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On Extended General Mittag–Leffler Functions and Certain Inequalities

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Abstract: In this paper, we introduce and investigate generalized fractional integral operators containing the new generalized Mittag–Leffler function of two variables. We establish several new refinements of Hermite–Hadamard-like inequalities via co-ordinated convex functions.

Keywords: co-ordinated convex function; Hermite–Hadamard inequalities; Mittag–Leffler function

MSC: 26D15; 26A51; 26A33; 33E12

1. Introduction and Preliminaries

The Hermite–Hadamard inequality states that if a function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then

$$\Psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \Psi(x)\,\mathrm{d}x \leq \frac{\Psi(a)+\Psi(b)}{2}, \tag{1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if Ψ is concave.

In recent years, many researchers have turned their attention to the Hermite–Hadamard inequality and have found many variations and generalizations of it via various types of convexity. Some of this research is related to functions that are convex on the coordinates (see, for instance, [1–5] and the references therein).

Coordinated convex functions are defined as:

Definition 1 ([3])**.** *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b, c < d$. A function $\Psi : \Delta \to \mathbb{R}$ will be called convex on the coordinates if the partial mappings $\Psi_y : [a,b] \to \mathbb{R}$, $\Psi_y(u) = \Psi(u,y)$ and $\Psi_x : [c,d] \to \mathbb{R}$, $\Psi_x(v) = \Psi(x,v)$ are convex where defined for all $y \in [c,d]$, and $x \in [a,b]$. Recall mapping $\Psi : \Delta \to \mathbb{R}$ is convex on the coordinates on Δ if the following inequality holds:*

$$\begin{aligned}\Psi\left(tx+(1-t)y, su+(1-s)w\right) &\leq ts\Psi(x,u)+t(1-s)\Psi(x,w)\\ &+s(1-t)\Psi(y,u)+(1-t)(1-s)\Psi(y,w),\end{aligned} \tag{2}$$

for all $(x,u), (y,w) \in \Delta$ and $t, s \in [0,1]$.

Discuss some preliminaries of fractional calculus.

Definition 2. *Let $\Psi \in L[a,b]$, where $a \geq 0$. The Riemann–Liouville integrals $J^{\nu}_{a+}\Psi$ and $J^{\nu}_{b-}\Psi$, of order $\nu > 0$, are defined by*

$$J^{\nu}_{a+}\Psi(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1}\Psi(t)\mathrm{d}t, \text{ for } x > a$$

and

$$J^{\nu}_{b-}\Psi(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1}\Psi(t)\mathrm{d}t, \text{ for } x < b,$$

respectively. Here, $\Gamma(\nu) = \int_0^{\infty} e^{-t}t^{\nu-1}\mathrm{d}t$ is the gamma function. We also make the convention

$$J^{0}_{a+}\Psi(x) = J^{0}_{b-}\Psi(x) = \Psi(x).$$

More details about the Riemann–Liouville fractional integrals may be found in [6].

Salim and Faraj [7] have defined the generalized fractional integral operators containing Mittag–Leffler functions:

Definition 3. *Let μ, ν, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag–Leffler function $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ and $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-}$ for a real-valued continuous function Ψ are defined by:*

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}\Psi\right)(x) = \int_a^x (x-t)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(x-t)^{\mu}\right)\Psi(t)\mathrm{d}t \tag{3}$$

and

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-}\Psi\right)(x) = \int_x^b (t-x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(t-x)^{\mu}\right)\Psi(t)\mathrm{d}t, \tag{4}$$

respectively, where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is a generalized Mittag–Leffler function defined as

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\mu n + \nu)} \frac{t^n}{(\delta)_{ln}},$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)\cdot \ldots \cdot (a+n-1)$, $(a)_0 = 1$.

Remark 1. *If $k = l = 1$ in (3), then the integral operator $\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,1,\omega,a^+}\Psi\right)$ reduces to an integral operator $\left(\varepsilon^{\gamma,\delta,1}_{\mu,\nu,l,\omega,a^+}\Psi\right)$ containing generalized Mittag–Leffler function $E^{\gamma,\delta,1}_{\mu,\nu,1}$ introduced by Srivastava and Tomovski in [8]. Along with $k = l = 1$, if $\delta = 1$, then (3) reduces to an integral operator defined by Prabhaker in [9] containing Mittag–Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega = 0$ in (3), the integral operator $\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}\Psi\right)$ reduces to the Riemann–Liouville fractional integral operator [7]. Note that $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ is absolutely convergent for all $t \in \mathbb{R}$, where $k < l + \mu$. Since $|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \leq \sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn}}{\Gamma(\mu n+\nu)}\frac{t^n}{(\delta)_{ln}}\right|$ with $\sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn}}{\Gamma(\mu n+\nu)}\frac{t^n}{(\delta)_{ln}}\right| = S$, we have $|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \leq S$.*

Inspired by Definition 3, we give the following new definition:

Definition 4. *Let μ, ν, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$, then*

$$\begin{aligned}&\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,c^+}\Psi\right)(x,y)\\ &= \int_a^x \int_c^y (x-t)^{\nu_1-1}(y-s)^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(x-t)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-s)^{\mu_2}\right)\Psi(t,s)\mathrm{d}s\mathrm{d}t,\end{aligned}$$

$x > a, y > c$;

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,d^-}\Psi\right)(x,y)$$
$$= \int_a^x \int_y^d (x-t)^{\nu_1-1}(s-y)^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(x-t)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-y)^{\mu_2}\right)\Psi(t,s)\mathrm{d}s\mathrm{d}t,$$

$x > a, y < d$;

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,c^+}\Psi\right)(x,y)$$
$$= \int_x^b \int_c^y (t-x)^{\nu_1-1}(y-s)^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-s)^{\mu_2}\right)\Psi(t,s)\mathrm{d}s\mathrm{d}t,$$

$x < b, y > d$ *respectively*

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,d^-}\Psi\right)(x,y)$$
$$= \int_x^b \int_y^d (t-x)^{\nu_1-1}(s-y)^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-y)^{\mu_2}\right)\Psi(t,s)\mathrm{d}s\mathrm{d}t,$$

$x < b, y < d$, *where* $\mu = (\mu_1,\mu_2), \nu = (\nu_1,\nu_2), \omega = (\omega_1,\omega_2), \gamma = (\gamma_1,\gamma_2), \delta = (\delta_1,\delta_2), k = (k_1,k_2), \mu,\nu,\omega,\gamma,\delta,k > (0,0)$.

Similar to Definition 4, we introduce the following fractional integrals

Definition 5. *Let* μ,ν,k,l,γ *be positive real numbers and* $\omega \in \mathbb{R}$, *then*

$$\left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1,a^+}\right)\Psi\left(x,\frac{c+d}{2}\right) = \int_a^x (x-t)^{\nu_1-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right)\Psi\left(t,\frac{c+d}{2}\right)\mathrm{d}t,$$
$$\left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1,b^-}\right)\Psi\left(x,\frac{c+d}{2}\right) = \int_x^b (t-x)^{\nu_1-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right)\Psi\left(t,\frac{c+d}{2}\right)\mathrm{d}t,$$
$$\left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2,c^+}\right)\Psi\left(\frac{a+b}{2},y\right) = \int_c^y (y-s)^{\nu_2-1} E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-s)^{\mu_2}\right)\Psi\left(\frac{a+b}{2},s\right)\mathrm{d}s,$$
$$\left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2,d^-}\right)\Psi\left(\frac{a+b}{2},y\right) = \int_y^d (s-y)^{\nu_2-1} E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-y)^{\mu_2}\right)\Psi\left(\frac{a+b}{2},s\right)\mathrm{d}s.$$

Definition 6. *A function* $t\mathcal{G} : \Delta \to \mathbb{R}$ *is said to be symmetric with respect to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *on the coordinates if*

$$t\mathcal{G}(x,y) = \begin{cases} t\mathcal{G}\left(a+b-x, c+d-y\right) \\ t\mathcal{G}\left(x, c+d-y\right) \\ t\mathcal{G}\left(a+b-x, y\right) \end{cases}$$

holds for all $x \in [a,b]$ *and* $y \in [c,d]$.

Lemma 1. *Let* $p \in \mathbb{R}\setminus\{0\}$, *and* $t\mathcal{G} : [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be integrable and* p*-symmetric with respect to* $\frac{a^p+b^p}{2}$, *then*

(i) If $p > 0$,

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} t\mathcal{G}\circ h\right)(b^p) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} t\mathcal{G}\circ h\right)(a^p)$$
$$= \frac{1}{2}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} t\mathcal{G}\circ h\right)(b^p) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} t\mathcal{G}\circ h\right)(a^p)\right],$$

with $h(x) = x^{1/p}$, $x \in [a^p, b^p]$.

(ii) *If* $p < 0$,

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} t\mathcal{G} \circ h\right)(a^p) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} t\mathcal{G} \circ h\right)(b^p)$$
$$= \frac{1}{2}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} t\mathcal{G} \circ h\right)(a^p) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} t\mathcal{G} \circ h\right)(b^p)\right],$$

with $h(x) = x^{1/p}$, $x \in [b^p, a^p]$.

For the applications of and related results containing Mittag–Leffler functions, see [10,11].

For details on Hermite–Hadamard-type inequalities involving fractional integrals via different classes of convex functions, see Kunt et al. [12], Mihai [13,14], Mihai and Mitroi [15], Nisan et al. [16], Noor et al. [17], Sarikaya and Yildirim [18], and others.

2. Main Results

Now we are in a position to present our main results.

Lemma 2. *If the function* $t\mathcal{G} : \Delta \to \mathbb{R}$ *is non-negative, integrable, and symmetric with respect to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *on the coordinates, then the following equalities hold:*

$$\begin{aligned}&\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,d^-} t\mathcal{G}\right)(a,c) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,c^+} t\mathcal{G}\right)(a,d) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,d^-} t\mathcal{G}\right)(b,c) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,c^+} t\mathcal{G}\right)(b,d)\\ &= \tfrac{1}{4}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,d^-} t\mathcal{G}\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,c^+} t\mathcal{G}\right)(a,d) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,d^-} t\mathcal{G}\right)(b,c)\right.\\ &\left.+ \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,c^+} t\mathcal{G}\right)(b,d)\right].\end{aligned} \tag{5}$$

Proof. Using $t\mathcal{G}$ symmetry with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on the coordinates and substitutions $x = a + b - t$, $y = c + d - s$, we have

$$\begin{aligned}&\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,c^+} t\mathcal{G}\right)(b,d)\\ &= \int_a^b \int_c^d (b-t)^{\nu_1-1}(d-s)^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(b-t)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(d-s)^{\mu_2}\right) t\mathcal{G}(t,s)\mathrm{d}s\mathrm{d}t\\ &= \int_a^b \int_c^d (b-t)^{\nu_1-1}(d-s)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(b-t)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(d-s)^{\mu_2}\right)\\ &\times t\mathcal{G}(a+b-t, c+d-s)\mathrm{d}s\mathrm{d}t\\ &= \int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(x-a)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-c)^{\mu_2}\right) t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x\\ &= \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,d^-} t\mathcal{G}\right)(a,c).\end{aligned} \tag{6}$$

$$\begin{aligned}&\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,d^-} t\mathcal{G}\right)(b,c)\\ &= \int_a^b \int_c^d (b-t)^{\nu_1-1}(s-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(b-t)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-c)^{\mu_2}\right) t\mathcal{G}(t,s)\mathrm{d}s\mathrm{d}t\\ &= \int_a^b \int_c^d (b-t)^{\nu_1-1}(s-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(b-t)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-c)^{\mu_2}\right)\\ &\times t\mathcal{G}(a+b-t, c+d-s)\mathrm{d}s\mathrm{d}t\\ &= \int_a^b \int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(x-a)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(d-y)^{\mu_2}\right) t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x\\ &= \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,c^+} t\mathcal{G}\right)(a,d).\end{aligned} \tag{7}$$

Now, using $t\mathcal{G}$ with respect to $\frac{a+b}{2}$ and substitution $u = a + b - t$, we obtain

$$
\begin{aligned}
&\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,c^+}t\mathcal{G}\right)(b,d)\\
&=\int_a^b\int_c^d(b-t)^{\nu_1-1}(d-s)^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(b-t)^{\mu_1}\right)\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(d-s)^{\mu_2}\right)t\mathcal{G}(t,s)\mathrm{d}s\mathrm{d}t\\
&=\int_a^b\int_c^d(b-t)^{\nu_1-1}(d-s)^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(b-t)^{\mu_1}\right)\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(d-s)^{\mu_2}\right)\\
&\times t\mathcal{G}(a+b-t,s)\mathrm{d}s\mathrm{d}t\\
&=\int_a^b\int_c^d(u-a)^{\nu_1-1}(d-s)^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(u-a)^{\mu_1}\right)\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(d-s)^{\mu_2}\right)t\mathcal{G}(u,s)\mathrm{d}s\mathrm{d}u\\
&=\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,c^+}t\mathcal{G}\right)(a,d).
\end{aligned}
\tag{8}
$$

Combining Equations (6)–(8), we get Equation (5) and the proof is complete. □

Lemma 3. *Let $\Psi:[a,b]\to\mathbb{R}$ be a convex function such that $\Psi\in L[a,b]$, and let $\mu,\nu,l,\gamma,\delta,k>0$, $\omega\in\mathbb{R}$. If the function $t\mathcal{G}:[a,b]\to\mathbb{R}$ is non-negative, integrable, and symmetric with respect to $\frac{a+b}{2}$, then the following inequalities for fractional integrals hold:*

$$
\begin{aligned}
&\Psi\left(\tfrac{a+b}{2}\right)\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-}t\mathcal{G}\right)(a)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+}t\mathcal{G}\right)(b)\right]\le\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-}fg\right)(a)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+}fg\right)(b)\right]\\
&\le\tfrac{\Psi(a)+\Psi(b)}{2}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-}t\mathcal{G}\right)(a)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+}t\mathcal{G}\right)(b)\right],
\end{aligned}
\tag{9}
$$

where $\omega'=\frac{\omega}{(b-a)^\mu}$.

Proof. Since Ψ is a convex function on $[a,b]$, we have for all $t\in[0,1]$

$$
\Psi\left(\frac{a+b}{2}\right)=\Psi\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right)\le\frac{\Psi(ta+(1-t)b)+\Psi(tb+(1-t)a)}{2}. \tag{10}
$$

Multiplying both sides of (10) by $2t^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega t^{\mu}\right)t\mathcal{G}(tb+(1-t)a)$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$
\begin{aligned}
&2\Psi\left(\frac{a+b}{2}\right)\int_0^1 t^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega t^{\mu}\right)t\mathcal{G}(tb+(1-t)a)\mathrm{d}t\\
&\le\int_0^1 t^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega t^{\mu}\right)\Psi(ta+(1-t)b)t\mathcal{G}(tb+(1-t)a)\mathrm{d}t\\
&+\int_0^1 t^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega t^{\mu}\right)\Psi(tb+(1-t)a)t\mathcal{G}(tb+(1-t)a)\mathrm{d}t.
\end{aligned}
$$

Setting $x=tb+(1-t)a$ and $\mathrm{d}x=(b-a)\mathrm{d}t$ gives

$$
\begin{aligned}
&\frac{2\Psi\left(\frac{a+b}{2}\right)}{(b-a)^\nu}\int_a^b(x-a)^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'(x-a)^{\mu}\right)t\mathcal{G}(x)\mathrm{d}x\\
&\le\frac{1}{(b-a)^\nu}\left[\int_a^b(x-a)^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'(x-a)^{\mu}\right)\Psi(a+b-x)t\mathcal{G}(x)\mathrm{d}x\right.\\
&\left.+\int_a^b(x-a)^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'(x-a)^{\mu}\right)\Psi(x)t\mathcal{G}(x)\mathrm{d}x\right]\\
&=\frac{1}{(b-a)^\nu}\left[\int_a^b(b-x)^{\nu-1}\mathcal{E}^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'(b-x)^{\mu}\right)\Psi(x)t\mathcal{G}(a+b-x)\mathrm{d}x\right.
\end{aligned}
$$

$$+\int_a^b (x-a)^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(x-a)^{\mu}\right)(fg)(x)\mathrm{d}x\Bigg]$$
$$=\frac{1}{(b-a)^{\nu}}\left[\int_a^b (b-x)^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(b-x)^{\mu}\right)\Psi(x)t\mathcal{G}(x)\mathrm{d}x\right.$$
$$\left.+\int_a^b (x-a)^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'(x-a)^{\mu}\right)(fg)(x)\mathrm{d}x\right].$$

So

$$2\Psi\left(\frac{a+b}{2}\right)\left(\varepsilon_{\mu,\nu,l,\omega',b^-}^{\gamma,\delta,k}t\mathcal{G}\right)(a)\le\left[\left(\varepsilon_{\mu,\nu,l,\omega',b^-}^{\gamma,\delta,k}fg\right)(a)+\left(\varepsilon_{\mu,\nu,l,\omega',a^+}^{\gamma,\delta,k}fg\right)(b)\right],$$

and using Lemma 1, we have

$$\Psi\left(\frac{a+b}{2}\right)\left[\left(\varepsilon_{\mu,\nu,l,\omega',b^-}^{\gamma,\delta,k}t\mathcal{G}\right)(a)+\left(\varepsilon_{\mu,\nu,l,\omega',a^+}^{\gamma,\delta,k}t\mathcal{G}\right)(b)\right]\le\left[\left(\varepsilon_{\mu,\nu,l,\omega',b^-}^{\gamma,\delta,k}fg\right)(a)+\left(\varepsilon_{\mu,\nu,l,\omega',a^+}^{\gamma,\delta,k}fg\right)(b)\right].$$

The first inequality is proved.

For the proof of the second inequality of (9), we first note that if Ψ is a convex function, then for all $t\in[0,1]$, it yields

$$\Psi(ta+(1-t)b)+\Psi((1-t)a+tb)\le\Psi(a)+\Psi(b). \tag{11}$$

Then, multiplying both sides of (11) by $t^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})\,t\mathcal{G}(tb+(1-t)a)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})\Psi(ta+(1-t)b)t\mathcal{G}(tb+(1-t)a)\mathrm{d}t$$
$$+\int_0^1 t^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})\Psi(tb+(1-t)a)t\mathcal{G}(tb+(1-t)a)\mathrm{d}t$$
$$\le(\Psi(a)+\Psi(b))\int_0^1 t^{\nu-1}\mathcal{E}_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})\,t\mathcal{G}(tb+(1-t)a)\mathrm{d}t.$$

That is,

$$\left[\left(\varepsilon_{\mu,\nu,l,\omega',b^-}^{\gamma,\delta,k}fg\right)(a)+\left(\varepsilon_{\mu,\nu,l,\omega',a^+}^{\gamma,\delta,k}fg\right)(b)\right]$$
$$\le\frac{\Psi(a)+\Psi(b)}{2}\left[\left(\varepsilon_{\mu,\nu,l,\omega',b^-}^{\gamma,\delta,k}t\mathcal{G}\right)(a)+\left(\varepsilon_{\mu,\nu,l,\omega',a^+}^{\gamma,\delta,k}t\mathcal{G}\right)(b)\right].$$

The proof is completed. □

Remark 2. *If in Lemma 3 we put $\omega=(0,0)$, we obtain [19] (Theorem 2.2).*

The next result is the Hermite–Hadamard-type inequality for coordinated convex functions containing the generalized Mittag–Leffler function.

Theorem 1. *Let $\Psi:\Delta\to\mathbb{R}$ be coordinated convex on $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b, c<d$ and $\Psi\in L[\Delta]$. Then, one has the inequalities*

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}1\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}1\right)(a,d)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}1\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}1\right)(b,d)\end{array}\right]\\
&\le\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}\Psi\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}\Psi\right)(a,d)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}\Psi\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}\Psi\right)(b,d)\end{array}\right]\\
&\le\tfrac{\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d)}{4}\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}1\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}1\right)(a,d)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}1\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}1\right)(b,d)\end{array}\right],
\end{aligned}\tag{12}$$

where $\mu=(\mu_1,\mu_2),\nu=(\nu_1,\nu_2),\omega'=(\omega_1',\omega_2'),\gamma=(\gamma_1,\gamma_2),\delta=(\delta_1,\delta_2),k=(k_1,k_2)$, $\mu,\nu,\omega',\gamma,\delta,k>(0,0)$ *with* $\omega_1'=\frac{\omega_1}{(b-a)^{\mu_1}},\omega_2'=\frac{\omega_2}{(d-c)^{\mu_2}}$.

Proof. Using (2) with $x=ta+(1-t)b, y=(1-t)a+tb$, $u=sc+(1-s)d, w=(1-s)c+td$ and $t=s=\frac{1}{2}$, we find that

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)\\
&\le[\Psi(ta+(1-t)b,sc+(1-s)d)+\Psi(ta+(1-t)b,(1-s)c+sd)\\
&+\Psi((1-t)a+tb,sc+(1-s)d)+\Psi((1-t)a+tb,(1-s)c+s)d)].
\end{aligned}\tag{13}$$

Thus, multiplying both sides of (13) by $t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})$ and by integrating with respect to (t,s) on $[0,1]\times[0,1]$, we obtain

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\mathrm{d}s\mathrm{d}t\\
&\le\frac{1}{4}\left[\begin{array}{l}\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\Psi(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\
+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\Psi(ta+(1-t)b,(1-s)c+sd)\mathrm{d}s\mathrm{d}t\\
+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\Psi((1-t)a+tb,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\
+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\Psi((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t\end{array}\right].
\end{aligned}\tag{14}$$

Using substitutions $u+ta+(1-t)b, v=sc+(1-s)d$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\mathrm{d}s\mathrm{d}t\\
&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\\
&\times\int_a^b\int_c^d(b-u)^{\nu_1-1}(d-v)^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(b-u)^{\mu_1}\right)\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(d-v)^{\mu_2}\right)\mathrm{d}v\mathrm{d}u\\
&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}1\right)(b,d),
\end{aligned}\tag{15}$$

with $\omega_1'=\frac{\omega_1}{(b-a)^{\nu_1-1}},\omega_2'=\frac{\omega_2}{(d-c)^{\nu_2-1}}$, and

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2s^{\mu_2})\,\Psi(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\
&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\\
&\times\int_a^b\int_c^d(b-u)^{\nu_1-1}(d-v)^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(b-u)^{\mu_1}\right)\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(d-v)^{\mu_2}\right)\Psi(u,v)\mathrm{d}v\mathrm{d}u\\
&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}\Psi\right)(b,d).
\end{aligned}\tag{16}$$

Analogously, we obtain

$$\begin{aligned}&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi(ta+(1-t)b,(1-s)c+sd)\mathrm{d}s\mathrm{d}t\\&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}\Psi\right)(b,c),\end{aligned}\tag{17}$$

$$\begin{aligned}&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi((1-t)a+tb,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}\Psi\right)(a,d),\end{aligned}\tag{18}$$

$$\begin{aligned}&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t\\&=\frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}\Psi\right)(a,c).\end{aligned}\tag{19}$$

Using Lemma 2, introducing relationships (15)–(19) in (14) and after multiplying with $(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}$, we get

$$\begin{aligned}&\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}1\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}1\right)(a,d)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}1\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}1\right)(b,d)\end{array}\right]\\&\le\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}\Psi\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}\Psi\right)(a,d)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}\Psi\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}\Psi\right)(b,d)\end{array}\right],\end{aligned}\tag{20}$$

by which the first inequality of (12) is proved.

For the proof of the second inequality in (12) using (2), we have

$$\begin{aligned}&\Psi(ta+(1-t)b,sc+(1-s)d)+\Psi(ta+(1-t)b,(1-s)c+sd)\\&+\Psi((1-t)a+tb,sc+(1-s)d)+\Psi((1-t)a+tb,(1-s)c+sd)\\&\le\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d).\end{aligned}\tag{21}$$

Then, multiplying both sides of (21) by $t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})$ and integrating with respect to (t,s) on $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi(ta+(1-t)b,(1-s)c+sd)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi((1-t)a+tb,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t\\&\le\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\left[\begin{array}{c}\Psi(a,c)+\Psi(b,c)\\+\Psi(a,d)+\Psi(b,d)\end{array}\right]\mathrm{d}s\mathrm{d}t.\end{aligned}$$

So, after multiplying with $(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}$ and using Lemma 2, we have

$$\begin{aligned}&\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}\Psi\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}\Psi\right)(a,d)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}\Psi\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}\Psi\right)(b,d)\end{array}\right]\\&\le\frac{\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d)}{4}\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-}1\right)(a,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+}1\right)(a,d)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-}1\right)(b,c)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+}1\right)(b,d)\end{array}\right],\end{aligned}$$

which finishes the proof. □

The following theorem establishes Hermite–Hadamard–Fejér-type inequalities for coordinated convex functions containing the generalized Mittag–Leffler function.

Theorem 2. *Let $\Psi : \Delta \to \mathbb{R}$ be coordinated convex on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b, c < d$ and $\Psi \in L[\Delta]$. If $t\mathcal{G} : \Delta \to \mathbb{R}$ is non-negative, integrable, and symmetric with respect to $\frac{a+b}{2}, \frac{c+d}{2}$ on the coordinates, then the following integral inequalities hold:*

$$\begin{aligned}
&\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\left[\begin{array}{c} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} t\mathcal{G}\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} t\mathcal{G}\right)(a,d) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} t\mathcal{G}\right)(b,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} t\mathcal{G}\right)(b,d) \end{array}\right] \\
&\leq \left[\begin{array}{c} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} fg\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} fg\right)(a,d) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} fg\right)(b,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} fg\right)(b,d) \end{array}\right] \\
&\leq \frac{\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d)}{4}\left[\begin{array}{c} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} t\mathcal{G}\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} t\mathcal{G}\right)(a,d) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} t\mathcal{G}\right)(b,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} t\mathcal{G}\right)(b,d) \end{array}\right],
\end{aligned} \tag{22}$$

where $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2), \omega' = (\omega'_1, \omega'_2), \gamma = (\gamma_1, \gamma_2), \delta = (\delta_1, \delta_2), k = (k_1, k_2)$, $\mu, \nu, \omega', \gamma, \delta, k > (0,0)$ with $\omega'_1 = \frac{\omega_1}{(b-a)^{\mu_1}}, \omega'_2 = \frac{\omega_2}{(d-c)^{\mu_2}}$.

Proof. Since Ψ is a convex function on Δ, then for all $t, s \in [0,1] \times [0,1]$, we can write

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right) = \Psi\left(\tfrac{ta+(1-t)b+(1-t)a+tb}{2}, \tfrac{sc+(1-s)d+(1-s)c+sd}{2}\right) \\
&\leq \tfrac{1}{4}\left[\begin{array}{c} \Psi(ta+(1-t)b, sc+(1-s)d) + \Psi(ta+(1-t)b, (1-s)c+sd) \\ +\Psi((1-t)a+tb, sc+(1-s)d) + \Psi((1-t)a+tb, (1-s)c+sd) \end{array}\right].
\end{aligned} \tag{23}$$

Then, multiplying both sides of (23) by $t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,t\mathcal{G}((1-t)a+tb,$ $(1-s)c+sd)$ and integrating with respect to (t,s) on $[0,1] \times [0,1]$, we obtain

$$\begin{aligned}
&\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\times \int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{4}\left[\begin{array}{c}
\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi(ta+(1-t)b, sc+(1-s)d) \\
\times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi(ta+(1-t)b, (1-s)c+sd) \\
\times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi((1-t)a+tb, sc+(1-s)d) \\
\times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,\Psi((1-t)a+tb, (1-s)c+sd) \\
\times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t
\end{array}\right].
\end{aligned} \tag{24}$$

Setting $x = (1-t)a+tb$ and $y = (1-s)c+sd$, we obtain

$$\begin{aligned}
I_1 &= \int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} t\mathcal{G}\right)(a,c)
\end{aligned} \tag{25}$$

and

$$
\begin{aligned}
I_2 = & \int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \Psi(ta+(1-t)b, sc+(1-s)d) \\
& t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
& \times \int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(x-a)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(y-c)^{\mu_2}\right) \\
& \times \Psi(a+b-x, c+d-y) t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

Now, using substitutions $u = a+b-x$, $v = c+d-y$ and the symmetry of function $t\mathcal{G}$, we have

$$
\begin{aligned}
I_2 = & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
& \times \int_a^b \int_c^d (b-u)^{\nu_1-1}(d-v)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(b-u)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(d-v)^{\mu_2}\right) \\
& \times \Psi(u,v) t\mathcal{G}(a+b-u, c+d-v)\mathrm{d}v\mathrm{d}u \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
& \times \int_a^b \int_c^d (b-u)^{\nu_1-1}(d-v)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(b-u)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(d-v)^{\mu_2}\right) \\
& \times \Psi(u,v) t\mathcal{G}(u,v)\mathrm{d}v\mathrm{d}u \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} fg\right)(b,d).
\end{aligned} \tag{26}
$$

Analogously, we get

$$
\begin{aligned}
I_3 = & \int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \Psi(ta+(1-t)b, (1-s)c+sd) \\
& \times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
& \times \int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(x-a)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(y-c)^{\mu_2}\right) \\
& \times \Psi(a+b-x, y) t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
& \times \int_a^b \int_c^d (b-u)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(b-u)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(y-c)^{\mu_2}\right) \\
& \times \Psi(u,y) t\mathcal{G}(a+b-u, y)\mathrm{d}y\mathrm{d}u \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
& \times \int_a^b \int_c^d (b-u)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'(b-u)^{\mu_1}\right) \mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'(y-c)^{\mu_2}\right) \\
& \times \Psi(u,y) t\mathcal{G}(u,y)\mathrm{d}y\mathrm{d}u \\
= & \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} fg\right)(b,c),
\end{aligned} \tag{27}
$$

$$\begin{aligned}
I_4 &= \int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} \mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1 t^{\mu_1}\right) \mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2 s^{\mu_2}\right) \Psi((1-t)a+tb, sc+(1-s)d) \\
&\times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
&\times \int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1'(x-a)^{\mu_1}\right) \mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2'(y-c)^{\mu_2}\right) \\
&\times \Psi(x,c+d-y)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
&\times \int_a^b \int_c^d (x-a)^{\nu_1-1}(d-v)^{\nu_2-1} \mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1'(x-a)^{\mu_1}\right) \mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2'(d-v)^{\mu_2}\right) \\
&\times \Psi(x,v)t\mathcal{G}(x,c+d-v)\mathrm{d}v\mathrm{d}x \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
&\times \int_a^b \int_c^d (x-a)^{\nu_1-1}(d-v)^{\nu_2-1} \mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1'(x-a)^{\mu_1}\right) \mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2'(d-v)^{\mu_2}\right) \\
&\times \Psi(x,v)t\mathcal{G}(x,v)\mathrm{d}v\mathrm{d}x \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \left(\varepsilon_{\mu,\nu,l,\omega',b^-,c^+}^{\gamma,\delta,k} fg\right)(a,d),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
I_5 &= \int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} \mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1 t^{\mu_1}\right) \mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2 s^{\mu_2}\right) \Psi((1-t)a+tb,(1-s)c+sd) \\
&\times t\mathcal{G}((1-t)a+tb,(1-s)c+sd)\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \\
&\times \int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1} \mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1'(x-a)^{\mu_1}\right) \mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2'(y-c)^{\mu_2}\right) \\
&\times \Psi(x,y)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}} \left(\varepsilon_{\mu,\nu,l,\omega',b^-,d^-}^{\gamma,\delta,k} fg\right)(a,c).
\end{aligned} \tag{29}$$

Introducing (25)–(29) in (24), multiplying the inequality with $(b-a)^{\nu_1-1}(d-c)^{\nu_2-1}$, and using Lemma 2, we have the first inequality of (22).

We shall prove the second inequality of (22). Since Ψ is a convex function on Δ, for all $(t,s) \in [0,1] \times [0,1]$, it yields

$$\begin{aligned}
&\Psi\left(ta+(1-t)b, sc+(1-s)d\right) + \Psi\left(ta+(1-t)b,(1-s)c+sd\right) \\
&+\Psi\left((1-t)a+tb, sc+(1-s)d\right) + \Psi\left((1-t)a+tb,(1-s)c+sd\right) \\
&\le \Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d).
\end{aligned} \tag{30}$$

Multiplying both sides of (30) by $t^{\nu_1-1}s^{\nu_2-1}\mathcal{E}_{\mu_1,\nu_1,l_1}^{\gamma_1,\delta_1,k_1}\left(\omega_1 t^{\mu_1}\right)\mathcal{E}_{\mu_2,\nu_2,l_2}^{\gamma_2,\delta_2,k_2}\left(\omega_2 s^{\mu_2}\right) t\mathcal{G}((1-t)a+tb,(1-s)c+sd)$ and integrating with respect to (t,s) on $[0,1] \times [0,1]$, we obtain

$$I_2+I_3+I_4+I_5 \le (\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d))I_1.$$

That is, using (25)–(29),

$$\begin{aligned}
&\frac{1}{4}\left[\begin{array}{c} \left(\varepsilon_{\mu,\nu,l,\omega',b^-,d^-}^{\gamma,\delta,k} fg\right)(a,c) + \left(\varepsilon_{\mu,\nu,l,\omega',b^-,c^+}^{\gamma,\delta,k} fg\right)(a,d) \\ + \left(\varepsilon_{\mu,\nu,l,\omega',a^+,d^-}^{\gamma,\delta,k} fg\right)(b,c) + \left(\varepsilon_{\mu,\nu,l,\omega',a^+,c^+}^{\gamma,\delta,k} fg\right)(b,d) \end{array}\right] \\
&\le \frac{\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d)}{4}\left(\varepsilon_{\mu,\nu,l,\omega',b^-,d^-}^{\gamma,\delta,k} t\mathcal{G}\right)(a,c).
\end{aligned}$$

Finally, by using Lemma 2 we get the second part of inequality (22), and the proof is complete. □

Remark 3. *If we take $\omega = (0,0)$ in Theorem 2, we have [20] (Theorem 7).*

Theorem 3. *Let $\Psi : \Delta \to \mathbb{R}$ be coordinated convex on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b, c < d$ and $\Psi \in L[\Delta]$. If $t\mathcal{G} : \Delta \to \mathbb{R}$ is non-negative, integrable, and symmetric with respect to $\frac{a+b}{2}, \frac{c+d}{2}$ on the coordinates, then the following integral inequalities hold:*

$$
\begin{aligned}
&\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \left[\begin{array}{c} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} t\mathcal{G}\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} t\mathcal{G}\right)(a,d) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} t\mathcal{G}\right)(b,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} t\mathcal{G}\right)(b,d) \end{array} \right] \\
&\leq \frac{1}{2} \left[\begin{array}{l}
\left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) \left(\Psi\left(b, \frac{c+d}{2}\right) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) t\mathcal{G}(b,d)\right) \\
+ \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) \left(\Psi\left(b, \frac{c+d}{2}\right) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) t\mathcal{G}(b,c)\right) \\
+ \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) \left(\Psi\left(a, \frac{c+d}{2}\right) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) t\mathcal{G}(a,d)\right) \\
+ \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) \left(\Psi\left(a, \frac{c+d}{2}\right) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) t\mathcal{G}(a,c)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) \left(\Psi\left(\frac{a+b}{2}, d\right) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) t\mathcal{G}(b,d)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) \left(\Psi\left(\frac{a+b}{2}, d\right) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) t\mathcal{G}(a,d)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) \left(\Psi\left(\frac{a+b}{2}, c\right) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) t\mathcal{G}(b,c)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) \left(\Psi\left(\frac{a+b}{2}, c\right) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) t\mathcal{G}(a,c)\right)
\end{array} \right] \\
&\leq \left[\begin{array}{c} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} fg\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} fg\right)(a,d) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} fg\right)(b,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} fg\right)(b,d) \end{array} \right] \qquad (31) \\
&\leq \frac{1}{2} \left[\begin{array}{l}
\left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) \left(\Psi(b,c) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) t\mathcal{G}(b,d)\right) \\
+ \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) \left(\Psi(b,d) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) t\mathcal{G}(b,c)\right) \\
+ \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) \left(\Psi(a,c) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) t\mathcal{G}(a,d)\right) \\
+ \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) \left(\Psi(a,d) \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) t\mathcal{G}(a,c)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) \left(\Psi(a,d) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) t\mathcal{G}(b,d)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+}\right) \left(\Psi(b,d) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) t\mathcal{G}(a,d)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) \left(\Psi(a,c) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+}\right) t\mathcal{G}(b,c)\right) \\
+ \left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-}\right) \left(\Psi(b,c) \left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-}\right) t\mathcal{G}(a,c)\right)
\end{array} \right] \\
&\leq \frac{\Psi(a,c) + \Psi(b,c) + \Psi(a,d) + \Psi(b,d)}{4} \left[\begin{array}{c} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} t\mathcal{G}\right)(a,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} t\mathcal{G}\right)(a,d) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} t\mathcal{G}\right)(b,c) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} t\mathcal{G}\right)(b,d) \end{array} \right].
\end{aligned}
$$

Proof. For an easier proof, we will use the following notations:
$\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',a^+} = \varepsilon_{a^+}$, $\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1',b^-} = \varepsilon_{b^-}$, $\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',c^+} = \varepsilon_{c^+}$, $\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2',d^-} = \varepsilon_{d^-}$,
$\mathcal{E}^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1} = \mathcal{E}_1$, $\mathcal{E}^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2} = \mathcal{E}_2$, $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,d^-} = \varepsilon_{b^-,d^-}$, $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',b^-,c^+} = \varepsilon_{b^-,c^+}$, $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,d^-} = \varepsilon_{a^+,d^-}$, and $\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',a^+,c^+} = \varepsilon_{a^+,c^+}$.

Since $\Psi : \Delta \to \mathbb{R}$ is convex on the coordinates, it follows that the mapping $\Psi_x : [c,d] \to \mathbb{R}, \Psi_x(y) = \Psi(x,y)$ is convex on $[c,d]$ and $t\mathcal{G}_x : [c,d] \to \mathbb{R}, t\mathcal{G}_x = t\mathcal{G}(x,y)$ is non-negative, integrable, and symmetric with respect to $\frac{c+d}{2}$, for all $x \in [a,b]$. Then, thanks to inequalities (9) we have

$$\begin{aligned}
&\Psi_x\left(\frac{c+d}{2}\right)\left[\left(\varepsilon_{c^+}t\mathcal{G}_x\right)(d) + \left(\varepsilon_{d^-}t\mathcal{G}_x\right)(c)\right] \le \left[\left(\varepsilon_{c^+}\Psi_x t\mathcal{G}_x\right)(d) + \left(\varepsilon_{d^-}\Psi_x t\mathcal{G}_x\right)(c)\right] \\
&\le \frac{\Psi_x(c)+\Psi_x(d)}{2}\left[\left(\varepsilon_{c^+}t\mathcal{G}_x\right)(d) + \left(\varepsilon_{d^-}t\mathcal{G}_x\right)(c)\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&\Psi\left(x, \frac{c+d}{2}\right)\left[\begin{array}{c} \int_c^d (d-y)^{\nu_2-1}\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})t\mathcal{G}(x,y)\mathrm{d}y \\ +\int_c^d (y-c)^{\nu_2-1}\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})t\mathcal{G}(x,y)\mathrm{d}y \end{array}\right] \\
&\le \left[\begin{array}{c} \int_c^d (d-y)^{\nu_2-1}\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(x,y)t\mathcal{G}(x,y)\mathrm{d}y \\ +\int_c^d (y-c)^{\nu_2-1}\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(x,y)t\mathcal{G}(x,y)\mathrm{d}y \end{array}\right] \qquad (32) \\
&\le \frac{\Psi(x,c)+\Psi(x,d)}{2}\left[\begin{array}{c} \int_c^d (d-y)^{\nu_2-1}\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})t\mathcal{G}(x,y)\mathrm{d}y \\ +\int_c^d (y-c)^{\nu_2-1}\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})t\mathcal{G}(x,y)\mathrm{d}y \end{array}\right].
\end{aligned}$$

Multiplying both sides of (32) by $(b-x)^{\nu_1-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})$ and $(x-a)^{\nu_1-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})$, and integrating with respect to x over $[a,b]$, respectively, we have

$$\begin{aligned}
&\int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi\left(x,\frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&+\int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})E_2(\omega_2'(y-c)^{\mu_2})\Psi\left(x,\frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&\le \int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(x,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \qquad (33) \\
&+\int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(x,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&\le \frac{1}{2}\left[\begin{array}{c} \int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(x,c)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(x,d)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right],
\end{aligned}$$

$$\begin{aligned}
&\int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi\left(x,\frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&+\int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi\left(x,\frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&\le \int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(x,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \qquad (34) \\
&+\int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(x,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&\le \frac{1}{2}\left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(x,c)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(x,d)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right].
\end{aligned}$$

For the mappings $\Psi_y : [a,b] \to \mathbb{R}, \Psi_y = \Psi(x,y)$ and $t\mathcal{G}_y : [a,b] \to \mathbb{R}, t\mathcal{G}_y = t\mathcal{G}(x,y)$, we use the same arguments as before. So we can state that

$$\begin{aligned}
&\int_a^b \int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi\left(\frac{a+b}{2},y\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&+\int_a^b \int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi\left(\frac{a+b}{2},y\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \qquad (35)\\
&\leq \frac{1}{2}\left[\begin{array}{c} \int_a^b \int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(a,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b \int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})\Psi(b,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi\left(\frac{a+b}{2},y\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\
&+\int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi\left(\frac{a+b}{2},y\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \qquad (36)\\
&\leq \frac{1}{2}\left[\begin{array}{c} \int_a^b \int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(a,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b \int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})\Psi(b,y)\,t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right].
\end{aligned}$$

Adding the inequalities (33)–(36), we can write

$$\begin{aligned}
&\varepsilon_{a^+}\left[\Psi\left(b,\frac{c+d}{2}\right)\varepsilon_{c^+}t\mathcal{G}(b,d)\right]+\varepsilon_{a^+}\left[\Psi\left(b,\frac{c+d}{2}\right)\varepsilon_{d^-}t\mathcal{G}(b,c)\right] \\
&+\varepsilon_{b^-}\left[\Psi\left(a,\frac{c+d}{2}\right)\varepsilon_{c^+}t\mathcal{G}(a,d)\right]+\varepsilon_{b^-}\left[\Psi\left(a,\frac{c+d}{2}\right)\varepsilon_{d^-}t\mathcal{G}(a,c)\right] \\
&+\varepsilon_{c^+}\left[\Psi\left(\frac{a+b}{2},d\right)\varepsilon_{a^+}t\mathcal{G}(b,d)\right]+\varepsilon_{c^+}\left[\Psi\left(\frac{a+b}{2},d\right)\varepsilon_{b^-}t\mathcal{G}(a,d)\right] \\
&+\varepsilon_{d^-}\left[\Psi\left(\frac{a+b}{2},d\right)\varepsilon_{a^+}t\mathcal{G}(b,c)\right]+\varepsilon_{d^-}\left[\Psi\left(\frac{a+b}{2},c\right)\varepsilon_{b^-}t\mathcal{G}(a,c)\right] \\
&\leq 2\left[(\varepsilon_{a^+c^+}fg)(b,d)+(\varepsilon_{a^+d^-}fg)(b,c)+(\varepsilon_{b^-c^+}fg)(a,d)+(\varepsilon_{b^-d^-}fg)(a,c)\right] \\
&\leq \frac{1}{2}\left[\begin{array}{c} \varepsilon_{a^+}[\Psi(b,c)\,\varepsilon_{c^+}t\mathcal{G}(b,d)]+\varepsilon_{a^+}[\Psi(b,d)\,\varepsilon_{d^-}t\mathcal{G}(b,c)] \\ +\varepsilon_{b^-}[\Psi(a,c)\,\varepsilon_{c^+}t\mathcal{G}(a,d)]+\varepsilon_{b^-}[\Psi(a,d)\,\varepsilon_{d^-}t\mathcal{G}(a,c)] \\ +\varepsilon_{c^+}[\Psi(a,d)\,\varepsilon_{a^+}t\mathcal{G}(b,d)]+\varepsilon_{c^+}[\Psi(b,d)\,\varepsilon_{b^-}t\mathcal{G}(a,d)] \\ +\varepsilon_{d^-}[\Psi(a,c)\,\varepsilon_{a^+}t\mathcal{G}(b,c)]+\varepsilon_{d^-}[\Psi(b,c)\,\varepsilon_{b^-}t\mathcal{G}(a,c)] \end{array}\right].
\end{aligned}$$

So,

$$\frac{1}{2}\left\{\begin{array}{c} \varepsilon_{a^+}\left[\Psi\left(b,\frac{c+d}{2}\right)\varepsilon_{c^+}t\mathcal{G}(b,d)\right]+\varepsilon_{a^+}\left[\Psi\left(b,\frac{c+d}{2}\right)\varepsilon_{d^-}t\mathcal{G}(b,c)\right] \\ +\varepsilon_{b^-}\left[\Psi\left(a,\frac{c+d}{2}\right)\varepsilon_{c^+}t\mathcal{G}(a,d)\right]+\varepsilon_{b^-}\left[\Psi\left(a,\frac{c+d}{2}\right)\varepsilon_{d^-}t\mathcal{G}(a,c)\right] \\ +\varepsilon_{c^+}\left[\Psi\left(\frac{a+b}{2},d\right)\varepsilon_{a^+}t\mathcal{G}(b,d)\right]+\varepsilon_{c^+}\left[\Psi\left(\frac{a+b}{2},d\right)\varepsilon_{b^-}t\mathcal{G}(a,d)\right] \\ +\varepsilon_{d^-}\left[\Psi\left(\frac{a+b}{2},d\right)\varepsilon_{a^+}t\mathcal{G}(b,c)\right]+\varepsilon_{d^-}\left[\Psi\left(\frac{a+b}{2},c\right)\varepsilon_{b^-}t\mathcal{G}(a,c)\right] \end{array}\right\}$$

$$\begin{aligned}
&\leq \left[(\varepsilon_{a^+c^+}fg)(b,d)+(\varepsilon_{a^+d^-}fg)(b,c)+(\varepsilon_{b^-c^+}fg)(a,d)+(\varepsilon_{b^-d^-}fg)(a,c)\right] \\
&\leq \frac{1}{4}\left[\begin{array}{c} \varepsilon_{a^+}[\Psi(b,c)\,\varepsilon_{c^+}t\mathcal{G}(b,d)]+\varepsilon_{a^+}[\Psi(b,d)\,\varepsilon_{d^-}t\mathcal{G}(b,c)] \\ +\varepsilon_{b^-}[\Psi(a,c)\,\varepsilon_{c^+}t\mathcal{G}(a,d)]+\varepsilon_{b^-}[\Psi(a,d)\,\varepsilon_{d^-}t\mathcal{G}(a,c)] \\ +\varepsilon_{c^+}[\Psi(a,d)\,\varepsilon_{a^+}t\mathcal{G}(b,d)]+\varepsilon_{c^+}[\Psi(b,d)\,\varepsilon_{b^-}t\mathcal{G}(a,d)] \\ +\varepsilon_{d^-}[\Psi(a,c)\,\varepsilon_{a^+}t\mathcal{G}(b,c)]+\varepsilon_{d^-}[\Psi(b,c)\,\varepsilon_{b^-}t\mathcal{G}(a,c)] \end{array}\right] \\
&\leq \frac{1}{2}\left[\begin{array}{c} \varepsilon_{a^+}[\Psi(b,c)\,\varepsilon_{c^+}t\mathcal{G}(b,d)]+\varepsilon_{a^+}[\Psi(b,d)\,\varepsilon_{d^-}t\mathcal{G}(b,c)] \\ +\varepsilon_{b^-}[\Psi(a,c)\,\varepsilon_{c^+}t\mathcal{G}(a,d)]+\varepsilon_{b^-}[\Psi(a,d)\,\varepsilon_{d^-}t\mathcal{G}(a,c)] \\ +\varepsilon_{c^+}[\Psi(a,d)\,\varepsilon_{a^+}t\mathcal{G}(b,d)]+\varepsilon_{c^+}[\Psi(b,d)\,\varepsilon_{b^-}t\mathcal{G}(a,d)] \\ +\varepsilon_{d^-}[\Psi(a,c)\,\varepsilon_{a^+}t\mathcal{G}(b,c)]+\varepsilon_{d^-}[\Psi(b,c)\,\varepsilon_{b^-}t\mathcal{G}(a,c)] \end{array}\right].
\end{aligned}$$

This gives the second and third inequalities in (31).

Now we will show the first inequality in (31).

Apply Lemma 3 for the functions $F_1 : [a,b] \to \mathbb{R}, F_1(x) = \Psi\left(x, \frac{c+d}{2}\right)$ and $t\mathcal{G}_y : [a,b] \to \mathbb{R}, t\mathcal{G}_y(x) = t\mathcal{G}(x,y)$:

$$F_1\left(\frac{a+b}{2}\right)\left[\left(\varepsilon_{b^-}t\mathcal{G}_y\right)(a) + \left(\varepsilon_{a^+}t\mathcal{G}_y\right)(b)\right] \le \left[\left(\varepsilon_{b^-}F_1 t\mathcal{G}_y\right)(a) + \left(\varepsilon_{a^+}F_1 t\mathcal{G}_y\right)(b)\right].$$

Therefore,

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right)\left[\begin{array}{c} \int_a^b (x-a)^{\nu_1-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})t\mathcal{G}(x,y)\mathrm{d}x \\ +\int_a^b (b-x)^{\nu_1-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})t\mathcal{G}(x,y)\mathrm{d}x \end{array}\right] \\
&\le \left[\begin{array}{c} \int_a^b (x-a)^{\nu_1-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\Psi\left(x, \frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}x \\ +\int_a^b (b-x)^{\nu_1-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\Psi\left(x, \frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}x \end{array}\right].
\end{aligned} \tag{37}$$

Multiplying both sides of (37) by $\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})(d-y)^{\nu_2-1}$ and $\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})(y-c)^{\nu_2-1}$ and integrating with respect to y over $[c,d]$, respectively, we have

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right)\left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right] \\
&\le \left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot\Psi\left(x, \frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot\Psi\left(x, \frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right],
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right)\left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right] \\
&\le \left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot\Psi\left(x, \frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot\Psi\left(x, \frac{c+d}{2}\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right].
\end{aligned} \tag{39}$$

The same way, we apply Lemma 3 for the functions $F_2 : [c,d] \to \mathbb{R}, F_1(x) = \Psi\left(\frac{a+b}{2}, y\right)$ and $t\mathcal{G}_x : [a,b] \to \mathbb{R}, t\mathcal{G}_x(y) = t\mathcal{G}(x,y)$, and we get:

$$\begin{aligned}
&\Psi\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right)\left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right] \\
&\le \left[\begin{array}{c} \int_a^b\int_c^d (x-a)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot\Psi\left(\frac{a+b}{2}, y\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (x-a)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot\Psi\left(\frac{a+b}{2}, y\right)t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right],
\end{aligned} \tag{40}$$

$$\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\left[\begin{array}{c} \int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right]$$
$$\leq \left[\begin{array}{c} \int_a^b\int_c^d (b-x)^{\nu_1-1}(y-c)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(y-c)^{\mu_2}) \\ \cdot\Psi\left(\frac{a+b}{2}, y\right) t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \\ +\int_a^b\int_c^d (b-x)^{\nu_1-1}(d-y)^{\nu_2-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})\mathcal{E}_2(\omega_2'(d-y)^{\mu_2}) \\ \cdot\Psi\left(\frac{a+b}{2}, y\right) t\mathcal{G}(x,y)\mathrm{d}y\mathrm{d}x \end{array}\right]. \tag{41}$$

Adding (38)–(41), we get the first inequality of (31).

For the last inequality of (31), we apply Lemma 3 to the functions

1. $F_3 : [a,b] \to \mathbb{R}, F_3(x) = \Psi(x,c)$ and $t\mathcal{G}_y : [a,b] \to \mathbb{R}, t\mathcal{G}_y(x) = t\mathcal{G}(x,y)$,
2. $F_4 : [a,b] \to \mathbb{R}, F_4(x) = \Psi(x,d)$ and $t\mathcal{G}_y : [a,b] \to \mathbb{R}, t\mathcal{G}_y(x) = t\mathcal{G}(x,y)$,
3. $F_5 : [c,d] \to \mathbb{R}, F_5(x) = \Psi(a,y)$ and $t\mathcal{G}_x : [c,d] \to \mathbb{R}, t\mathcal{G}_x(y) = t\mathcal{G}(x,y)$
4. $F_6 : [c,d] \to \mathbb{R}, F_6(x) = \Psi(b,y)$ and $t\mathcal{G}_x : [c,d] \to \mathbb{R}, t\mathcal{G}_x(y) = t\mathcal{G}(x,y)$,

and we obtain

$$\left(\varepsilon_{b^-}F_3t\mathcal{G}_y\right)(a) + \left(\varepsilon_{a^+}F_3t\mathcal{G}_y\right)(b) \leq \frac{F_3(a)+F_3(b)}{2}\left[\left(\varepsilon_{b^-}t\mathcal{G}_y\right)(a) + \left(\varepsilon_{a^+}t\mathcal{G}_y\right)(b)\right], \tag{42}$$

$$\left(\varepsilon_{b^-}F_4t\mathcal{G}_y\right)(a) + \left(\varepsilon_{a^+}F_4t\mathcal{G}_y\right)(b) \leq \frac{F_4(a)+F_4(b)}{2}\left[\left(\varepsilon_{b^-}t\mathcal{G}_y\right)(a) + \left(\varepsilon_{a^+}t\mathcal{G}_y\right)(b)\right], \tag{43}$$

$$\left(\varepsilon_{d^-}F_5t\mathcal{G}_x\right)(c) + \left(\varepsilon_{c^+}F_5t\mathcal{G}_x\right)(d) \leq \frac{F_5(c)+F_5(d)}{2}\left[\left(\varepsilon_{d^-}t\mathcal{G}_x\right)(c) + \left(\varepsilon_{c^+}t\mathcal{G}_x\right)(d)\right], \tag{44}$$

and

$$\left(\varepsilon_{d^-}F_6t\mathcal{G}_x\right)(c) + \left(\varepsilon_{c^+}F_6t\mathcal{G}_x\right)(d) \leq \frac{F_6(c)+F_6(d)}{2}\left[\left(\varepsilon_{d^-}t\mathcal{G}_x\right)(c) + \left(\varepsilon_{c^+}t\mathcal{G}_x\right)(d)\right]. \tag{45}$$

Then we have inequality

1. (42) and (43) by $(y-c)^{\nu_2-1}\mathcal{E}_2(\omega_2'(y-c)^{\mu_2})$, respectively $(d-y)^{\nu_2-1}\mathcal{E}_2(\omega_2'(d-y)^{\mu_2})$ and integrating with respect to y over $[c,d]$,
2. (44) and (45) by $(x-a)^{\nu_1-1}\mathcal{E}_1(\omega_1'(x-a)^{\mu_1})$, respectively $(b-x)^{\nu_1-1}\mathcal{E}_1(\omega_1'(b-x)^{\mu_1})$ and integrating with respect to x over $[a,b]$.

We get four inequalities that we are adding, and taking into account the symmetry of $t\mathcal{G}$, we obtain the last inequality of (31). The proof is complete. □

Remark 4. *If in Theorem 3 we put $\omega = (0,0)$, we obtain [20] (Theorem 8).*

3. Conclusions

First, we introduced and studied generalized fractional integral operators containing the new extended general Mittag–Leffler function of two variables. We then obtained several new two-dimensional versions of trapezium-like inequalities via coordinated convex functions. We also discussed the linkage of the obtained results with previously known results by considering some special cases. It is expected that the ideas and techniques of this paper may stimulate further research.

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