynamics (CFD) studies of blood flow problems, bacterial self-regularization models, for instance, see [3–5].*

The topic of fractional order boundary value problems has been of great interest in recent years and many researchers contributed to it by contributing a variety of results involving different kinds of boundary conditions. The literature on this subject is now quite enriched and varies from the existence theory to the methods of solution for these problems [6–21]. Fractional order differential and integral operators are found to be of great utility in enhancing the mathematical modeling of dynamical systems involving fractals and chaos. It has been mainly due to the nonlocal nature of these operators, which accounts for hereditary characteristics of many materials and processes in contrast to their integer-order counterparts. For application details of fractional differential equations, we refer the reader to the works [22–28], while the theoretical aspects of fractional calculus can be found in the texts [29–31].

In Section 2, we outline the basic concepts of fractional calculus and prove an auxiliary lemma. Section 3 contains the main results for the problem (1) and (2) and illustrative examples for the obtained results. In Section 4, we present some variants of the problem (1) and (2).

2. Preliminaries

Before presenting some auxiliary results, let us recall some preliminary concepts of fractional calculus [30].

Definition 1. *Let ζ be a locally integrable real-valued function on $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional integral I_a^α of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) is defined as*

$$I_a^\alpha \zeta(t) = (\zeta * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \zeta(s) ds,$$

where $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ denotes the Euler gamma function.

Definition 2. *Let $\zeta, \zeta^{(m)} \in L^1[a, b]$ for $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional derivative D_a^α of order $\alpha > 0$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as*

$$D_a^\alpha \zeta(t) = \frac{d^m}{dt^m} I_a^{1-\alpha} \zeta(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} \zeta(s) ds.$$

Definition 3. *Let $\zeta \in C^m[a, b]$. Then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as*

$${}^cD_a^\alpha \zeta(t) = I_a^{1-\alpha} \zeta^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \zeta^{(m)}(s) ds.$$

Remark 2. *The Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) can be expressed in the following equivalent form*

$${}^cD_a^\alpha \zeta(t) = D_a^\alpha \left[\zeta(t) - \zeta(a) - \zeta'(a) \frac{(t-a)}{1!} - \ldots - \zeta^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

In the present work, we denote the Riemann-Liouville fractional integral I_a^α and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a = 0$ by I^α and ${}^cD^\alpha$ respectively.

Definition 4. *A function $x \in C^2[0,1]$ satisfying the problem (1) and (2) is called its solution on $[0,1]$.*

Associated with the linear variant of problem (1) and (2), we consider the following lemma.

Lemma 1. *Let $h, k \in C([0,1], \mathbb{R})$, the unique solution of the linear fractional differential equation*

$$ {}^cD^{p+q}x(t) + {}^cD^p h(t)) = k(t), \ \ 0 < p, q \leq 1, \tag{3}$$

supplemented with the boundary conditions (2) is given by

$$\begin{aligned} x(t) \ &= \ -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} k(s)ds \\ &-\lambda_1(t)\Bigg[b\int_0^1 \left(\frac{(1-s)^{q-1}}{\Gamma(q)} h(s) - \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} k(s) \right) ds \\ &+a\int_0^1 \left(\int_0^s \left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)} k(u) - \frac{(s-u)^{q-1}}{\Gamma(q)} h(u) \right) du \right) dH(s) \\ &+\sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} \left(\int_0^s \left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)} k(u) - \frac{(s-u)^{q-1}}{\Gamma(q)} h(u) \right) du \right) ds \Bigg] \\ &+\lambda_2(t) \sum_{j=1}^m \beta_j \left(\int_0^{\sigma_j} \frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} h(s)ds - \int_0^{\sigma_j} \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} k(s)ds \right), \end{aligned} \tag{4}$$

where

$$\lambda_1(t) = \frac{1}{\kappa}\Big(\rho_1 - \frac{\rho_2 t^q}{\Gamma(q+1)}\Big), \qquad \lambda_2(t) = \frac{1}{\kappa}\Big(\rho_3 - \frac{\rho_4 t^q}{\Gamma(q+1)}\Big),$$

$$\begin{aligned} &\rho_1 = \sum_{j=1}^m \beta_j \frac{\sigma_j^q}{\Gamma(q+1)}, \ \ \rho_2 = \sum_{j=1}^m \beta_j - 1, \\ &\rho_3 = \frac{1}{\Gamma(q+1)} \left(b - a\int_0^1 s^q dH(s) - \sum_{i=1}^n \alpha_i \frac{(\eta_i^{q+1} - \xi_i^{q+1})}{q+1} \right), \\ &\rho_4 = b - a\int_0^1 dH(s) - \sum_{i=1}^n \alpha_i(\eta_i - \xi_i), \end{aligned} \tag{5}$$

$$\kappa = \rho_2\rho_3 - \rho_1\rho_4 \neq 0. \tag{6}$$

Proof. Applying the integral operator I^p on (3), and then I^q on the resulting equation together with Lemma 2.22 in [29], we get

$$x(t) = -I^q h(t) + I^{q+p}k(t) + c_0 \frac{t^q}{\Gamma(q+1)} + c_1, \tag{7}$$

where c_0, c_1 are arbitrary constants. Using the boundary condition (2) in (7), we obtain

$$\sum_{j=1}^m \beta_j \frac{\sigma_j^q}{\Gamma(q+1)} c_0 + \Big(\sum_{j=1}^m \beta_j - 1 \Big) c_1 = \sum_{j=1}^m \beta_j I^q h(\sigma_j) - \sum_{j=1}^m \beta_j I^{q+p} k(\sigma_j), \tag{8}$$

$$\frac{1}{\Gamma(q+1)}\Big(b-a\int_0^1 s^q dHs-\sum_{i=1}^n \alpha_i \frac{(\eta_i^{q+1}-\xi_i^{q+1})}{q+1}\Big)c_0+\Big(b-a\int_0^1 dHs-\sum_{i=1}^n \alpha_i(\eta_i-\xi_i)\Big)c_1$$
$$=bI^q h(1)-bI^{q+p}k(1)+a\int_0^1\Big(-I^q h(s)+I^{q+p}k(s)\Big)dH(s)$$
$$+\sum_{i=1}^n \alpha_i\int_{\xi_i}^{\eta_i}\Big(I^{q+p}k(s)-I^q h(s)\Big)ds. \tag{9}$$

For the sake of convenience, we use the notations (5) in (8) and (9) to find the following system of equations

$$\begin{cases}\rho_1 c_0+\rho_2 c_1=\rho_5,\\ \rho_3 c_0+\rho_4 c_1=\rho_6,\end{cases} \tag{10}$$

where

$$\begin{aligned}\rho_5 &= \sum_{j=1}^m \beta_j I^q h(\sigma_j)-\sum_{j=1}^m \beta_j I^{q+p}k(\sigma_j),\\ \rho_6 &= bI^q h(1)-bI^{q+p}k(1)+a\int_0^1\left(-I^q h(s)+I^{q+p}k(s)\right)dH(s)\\ &\quad+\sum_{i=1}^n \alpha_i\int_{\xi_i}^{\eta_i}\left(I^{q+p}k(s)-I^q h(s)\right)ds.\end{aligned}$$

Solving the system (10) for c_0 and c_1, we get

$$c_0=(\rho_2\rho_6-\rho_5\rho_4)/\kappa,\ \ c_1=(\rho_3\rho_5-\rho_1\rho_6)/\kappa,$$

where κ is given by (6). Substituting the values of c_0, and c_1 in (7) together with the notations (5), we get the solution (4). By direct computation, one can obtain the converse of the lemma. This completes the proof. □

3. Existence and Uniqueness Results

In view of Lemma 1, we transform the problem (1) and (2) into a fixed point problem as $x=\mathcal{G}x$, where the operator $\mathcal{G}: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is defined by

$$\begin{aligned}\mathcal{G}x(t) &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds\\ &\quad-\lambda_1(t)\Bigg[b\int_0^1\left(\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))-\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))\right)ds\\ &\quad+a\int_0^1\int_0^s\left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}g(u,x(u))-\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))\right)du\,dH(s)\\ &\quad+\sum_{i=1}^n\alpha_i\int_{\xi_i}^{\eta_i}\int_0^s\left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}g(u,x(u))-\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))\right)du\,ds\Bigg]\\ &\quad+\lambda_2(t)\sum_{j=1}^m\beta_j\left(\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds-\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds\right).\end{aligned} \tag{11}$$

Note that $C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions $x:[0,1]\to\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup\{|x(t)|,t\in[0,1]\}$.

For the sake of computational convenience, we set

$$\begin{aligned}\Lambda \;=\;& \frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\overline{\lambda}_1\Bigg[\frac{|b|}{\Gamma(q+1)}+\frac{|b|}{\Gamma(q+p+1)}\\ &+|a|\int_0^1\left(\frac{s^q}{\Gamma(q+1)}+\frac{s^{q+p}}{\Gamma(q+p+1)}\right)dH(s)\\ &+\sum_{i=1}^{n-2}|\alpha_i|\left(\frac{(\eta_i^{q+1}-\xi_i^{q+1})}{\Gamma(q+2)}+\frac{(\eta_i^{q+p+1}-\xi_i^{q+p+1})}{\Gamma(q+p+2)}\right)\Bigg]\\ &+\overline{\lambda}_2\sum_{j=1}^{m}|\beta_j|\left(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\right), \end{aligned} \tag{12}$$

and

$$\Lambda_1=\Lambda-\frac{1}{\Gamma(q+1)}-\frac{1}{\Gamma(q+p+1)}, \tag{13}$$

where

$$\overline{\lambda}_1=\max_{t\in[0,1]}|\lambda_1(t)|=\frac{1}{|\kappa|}\Big(\frac{|\rho_2|}{\Gamma(q+1)}+|\rho_1|\Big),\qquad \overline{\lambda}_2=\max_{t\in[0,1]}|\lambda_2(t)|=\frac{1}{|\kappa|}\Big(\frac{|\rho_4|}{\Gamma(q+1)}+|\rho_3|\Big).$$

Now we present the existence and uniqueness results in the subsequent subsections.

3.1. Existence Result Via Leray–Schauder Nonlineear Alternative

Lemma 2. *(Nonlinear alternative for single valued maps [32]) Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or (ii) there is a $u \in \partial U$(the boundary of U in C) and $\varepsilon \in (0,1)$ with $u = \varepsilon F(u)$.

Theorem 1. *Let $f, g : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that:*

- (A_1) *There exist functions $p_1, p_2 \in C([0,1],\mathbb{R}^+)$, with $p = \max\{p_1, p_2\}$ and nondecreasing functions $\psi_1,\psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi = \max\{\psi_1,\psi_2\}$ such that $|f(t,x)| \le p_1\psi_1(\|x\|)$ and $|g(t,x)| \le p_2\psi_2(\|x\|)$, for all $(t,x) \in [0,1]\times\mathbb{R}$.*
- (A_2) *There exists a constant $M > 0$ such that*

$$\frac{M}{\|p\|\psi(M)\Lambda}>1.$$

Then the boundary value problem (1) and (2) has at least one solution on $[0,1]$.

Proof. Let us first show that the operator $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by (11) maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, in view of the assumption (A_1), we have

$$\begin{aligned}|(\mathcal{G}x)(t)| \;\le\;& \|p_1\|\psi_1(\|x\|)\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds+|\lambda_1(t)|\Bigg[|b|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds\\ &+|a|\int_0^1\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}du\right)dH(s)+\sum_{i=1}^{n}|\alpha_i|\int_{\xi_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}du\Big)ds\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}ds\Bigg\}\\
&+\|p_2\|\psi_2(\|x\|)\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}ds+|\lambda_1(t)|\Bigg[|b|\int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}ds\\
&+|a|\int_0^1\Big(\int_0^s\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}du\Big)dH(s)+\sum_{i=1}^{n}|\alpha_i|\int_{\xi_i}^{\eta_i}\Big(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}\Big)du\Big)ds\Bigg]\\
&+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}ds\Bigg\}\\
\le\;&\|p_1\|\psi_1(\|x\|)\sup_{t\in[0,1]}\Bigg\{\frac{t^q}{\Gamma(q+1)}+|\lambda_1(t)|\Bigg[\frac{|b|}{\Gamma(q+1)}+|a|\int_0^1\frac{s^q}{\Gamma(q+1)}dH(s)\\
&+\sum_{i=1}^{n}|\alpha_i|\Big(\frac{\eta_i^{q+1}-\xi_i^{q+1}}{\Gamma(q+2)}\Big)\Bigg]+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\frac{\sigma_j^q}{\Gamma(q+1)}\Bigg\}\\
&+\|p_2\|\psi_2(\|x\|)\sup_{t\in[0,1]}\Bigg\{\frac{t^{q+p}}{\Gamma(q+p+1)}+|\lambda_1(t)|\Bigg[\frac{b}{\Gamma(q+p+1)}\\
&+|a|\int_0^1\frac{s^{q+p}}{\Gamma(q+p+1)}dH(s)+\sum_{i=1}^{n}|\alpha_i|\Big(\frac{\eta_i^{q+p+1}-\xi_i^{q+p+1}}{\Gamma(q+p+2)}\Big)\Bigg]\\
&+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\Bigg\}\\
\le\;&\|p\|\psi(\|r\|)\Bigg\{\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\overline{\lambda}_1\Bigg[|b|\Big(\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}\Big)\\
&+|a|\int_0^1\Big(\frac{s^{q+p}}{\Gamma(q+p+1)}+\frac{s^q}{\Gamma(q+1)}\Big)dH(s)\\
&+\sum_{i=1}^{n}|\alpha_i|\Big(\frac{\eta_i^{q+1}-\xi_i^{q+1}}{\Gamma(q+2)}+\frac{\eta_i^{q+p+1}-\xi_i^{q+p+1}}{\Gamma(q+p+2)}\Big)\Bigg]\\
&+\overline{\lambda}_2\sum_{j=1}^{m}|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\Big)\Bigg\}.
\end{aligned}
$$

Consequently, using the notation (12), we have

$$
\|\mathcal{G}x\|\le\|p\|\psi(\|r\|)\Lambda.
$$

Next we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$ and $x\in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&|\mathcal{G}x(\tau_2)-\mathcal{G}x(\tau_1)|\\
\le\;&\Big|\int_0^{\tau_1}\frac{(\tau_1-s)^{q-1}-(\tau_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big|\\
&+\Big|\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big|+\Big|\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds\Big|\\
&+\Big|\int_0^{\tau_1}\frac{(\tau_2-s)^{q+p-1}-(\tau_1-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+|\lambda_1(\tau_2)-\lambda_1(\tau_1)|\Bigg[|b|\int_0^1\left(\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|+\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|\right)ds\\
&+|a|\int_0^1\left(\int_0^s\left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}|g(u,x(u))|+\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|\right)du\right)dH(s)\\
&+\sum_{i=1}^n|\alpha_i|\int_{\xi_i}^{\eta_i}\Big(\int_0^s\Big(\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|+\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}|g(u,x(u))|\Big)du\Big)ds\Bigg]\\
&+|\lambda_2(\tau_2)-\lambda_2(\tau_1)|\sum_{j=1}^m|\beta_j|\int_0^{\sigma_j}\Big(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|\Big)+\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|\Big)ds\\
\le\;& \|p\|\psi(\|x\|)\Bigg\{\frac{|\tau_1^q-\tau_2^q|+2(\tau_2-\tau_1)^q}{\Gamma(q+1)}+\frac{|\tau_1^{q+p}-\tau_2^{q+p}|+2(\tau_2-\tau_1)^{q+p}}{\Gamma(q+p+1)}\\
&+\Big|\frac{\rho_2(\tau_2^q-\tau_1^q)}{\kappa\Gamma(q+1)}\Big|\Bigg[|b|\left(\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}\right)+|a|\int_0^1\left(\frac{s^{q+p}}{\Gamma(q+p+1)}+\frac{s^q}{\Gamma(q+1)}\right)dH(s)\\
&+\sum_{i=1}^n|\alpha_i|\Big(\frac{\eta_i^{q+1}-\xi_i^{q+1}}{\Gamma(q+2)}+\frac{\eta_i^{q+p+1}-\xi_i^{q+p+1}}{\Gamma(q+p+2)}\Big)\Bigg]\\
&+\Big|\frac{\rho_4(\tau_2^q-\tau_1^q)}{\kappa\Gamma(q+1)}\Big|\sum_{j=1}^m|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q)}+\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\Big)\Bigg\}.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_2 - \tau_1 \to 0$. As $\mathcal{G}$ satisfies the above assumptions, therefore it follows by the Arzelá–Ascoli theorem that $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The conclusion of the Leray–Schauder nonlinear alternative (Lemma 2) will apply once we establish the boundedness of the set of all solutions to equations $x = \varepsilon\mathcal{G}x$, for $\varepsilon \in (0,1)$. Let x be a solution of (1) and (2). Then, following the computation used in proving the boundedness of $\mathcal{G}$, we get

$$
\begin{aligned}
|x(t)| \;\le\;& \|p\|\psi(\|x\|)\Bigg\{\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\overline{\lambda}_1\Bigg[|b|\left(\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}\right)\\
&+|a|\int_0^1\left(\frac{s^{q+p}}{\Gamma(q+p+1)}+\frac{s^q}{\Gamma(q+1)}\right)dH(s)\\
&+\sum_{i=1}^n|\alpha_i|\Big(\frac{\eta_i^{q+1}-\xi_i^{q+1}}{\Gamma(q+2)}+\frac{\eta_i^{q+p+1}-\xi_i^{q+p+1}}{\Gamma(q+p+2)}\Big)\Bigg]\\
&+\overline{\lambda}_2\sum_{j=1}^m|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\Big)\Bigg\},
\end{aligned}
$$

which, on taking the norm for $t \in [0,1]$ and using (12), takes the form

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Lambda} \le 1.$$

By the condition (A_2), we can find a positive number M such that $\|x\| \neq M$. Let us define a set $Y = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}$ and note that the operator $\mathcal{G} : \overline{Y} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of Y, there is no $x \in \partial Y$ such that $x = \varepsilon\mathcal{G}(x)$ for some $\varepsilon \in (0,1)$. In consequence, we deduce by the nonlinear alternative of Leray–Schauder type (Lemma 2) that the operator $\mathcal{G}$ has a fixed point $x \in \overline{Y}$ which is a solution of the problem (1) and (2). The proof is completed. □

3.2. Existence Result via Krasnoselskii's Fixed Point Theorem

Lemma 3. *(Krasnoselskii's fixed point theorem [33]). Let $\mathcal{X}$ be a bounded, closed, convex, and nonempty subset of a Banach space $\mathcal{Y}$. Let φ_1, φ_2 be the operators mapping $\mathcal{X}$ into $\mathcal{Y}$, such that (i) $\varphi_1 x_1 + \varphi_2 x_2 \in \mathcal{X}$ whenever $x_1, x_2 \in \mathcal{X}$; (ii) φ_1 is compact and continuous; (iii) φ_2 is a contraction mapping. Then there exists $x_3 \in \mathcal{X}$ such that $x_3 = \varphi_1 x_3 + \varphi_2 x_3$.*

Theorem 2. *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the conditions:*

(A_3) *$|f(t,x) - f(t,y)| \le L_1|x-y|$, and $|g(t,x) - g(t,y)| \le L_2|x-y|$ for all $t \in [0,1]$, $L > 0$, $x, y \in \mathbb{R}$, with $L < 1/\Lambda_1$, where Λ_1 is given by (13), and $L = \max\{L_1, L_2\}$.*

(A_4) *$|f(t,x)| \le \mu_1(t)$, $|g(t,x)| \le \mu_2(t)$, for all $(t,x) \in [0,1] \times \mathbb{R}$, $\mu_1, \mu_2 \in C([0,1], \mathbb{R}^+)$ and $\mu = \max\{\mu_1, \mu_2\}$.*

Then the boundary value problem (1) and (2) has at least one solution on $[0,1]$.

Proof. By the assumption (A_4) and (12), we fix $\bar{r} \ge \Lambda\|\mu\|$ and consider the closed ball $B_{\bar{r}} = \{x \in C : \|x\| \le \bar{r}\}$. Next we define operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on $B_{\bar{r}}$ as follows

$$(\mathcal{G}_1 x)(t) = -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds, \ \ t \in [0,1],$$

$$\begin{aligned}
(\mathcal{G}_2 x)(t) \ \ = \ \ & -\lambda_1(t)\Bigg[b\int_0^1 \left(\frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s)) - \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s)) \right) ds \\
& + a\int_0^1 \int_0^s \left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)} g(u,x(u)) - \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u)) \right) du\, dH(s) \\
& + \sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)} g(u,x(u)) - \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u)) \right) du\, ds \Bigg] \\
& + \lambda_2(t) \sum_{j=1}^m \beta_j \Bigg(\int_0^{\sigma_j} \frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& - \int_0^{\sigma_j} \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds \Bigg), \ \ t \in [0,1].
\end{aligned}$$

For $x, y \in B_{\bar{r}}$, we find that

$$\begin{aligned}
\|\mathcal{G}_1 x + \mathcal{G}_2 y\| \ \ = \ \ & \sup_{t \in [0,1]} |\mathcal{G}_1 x + \mathcal{G}_2 y| \\
\le \ \ & \|\mu\| \Bigg\{ \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \bar{\lambda}_1 \Bigg[\frac{|b|}{\Gamma(q+1)} + \frac{|b|}{\Gamma(q+p+1)} \\
& + |a| \int_0^1 \left(\frac{s^q}{\Gamma(q+1)} + \frac{s^{q+p}}{\Gamma(q+p+1)} \right) dH(s) \\
& + \sum_{i=1}^{n-2} |\alpha_i| \left(\frac{(\eta_i^{q+1} - \xi_i^{q+1})}{\Gamma(q+2)} + \frac{(\eta_i^{q+p+1} - \xi_i^{q+p+1})}{\Gamma(q+p+2)} \right) \Bigg] \\
& + \bar{\lambda}_2 \sum_{j=1}^m |\beta_j| \left(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{q+p}}{\Gamma(q+p+1)} \right) \Bigg\} \\
= \ \ & \|\mu\|\Lambda \le \bar{r}.
\end{aligned}$$

This shows that $\mathcal{G}_1 x + \mathcal{G}_2 y \in B_{\bar{r}}$. Next we establish that $\mathcal{G}_2$ is a contraction mapping. For $x, y \in C([0,1], \mathbb{R})$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|\mathcal{G}_2 x - \mathcal{G}_2 y\| \leq\; & \sup_{t\in[0,1]} \Bigg\{ |\lambda_1(t)| \Big[|b| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
& + |b| \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} |g(s,x(s)) - g(s,y(s))| ds \\
& + |a| \int_0^1 \left(\int_0^s \frac{(s-u)^{q+p-1}}{\Gamma(q+p)} |g(u,x(u)) - g(u,y(u))| du \right) dH(s) \\
& + |a| \int_0^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u)) - f(u,y(u))| du \right) dH(s) \\
& + \sum_{i=1}^n |\alpha_i| \int_{\xi_i}^{\eta_i} \Big(\int_0^s \Big(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)} |g(u,x(u)) - g(u,y(u))| \\
& + \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u)) - f(u,y(u))| \Big) du \Big) ds \Big] \\
& + |\lambda_2(t)| \sum_{j=1}^m |\beta_j| \Big(\int_0^{\sigma_j} \frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
& + \int_0^{\sigma_j} \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} |g(s,x(s)) - g(s,y(s))| \Big) ds \Bigg\} \\
\leq\; & L \Bigg\{ \overline{\lambda}_1 \Big[\frac{|b|}{\Gamma(q+1)} + \frac{|b|}{\Gamma(q+p+1)} \\
& + |a| \int_0^1 \left(\frac{s^{q+p}}{\Gamma(q+p+1)} + \frac{s^q}{\Gamma(q+1)} \right) dH(s) \\
& + \sum_{i=1}^n |\alpha_i| \Big(\frac{\eta_i^{q+p+1} - \xi_i^{q+p+1}}{\Gamma(q+p+2)} + \frac{\eta_i^{q+1} - \xi_i^{q+1}}{\Gamma(q+2)} \Big) \Big] \\
& + \overline{\lambda}_2 \sum_{j=1}^m |\beta_j| \Big(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{q+p}}{\Gamma(q+p+1)} \Big) \Bigg\} \|x - y\| \\
\leq\; & L \Lambda_1 \|x - y\|,
\end{aligned}
$$

which is a contraction mapping by assumption $L\Lambda_1 < 1$ (Λ_1 is given by (13)).

Continuity of f, g implies that the operator $\mathcal{G}_1$ is continuous. Also, $\mathcal{G}_1$ is uniformly bounded on $B_{\bar{r}}$ as

$$
\begin{aligned}
\|\mathcal{G}_1 x\| \leq\; & \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} |g(s,x(s))| ds \right\} \\
& \leq \|\mu\| \left(\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} \right).
\end{aligned}
$$

Now we prove the compactness of the operator $\mathcal{G}_1$. In view of (A_3), we define

$$
\sup_{(t,x)\in[0,1]\times B_{\bar{r}}} |f(t,x)| = \overline{f}, \qquad \sup_{(t,x)\in[0,1]\times B_{\bar{r}}} |g(t,x)| = \overline{g}.
$$

Consequently, for $0 \leq t_2 < t_1 \leq 1$, we have

$$
|\mathcal{G}_1 x(t_1) - \mathcal{G}_1 x(t_2)|
$$

$$\leq \frac{\overline{f}}{\Gamma(q+1)}[|t_2^q - t_1^q| + 2(t_1-t_2)^q] + \frac{\overline{g}}{\Gamma(q+p+1)}[|t_2^{q+p} - t_1^{q+p}| + 2(t_1-t_2)^{q+p}] \to 0,$$

as $t_1 - t_2 \to 0$, independent of x. Thus, $\mathcal{G}_1$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá–Ascoli theorem, $\mathcal{G}_1$ is compact on $B_{\bar{r}}$. Thus the hypotheses of Lemma 3 are satisfied. Hence we deduce by the conclusion of Lemma 3 that the problem (1) and (2) has at least one solution on $[0,1]$. □

3.3. Existence and Uniqueness Result

Theorem 3. *Assume that $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the assumption (A_3). Then the problem (1) and (2) has a unique solution on $[0,1]$ if $L\Lambda < 1$, where Λ is given by (12).*

Proof. Define $M = \max\{M_1, M_2\}$, where M_1 and M_2 are positive numbers such that $\sup_{t\in[0,1]} |f(t,0)| = M_1$ and $\sup_{t\in[0,1]} |g(t,0)| = M_2$. Fixing $r \geq \dfrac{M\Lambda}{1-L\Lambda}$, we consider $B_r = \{x \in C : \|x\| \leq r\}$. Then, in view of the assumption (A_3), we have

$$|f(t,x)| = |f(t,x) - f(t,0) + f(t,0)| \leq |f(t,x) - f(t,0)| + |f(t,0)| \leq L_1\|x\| + M_1 \leq L_1 r + M_1.$$

Similarly one can obtain that $|g(t,x)| \leq L_2 r + M_2$. In the first step, we show that $\mathcal{G}B_r \subset B_r$. For any $x \in B_r$, we have

$$\begin{aligned}
\|\mathcal{G}x\| &= \sup_{t\in[0,1]} |\mathcal{G}x(t)| \\
&\leq (Lr+M)\Bigg\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \overline{\lambda}_1\Bigg[\frac{|b|}{\Gamma(q+1)} + \frac{|b|}{\Gamma(q+p+1)} \\
&\quad + |a|\int_0^1 \left(\frac{s^q}{\Gamma(q+1)} + \frac{s^{q+p}}{\Gamma(q+p+1)}\right) dH(s) \\
&\quad + \sum_{i=1}^n |\alpha_i| \left(\frac{(\eta_i^{q+1} - \xi_i^{q+1})}{\Gamma(q+2)} + \frac{(\eta_i^{q+p+1} - \xi_i^{q+p+1})}{\Gamma(q+p+2)}\right)\Bigg] \\
&\quad + \overline{\lambda}_2 \sum_{j=1}^m |\beta_j| \left(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\right)\Bigg\} \\
&= (Lr+M)\Lambda \leq r,
\end{aligned}$$

which implies that $\mathcal{G}B_r \subset B_r$. Next, for $x, y \in C([0,1], \mathbb{R})$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
\|\mathcal{G}x - \mathcal{G}y\| &\leq \sup_{t\in[0,1]} \Bigg\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
&\quad + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} |g(s,x(s)) - g(s,y(s))| ds \\
&\quad + |\lambda_1(t)| \Bigg[|b| \int_0^1 \Big(\frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| \\
&\quad + \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} |g(s,x(s)) - g(s,y(s))|\Big) ds \\
&\quad + |a| \int_0^1 \Big(\int_0^s \frac{(s-u)^{q+p-1}}{\Gamma(q+p)} |g(u,x(u)) - g(u,y(u))| \\
&\quad + \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u)) - f(u,y(u))| du\Big) dH(s)
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{i=1}^{n}|\alpha_i|\int_{\xi_i}^{\eta_i}\Big(\int_0^s\Big(\frac{(s-u)^{q+p-1}}{\Gamma(q+p)}|g(u,x(u))-g(u,y(u))| \\
& +\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|\Big)du\Big)ds\Big] \\
& +|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\Big(\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))-g(s,y(s))| \\
& +\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\Big)\Big\} \\
\leq\ & L\Lambda\|x-y\|.
\end{aligned}
$$

From the above inequality together with the given condition $L\Lambda < 1$ (Λ is given by (12)), it follows that the operator $\mathcal{G}$ is a contraction by means of the contraction mapping principle (Banach fixed point theorem). Therefore, there exists a unique solution for the problem (1) and (2) on $[0,1]$. □

Remark 3. *In Theorem 2, we proved the existence of solutions for the problem (1) and (2) under the assumption that $L\Lambda_1 < 1$ ((A_3)) by applying Krasnoselskii's fixed point theorem, which is a hybrid fixed point theorem combining two well known theorems (algebraic (Banach) and one topological (Schauder)). It gives a fixed point for the sum of two operators; one of them is a contraction while the other one is completely continuous. It is well known that the application of Krasnoselskii fixed point theorem only provides an existence result as only a part of the associated operator is shown to be a contraction. In other words, the entire operator is not a contraction. Indeed, Theorem 3 is an existence–uniqueness result obtained by applying Banach contraction mapping principle under the condition $L\Lambda < 1$. Moreover, $L\Lambda < 1$ implies that $L\Lambda_1 < 1$, where $\Lambda = \Lambda_1 + \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)}$. This means that an increase of $\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)}$ to the value of Λ_1 will lead to the condition $L\Lambda < 1$, ensuring the uniqueness of solutions. This provides the relationship between the contractive conditions imposed in Theorems 2 and 3. On the other hand, interchanging the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the proof of Theorem 2, the difference of the values of Λ and Λ_1 is*

$$
\Lambda - \Lambda_1 = \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)}.
$$

Thus, Theorem 2 provides a precise estimate to extend the contractive condition required for the existence of solutions to the one needed to ensure the uniqueness of solutions in Theorem 3 for the problem at hand.

Example 1. *Consider the following boundary value problem:*

$$
\begin{aligned}
& {}^cD^{1/2}({}^cD^{1/2}x(t) + f(t,x(t))) = g(t,x(t)), \ \ t \in [0,1], \\
& x(0) = \sum_{j=1}^{3}\beta_j x(\sigma_j), \ \ x(1) = \int_0^1 x(s)dH(s) + \sum_{i=1}^{3}\alpha_i\int_{\xi_i}^{\eta_i}x(s)ds,
\end{aligned}
\tag{14}
$$

where $p = q = 1/2, a = 1, b = 1, H(s) = s, \sigma_1 = 1/32, \sigma_2 = 1/26, \sigma_3 = 1/16, \xi_1 = 1/7, \xi_2 = 3/7, \xi_3 = 5/7, \eta_1 = 2/7, \eta_2 = 4/7, \eta_3 = 6/7, \alpha_1 = 1/12, \alpha_2 = 1/6, \alpha_3 = 1/4, \beta_1 = 1/15, \beta_2 = 1/10, \beta_3 = 1/5$, and $f(t,x)$ and $g(t,x)$ will be a fixed later. Using the give values, we find that $\Lambda \approx 16.905854$ and $\Lambda_1 \approx 14.777475$ (Λ and Λ_1 are respectively given by (12) and (13)).

In order to illustrate Theorem 1, we take

$$
f(t,x) = \frac{2e^{-t}}{36\pi}\tan^{-1}x + \frac{1}{t^2+36}, \ \ g(t,x) = \frac{e^{-2t}}{\sqrt{289+t^2}}\left(\frac{|x|}{2(1+|x|)} + \sin x + \frac{1}{2}\right).
\tag{15}
$$

Clearly $|f(t,x)| \leq [e^{-t}/36 + 1/(t^2+36)]$, $|g(t,x)| \leq e^{-2t}(1+\|x\|)/\sqrt{289+t^2}$ with $p_1(t) = e^{-t}/36 + 1/(t^2+36)$ ($\|p_1\| = 1/18$), $\psi_1(\|x\|) = 1$, $p_2(t) = e^{-2t}/\sqrt{289+t^2}$ ($\|p_2\| = 1/17$), $\psi_2(\|x\|) = 1 + \|x\|$,

$p = \max\{1/18, 1/17\} = 1/17$ *and* $\psi = \max\{1, 1 + \|x\|\} = 1 + \|x\|$. *From the assumption* (A_2), *we find that* $M > 179.570603$. *As all the conditions of Theorem 1 are satisfied, there exists at least one solution on* $[0,1]$ *for the problem (14) with* $f(t,x)$ *and* $g(t,x)$ *given by (15).*

Next we explain Theorem 2 by choosing the following functions in the problem (14):

$$f(t,x) = \frac{1}{20}\sin x + e^{-t}\cos t, \;\; g(t,x) = \frac{1}{19}\Big(\frac{|x|}{1+|x|}\Big) + 6t. \tag{16}$$

Notice that $L_1 = 1/20, L_2 = 1/19$ *as*

$$|f(t,x) - f(t,y)| \leq \frac{1}{20}|x-y|, \; |g(t,x) - g(t,y)| \leq \frac{1}{19}|x-y|.$$

Moreover

$$|f(t,x)| \leq \frac{1}{20}|\sin x| + e^{-t}|\cos t| \leq \frac{1}{20} + e^{-t}\cos t = \mu_1(t), |g(t,x)| \leq \frac{1}{19} + 6t = \mu_2(t).$$

Obviously $\|\mu_1\| = 21/20, \|\mu_2\| = 115/19$, $L = \max\{1/20, 1/19\} = 1/19$, $\mu = \max\{21/20, 115/19\} = 115/19$, $L\Lambda_1 \approx 0.777762 < 1$ *and* $L\Lambda \approx 0.889782 < 1$. *Clearly all the assumptions of Theorem 2 are satisfied. Hence, by the conclusions of Theorem 2, we deduce that there exists at least one solution for the problem (14) on* $[0,1]$ *with* $f(t,x(t))$ *and* $g(t,x(t))$ *given by (16).*

Finally one can notice that the problem (14) with $f(t,x(t))$ *and* $g(t,x(t))$ *given by (16) has a unique solution on* $[0,1]$ *as the hypothesis of Theorem 3 holds true.*

Remark 4. *Several new results for the fractional differential equation with mixed nonlinearities (1) subject to different boundary conditions follow as special cases by fixing the parameters in (2). For example, our results correspond to (i) Dirichlet boundary conditions if we take* $\beta_j = 0, \forall j = 1, \ldots, m, a = 0, b \neq 0, \alpha_i = 0, \forall i = 1, \ldots, n$; *(ii) multi-point Riemann-Stieltjes integral boundary conditions if we take* $\alpha_i = 0, \forall i = 1, \ldots, n$; *(iii) multi-point and multi-strip conditions if we take* $a = 0$; *etc.*

4. Analogue Problems

In this section, we discuss variants of the problem (1) and (2). As a first problem we consider

$$\begin{aligned} &{}^cD^{p+q}x(t) + {}^cD^{p}f(t,x(t))) = g(t,x(t)), \;\; 0 < t < 1, \;\; 0 < p, q \leq 1, \\ &x(0) = \sum_{j=1}^{m}\beta_j x(\sigma_j), \; a\int_0^1 x(s)ds - \sum_{i=1}^{n}\alpha_i\int_{\xi_i}^{\eta_i} x(s)ds = \delta, \end{aligned} \tag{17}$$

$0 < \sigma_j < \xi_i < \eta_i < 1, i = 1, 2, \ldots, n, a, \delta \in \mathbb{R}$.

As argued for the problem (1) and (2), we can transform the problem (17) into a fixed point problem with associated operator $\mathcal{W} : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ defined by

$$\begin{aligned} (\mathcal{W}x)(t) \;\; = \;\; & -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds \\ & -\omega_1(t)\Bigg[a\int_0^1\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))du\Big)ds \\ & -a\int_0^1\Big(\int_0^s \frac{(s-u)^{q+p-1}}{\Gamma(q+p)} g(u,x(u))du\Big)ds \\ & -\sum_{i=1}^{n}\alpha_i\int_{\xi_i}^{\eta_i}\Big(\int_0^s\Big(\frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u)) - \frac{(s-u)^{q+p-1}}{\Gamma(q+p)} g(u,x(u))\Big)du\Big)ds + \delta\Bigg] \end{aligned}$$

$$+\omega_2(t)\sum_{j=1}^{m}\beta_j\Big(\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds-\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds\Big),$$

where

$$\omega_1(t)=\frac{1}{\widehat{\kappa}\Gamma(q+1)}\Big(\sum_{j=1}^{m}\beta_j\sigma_j^q-\sum_{j=1}^{m}(\beta_j-1)t^q\Big),$$
$$\omega_2(t)=\frac{1}{\widehat{\kappa}\Gamma(q+2)}\Big(\Big(a-\sum_{i=1}^{n}\alpha_i(\eta_i^{q+1}-\xi_i^{q+1})\Big)-(q+1)(a-\sum_{i=1}^{n}\alpha_i(\eta_i-\xi_i))t^q\Big),$$

and

$$\begin{aligned}\widehat{\kappa} &= \frac{1}{\Gamma(q+2)}\Big[\Big(\sum_{j=1}^{m}\beta_j-1\Big)\Big(a-\sum_{i=1}^{n}\alpha_i(\eta_i^{q+1}-\xi_i^{q+1})\Big)\\ &\quad -(q+1)\sum_{j=1}^{m}\beta_j\sigma_j^q\Big(a-\sum_{i=1}^{n}\alpha_i(\eta_i-\xi_i)\Big)\Big]\neq 0.\end{aligned}$$

In relation to the problem (17), we define

$$\begin{aligned}\widehat{\Lambda} &= \frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\widehat{\omega}_1\Big[\frac{|a|}{\Gamma(q+2)}+\frac{|a|}{\Gamma(q+p+2)}\\ &\quad+\sum_{i=1}^{n-2}|\alpha_i|\Big(\frac{(\eta_i^{q+1}-\xi_i^{q+1})}{\Gamma(q+2)}+\frac{(\eta_i^{q+p+1}-\xi_i^{q+p+1})}{\Gamma(q+p+2)}\Big)+|\delta|\Big]\\ &\quad+\widehat{\omega}_2\sum_{j=1}^{m}|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\Big),\end{aligned}$$

$$\widehat{\Lambda}_1 = \widehat{\Lambda}-\frac{1}{\Gamma(q+1)}-\frac{1}{\Gamma(q+p+1)},$$

where

$$\widehat{\omega}_1=\max_{t\in[0,1]}|\omega_1(t)|=\frac{1}{|\widehat{\kappa}|}\Big[\Big|\sum_{j=1}^{m}\beta_j-1\Big|+\Big|\sum_{j=1}^{m}\beta_j\frac{\sigma_j^q}{\Gamma(q+1)}\Big|\Big],$$
$$\widehat{\omega}_2=\max_{t\in[0,1]}|\omega_2(t)|=\frac{1}{|\widehat{\kappa}|}\Big[\Big|a-\sum_{i=1}^{n}\alpha_i(\eta_i-\xi_i)\Big|+\Big|\frac{1}{\Gamma(q+2)}\Big(a-\sum_{i=1}^{n}\alpha_i(\eta_i^{q+1}-\xi_i^{q+1})\Big)\Big|\Big].$$

The existence and uniqueness results for the problem (17), analogue to the ones for the problem (1) and (2) obtained in Section 3, can be obtained in a similar manner.

As a second variant of the problem (1) and (2), we consider

$$\begin{aligned}&{}^cD^{p+q}x(t)+{}^cD^{p}f(t,x(t)))=g(t,x(t)),\ \ 0<t<1,\ \ 0<p,q\leq 1,\\ &x(0)=\sum_{j=1}^{p}\beta_j x(\sigma_j),\ \ x(1)=\sum_{i=1}^{n-2}\alpha_i\int_{\xi_i}^{\eta_i}x(s)ds,\ 0<\sigma_j<\xi_i<\eta_i<1.\end{aligned}\tag{18}$$

In relation to the problem (18), the fixed point operator $\mathcal{V}: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is defined by

$$
\begin{aligned}
(\mathcal{V}x)(t) = \; & -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds \\
& -\nu_1(t)\Bigg[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds \\
& -\sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} \Big(\int_0^s \Big(\frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u)) - \frac{(s-u)^{q+p-1}}{\Gamma(q+p)} g(u,x(u)) \Big) du \Big) ds \Bigg] \\
& +\nu_2(t)\Bigg[\sum_{j=1}^m \beta_j \Big(\int_0^{\sigma_j} \frac{(\sigma_j-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \int_0^{\sigma_j} \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds \Big) \Bigg],
\end{aligned} \tag{19}
$$

where

$$
\begin{aligned}
\nu_1(t) &= \frac{1}{\Omega\Gamma(q+1)} \Big[\sum_{j=1}^m \beta_j \sigma_j^q - \sum_{j=1}^m (\beta_j - 1) t^q \Big], \\
\nu_2(t) &= \frac{1}{\Omega\Gamma(q+2)} \Big[\Big(q+1 - \sum_{i=1}^n \alpha_i (\eta_i^{q+1} - \xi_i^{q+1}) \Big) - (q+1)(1 - \sum_{i=1}^n \alpha_i(\eta_i - \xi_i)) t^q \Big], \\
\Omega &= \frac{1}{\Gamma(q+2)} \Big[\Big(\sum_{j=1}^m \beta_j - 1 \Big) \Big(q+1 - \sum_{i=1}^n \alpha_i (\eta_i^{q+1} - \xi_i^{q+1}) \Big) \\
&\quad -(q+1) \sum_{j=1}^m \beta_j \sigma_j^q \Big(1 - \sum_{i=1}^n \alpha_i (\eta_i - \xi_i) \Big) \Big] \neq 0.
\end{aligned}
$$

Moreover, we set

$$
\begin{aligned}
\varrho &= \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \overline{\nu}_1 \Big(\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} \\
&\quad + \sum_{i=1}^{n-2} |\alpha_i| \Big(\frac{(\eta_i^{q+1} - \xi_i^{q+1})}{\Gamma(q+2)} + \frac{(\eta_i^{q+p+1} - \xi_i^{q+p+1})}{\Gamma(q+p+2)} \Big) \Big) \\
&\quad + \overline{\nu}_2 \sum_{j=1}^m |\beta_j| \Big(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{q+p}}{\Gamma(q+p+1)} \Big), \quad \overline{\nu}_i = \max_{t\in[0,1]} |\nu_i(t)|, i = 1,2, \\
\overline{\varrho} &= \varrho - \frac{1}{\Gamma(q+1)} - \frac{1}{\Gamma(q+p+1)}.
\end{aligned}
$$

As in Section 3, we can obtain the existence and uniqueness results for the problem (18) with the aid of the operator $\mathcal{V}$ and the parameters-dependent quantities ϱ and $\overline{\varrho}$ (defined above).

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