

Article

# Which Derivative?

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**Abstract:** The actual state of interplay between Fractional Calculus, Signal Processing, and Applied Sciences is discussed in this paper. A framework for compatible integer and fractional derivatives/integrals in signals and systems context is described. It is shown how suitable fractional formulations are really extensions of the integer order definitions currently used in Signal Processing. The particular case of fractional linear systems is considered and the problem of initial conditions is tackled.

**Keywords:** fractional derivatives; compatibility; fractional linear systems

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## 1. Introduction

Fractional derivative (FD) is the name assigned to several mathematical operators, namely the Grünwald-Letnikov (GL), Liouville (L), Riemann-Liouville (RL), Caputo (C), Marchaud, Hadamard, Riesz, and other definitions [1–7]. Nevertheless, there is a long standing debate about the pros and cons of each formulation and some researchers question which can be considered as FD. In [8] a methodology for deciding if a given operator is a FD was proposed. Several authors addressed the problems about the FD definitions and interpretation, but the debate is far from concluded and we verify the emergence of new alternative definitions of FD. This state of affairs poses additional difficulties for the direct adoption of FD tools in Signals and Systems and Applied Sciences.

In the scope of FD operators we must consider the formulations that can be useful for the introduction of fractional linear systems and their characterisation in the perspective of compatibility with integer order definitions. This problem had a first review in [9] where the role of Riemann-Liouville and Caputo derivatives for dealing with fractional linear systems was discussed. In [10] this problem was studied in the context of the periodic functions and it was verified that we should use fractional derivatives defined on the whole real line. Here we revise the topic in a more general perspective. Similar problems appear in Applied Sciences where proposed formulations are frequently not compatible with classic integer order counterparts. This leads to the frequently posed question: “Are mathematical models with FD consistent with the laws of physics?” [11]. The answer is probably “no”, because we find fractional-order models that are proposed without considering classic laws. Moreover, some models lead to uncommon results motivating the term “metaphysical derivatives” suggested by Stéphane Dugowson [12]. These ideas are in the fringe of present day scientific reasoning. However, it is ubiquitous in science that some controversial concepts led often to new directions of research. Such kind of considerations motivate questions like: Which fractional derivatives are consistent with the classic integer order laws? Or, which FD are suitable for dealing with practical systems in a compatibility with integer order definitions? It is this exercise that we will approach in this paper, trying to unravel the “metaphysical” character of some FD, and developing a discussion to answer to some open questions. This will show what FD formulations should should

be adopted in Applied Sciences in order to keep coherence with the classic results. In fact this is a challenging task that has been considered by several researchers [3–5,7,13,14], but the topic is far from reaching a consensus leading to assertive directions of study.

The main aim in this paper is to show that there is a coherent basis for establishing fractional operators compatible the corresponding classic integer order. In particular, the formulation developed in the sequel allows the currently used tools like, Impulse Response, Transfer Function and Frequency Response in a very general scope that includes the classic operators obtained when the orders become integers.

Having these ideas in mind, this paper is organized as follows. Section 2 starts by recalling some classic results. Section 3 analyses the backward compatibility for fractional calculus (FC). Section 4 exemplifies the use of such derivatives in the of linear systems defined by differential equations. Finally, Section 5 outlines some conclusions.

### Remarks

We assume that

- We work on  $\mathbf{R}$ .
- We use the two-sided Laplace transform (LT):

$$F(s) = \int_{\mathbf{R}} f(t)e^{-st} dt \quad (1)$$

where  $f(t)$  is any function defined on  $\mathbf{R}$  and  $F(s)$  is its transform, provided it has a non empty region of convergence

- The Fourier transform (FT) is obtained from the LT through the substitution  $s = i\omega$  with  $\omega \in \mathbf{R}$
- The functions and distributions have Laplace and/or Fourier transforms
- Current properties of the Dirac delta distribution and its derivatives will be used
- We will work with the usual convolution

$$f(t) * g(t) = \int_{\mathbf{R}} f(\tau)g(t - \tau)d\tau \quad (2)$$

- The fractional derivative order is assumed to be any real number
- The multi-valued expressions  $s^\alpha$  and  $(-s)^\alpha$  will frequently be used. To obtain functions from them we will fix for branch-cut lines the negative real half axis for the first and the positive real half axis for the second; for both the first Riemann surface is chosen.

## 2. Some Classic Results

### 2.1. The Derivative Operators and Their Inverses

To look forward FD formulations consistent with the laws of physics we recall the most important results from the classic calculus. The standard definition of derivative is

$$Df(t) = f'(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t - h)}{h}, \quad (3)$$

or

$$Df(t) = f'(t) = \lim_{h \rightarrow 0} \frac{f(t + h) - f(t)}{h}. \quad (4)$$

Substituting  $-h$  for  $h$  interchanges the definitions, meaning that we only have to consider  $h > 0$ . In this situation, (3) uses the present and past values while (4) uses present and future values. In the following we will distinguish the two cases by using the subscripts  $f$  (forward – in the sense that we go from past into future, a direct time flow) and  $b$  (backward – meaning a reverse time flow).

It is straightforward to invert the above equations to obtain

$$D_f^{-1}f(t) = \lim_{h \rightarrow 0} \sum_{n=0}^{\infty} f(t - nh) \cdot h, \quad (5)$$

and

$$D_b^{-1}f(t) = \lim_{h \rightarrow 0} \sum_{n=0}^{\infty} f(t + nh) \cdot h. \quad (6)$$

These relations motivate the following comments:

- The different time flow shows its influence: the causality (anti-causality) is clearly stated,
- We have  $D_f^{-1}D_f f(t) = D_f D_f^{-1}f(t) = f(t)$  and  $D_b^{-1}D_b f(t) = D_b D_b^{-1}f(t) = f(t)$ . We will call  $D^{-1}$  “anti-derivative” [6].

High order derivatives and corresponding inverses are usually done recursively, but closed formulae can also be established. Considering  $N$  a positive integer, such expressions assume the form

$$D_f^{\pm N}f(t) = \lim_{h \rightarrow 0^+} \frac{\sum_{n=0}^{\infty} \frac{(\mp N)_n}{n!} f(t - nh)}{h^{\pm N}}, \quad (7)$$

and

$$D_b^{\pm N}f(t) = (-1)^N \lim_{h \rightarrow 0^+} \frac{\sum_{n=0}^{\infty} \frac{(\mp N)_n}{n!} f(t + nh)}{h^{\pm N}}, \quad (8)$$

where  $(a)_k = a(a+1)(a+2)\dots(a+k-1)$  denotes the Pochhammer symbol.

These relations motivate the comments:

- The one-step computation of derivatives using (7) and (8) may not give the result that we would obtain in a recursive procedure. However, in the conditions stated at the Introduction they give the same result,
- In the derivative case  $(+N)$  the summation goes only to  $N$ , since the  $(-N)_n$  becomes null for  $n > N$ ,
- Let  $n_1$  and  $n_2$  be two integer values. With (7) and under the assumed functional space we can write

$$D_f^{n_1} D_f^{n_2} f(t) = D_f^{n_2} D_f^{n_1} f(t) = D_f^{n_1+n_2} f(t) \quad (9)$$

For the backward derivatives the situation is similar. This result is straightforward using the properties of the binomial coefficients or the  $\mathcal{Z}$  transform [15]

- It is possible to combine the forward and backward derivatives to obtain the two-sided (centred) derivatives [6,16].

Let us introduce now two fundamental functions:

- The constant function

$$c(t) = 1, \quad t \in \mathbf{R}, \quad (10)$$

- The Heaviside, or unit step, function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}, \quad t \in \mathbf{R}. \quad (11)$$

These functions have the same future, but a different past. Therefore, their derivatives must be distinct as well. It is well-known that

$$\begin{cases} Dc(t) = 0 \\ Du(t) = \delta(t) \end{cases}, \quad t \in \mathbf{R}, \tag{12}$$

both for the forward and backward derivatives.

### 2.2. System Interpretation

From a system point of view, expressions (3) and (4) can be represented by the transfer function (TF) given by  $H(s) = s$ , analytic on the whole complex plane. The corresponding impulse response is  $h(t) = \delta'(t)$ , where  $\delta(t)$  is the Dirac's impulse distribution.

From this perspective, we consider the system approach with the integer order derivatives/anti-derivatives formulated by means of the two-sided LT property  $\mathcal{L} [f^{(\pm n)}(t)] = s^{\pm n} L[f(t)]$  where  $n \in \mathbf{N}$ . Therefore, the  $n^{\text{th}}$  order derivative is represented in the transform domain by an operator with TF given by  $H(s) = s^{\pm n}$ , in such a way that we can write the sequence in the Laplace domain:

$$\dots s^{-n} \dots s^{-2} s^{-1} 1 s^1 s^2 \dots s^n \dots \tag{13}$$

We observe that:

- The terms with negative exponents represent two TF corresponding to two disjoint regions of convergence, namely  $\text{Re}(s) > 0$  (causal system) and  $\text{Re}(s) < 0$  (anti-causal system),
- The terms with positive or null exponents are analytic on the whole complex plane and consequently there is no causality involved.

When inverting the powers with negative exponents in (13) towards time domain we obtain the causal and anti-causal solutions, leading to the sequence:

$$\dots \pm \frac{t^{n-1}}{(n-1)!} u(\pm t) \dots \pm \frac{t^2}{2!} u(\pm t) \pm \frac{t^1}{1!} u(\pm t) \pm u(\pm t) \delta(t) \delta'(t) \delta''(t) \dots \delta^{(n)}(t) \dots \tag{14}$$

These distributions are the impulse responses of causal (+) and anti-causal (-) systems. Their convolution with a given function  $f(t)$ , produce outputs that are the  $n$ -th order derivative/anti-derivative

$$f^{(n)}(t) = \pm \begin{cases} \int_{-\infty}^{\infty} \delta^{(n)}(\tau) f(t \pm \tau) d\tau, & n \geq 0 \\ \int_0^{\infty} \frac{\tau^{-n-1}}{(-n-1)!} f(t \pm \tau) d\tau, & n < 0 \end{cases}, \quad n \in \mathbf{Z}. \tag{15}$$

In the sequel the causal solution it is adopted, since for the anti-causal the procedures and the results are similar.

Let  $n$  be a positive integer. From the above sequence, it is straightforward to conclude that

$$D^k \left[ \frac{t^n}{n!} u(t) \right] = \frac{t^{n-k}}{(n-k)!} u(t)$$

and

$$D^k [t^n u(t)] = \frac{\Gamma(n+1)}{\Gamma(n-k+1)} t^{n-k} u(t), \tag{16}$$

provided that  $n \geq k$ .

This equation limits our view since it cannot be used for  $n < k$ . Therefore, we recall the relation derived by Gel'fand and Shilov [17]:

$$\delta^{(n)}(t) = \frac{t^{-n-1}}{(-n-1)!} u(t), \quad n \geq 0. \quad (17)$$

With this expression we can extend the validity of (16) to any integer order and from (15) we obtain

$$f^{(n)}(t) = \int_0^\infty \frac{\tau^{-n-1}}{(-n-1)!} f(t-\tau) d\tau = \int_0^\infty \frac{\tau^{-n-1}}{\Gamma(-n)} f(t-\tau) d\tau. \quad (18)$$

In the case of positive values of  $n$ , this representation gives a unique formula for expressing the derivatives and anti-derivatives. This result means that there is a general linear system (sometimes called differintegrator) with TF

$$H(s) = s^n$$

and impulse response

$$h(t) = \frac{t^{-n-1}}{\Gamma(-n)} u(t).$$

With this impulse response we obtain the simple integral representation (18) for the derivatives/anti-derivatives.

### 2.3. Other Important Results

#### 1. Derivative eigenfunctions

Returning back to (18)

$$D^n e^{zt} = z^n e^{zt}, \quad n \in \mathbf{Z}, t \in \mathbf{R}, z \in \mathbf{C}. \quad (19)$$

This result, for the particular case of  $z = i\omega$ ,  $\omega \in \mathbf{R}^+$ ,  $i = \sqrt{-1}$ , yields

$$\begin{cases} D^n \cos(\omega t) = \omega \cos(\omega t + n \cdot \frac{\pi}{2}) \\ D^n \sin(\omega t) = \omega \sin(\omega t + n \cdot \frac{\pi}{2}) \end{cases}, \quad t \in \mathbf{R}. \quad (20)$$

These results are also valid for negative  $n$ .

#### 2. The Leibniz relation for the product

The classic Leibniz relation gives the derivative of the product of two functions and can be written as

$$D_f^n [f(t)g(t)] = \sum_{k=0}^n \binom{n}{k} D_f^k f(t) D_f^{n-k} g(t), \quad (21)$$

where  $n$  is a positive integer. For the backward the formula is identical. The Leibniz relation is very important and its generalization to the fractional case is a characteristic of the FD [18,19].

## 3. Backward Compatibility in Fractional Calculus

### 3.1. Some Considerations

The idea of differentiation of non integer order seems to result from some reflections expressed by Leibniz in a letter addressed to J. Bernoulli [12]. The subject was first discussed between the two and later with other mathematicians, such as L'Hospital, but no formula or procedure for the computation was proposed. As a "by-product" of this discussion the famous "Leibniz formula" for the  $n^{\text{th}}$  order derivative of a product appeared for the first time. The first important contribution in FC was accomplished by J. Liouville that proposed several formulae for computing the derivatives of any order [20–24]. However, Liouville based his theory on the development of the functions as a sum of exponentials, what created several technical difficulties because at that time the Bromwich

integral was not yet formulated. Among the proposed formulae he proposed generalizations of the incremental ratio. However, the importance of this result was overlooked and only later Grünwald [25] and Letnikov [26] returned to the subject, but considering only the particular case of functions defined on  $\mathbf{R}^+$ . The name “Grünwald-Letnikov derivatives” is currently adopted in derivatives based on the incremental ratio. The possibility of using the incremental ratio for the formulation of FC was considered in [27]. It was clear that a relation with integral formulations was relevant and such viewpoint was explored in [28].

### 3.2. Causal FC Based on the Incremental Ratio

From the of applied sciences point of view, the Grünwald-Letnikov FD seem to be a natural way for generalizing the notion of derivative. They have a solid meaning for any real or complex order and we recover the classic results for integer orders.

Let the function  $f(t)$  to be defined on  $\mathbf{R}$  and  $\alpha$  any real number. We have the forward and backward derivatives given by:

$$D_f^\alpha f(t) = \lim_{h \rightarrow 0^+} \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t - nh)}{h^\alpha}, \quad (22)$$

$$D_b^\alpha f(t) = e^{-i\pi\alpha} \lim_{h \rightarrow 0^+} \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t + nh)}{h^\alpha}. \quad (23)$$

Necessary conditions for the existence of the above derivatives were formulated in, where it was highlighted the fact that we must take into account the behaviour of the function  $f(t)$  along the half straight line  $t \pm nh$ , with  $n \in \mathbf{Z}^+$ . There are several properties exhibited by (22) and (23) [8]:

- Linearity
- Additivity and Commutativity of the orders. If we apply (22) twice for any two orders, we have

$$D_f^\alpha D_f^\beta f(t) = D_f^\beta D_f^\alpha f(t) = D_f^{\alpha+\beta} f(t). \quad (24)$$

- Neutral element

$$D_f^\alpha D_f^{-\alpha} f(t) = D_f^0 f(t) = f(t). \quad (25)$$

From (25) we conclude that there is always an inverse element, that is, for every  $\alpha$  there is always the  $-\alpha$  order derivative.

- Backward compatibility ( $n \in \mathbf{N}$ )

If  $\alpha = n$ , then:

$$D_f^n f(t) = \lim_{h \rightarrow 0} \frac{\sum_{k=0}^n (-1)^k \binom{n}{k} f(t - kh)}{h^n}$$

We obtain this expression repeating the first order derivative.

If  $\alpha = -n$ , then:

$$D_f^{-n} f(t) = \lim_{h \rightarrow 0} \sum_{k=0}^n \frac{\binom{n}{k}}{k!} f(t - kh) \cdot h^n,$$

that corresponds to a  $n$ -th repeated summation [6].

- We can apply the two-sided LT to (22) and (23) to obtain

$$\mathcal{L} \left[ D_f^\alpha f(t) \right] = s^\alpha \mathcal{L} [f(t)], \quad (26)$$

with  $Re(s) > 0$  in the first and  $Re(s) < 0$  in the second. This means that there are two systems (one causal and one anti-causal differintegrator) with TF given by  $H(s) = s^\alpha$ . This result agrees with the discussion in Section 2

- The generalized Leibniz rule for the product  
The generalized Leibniz rule gives the FD of the product of two functions. It is one of the most important characteristics of the FD [18,19] and assumes the format [1]

$$D_f^\alpha [f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_f^k f(t) D_f^{\alpha-k} g(t). \quad (27)$$

For the backward the formula is identical.

### 3.3. Some Examples

We now consider the particular cases mentioned in Section 2:

#### 1. Constant function

If  $f(t) = 1$ , for every  $t \in \mathbf{R}$  and  $\alpha \in \mathbf{R}$ , then we have

$$D^\alpha f(t) = \lim_{h \rightarrow 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k}{h^\alpha} = \begin{cases} 0, & \text{if } \alpha > 0 \\ \infty, & \text{if } \alpha < 0 \end{cases}. \quad (28)$$

#### 2. Causal power function

We calculate the FD of the Heaviside function. Starting from [1]:

$$\sum_{k=0}^n \binom{\alpha}{k} (-1)^k = \binom{\alpha-1}{n} (-1)^n = \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(-\alpha+n+1)}{\Gamma(n+1)}, \quad (29)$$

we can show that

$$D^\alpha u(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u(t). \quad (30)$$

As  $\delta(t) = Du(t)$ , we deduce using (24) that

$$D^\alpha \delta(t) = \frac{t^{-\alpha-1}}{\Gamma(\alpha+1)} u(t). \quad (31)$$

With (30) it is possible to obtain the derivative of any order of the continuous function  $p(t) = t^\beta u(t)$ , with  $\beta > 0$ . The LT of  $p(t)$ , yields  $P(s) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$ , for  $\text{Re}(s) > 0$  and the FD of order  $\alpha$  is given by  $s^\alpha \frac{\Gamma(\beta+1)}{s^{\beta+1}}$ . Therefore, the expression

$$D^\alpha t^\beta u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} u(t), \quad (32)$$

generalizes the integer order formula (see Section 2) for any  $\alpha \in \mathbf{R}$  [2]. In particular, with  $\beta = \alpha - 1$  it results:

$$D^\alpha \frac{t^{\alpha-1}}{\Gamma(\alpha+1)} u(t) = \frac{t^{-1}}{\Gamma(0)} u(t) = \delta(t). \quad (33)$$

### 3.4. Obtaining Integral Formulations

In (31) we obtained the impulse response of the causal differintegrator. So, its output corresponding to a given function,  $f(t)$  is given by a convolution

$$D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \tau^{-\alpha-1} f(t-\tau) d\tau. \quad (34)$$

Expression (34) is not as handy as (22) due to the singularity of  $h(t)$  at the origin when  $\alpha > 0$ , but it can be regularized. Let  $N = \lfloor \alpha \rfloor + 1$ ; then [29]

$$D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \tau^{-\alpha-1} \left[ f(t-\tau) - \sum_0^{N-1} \frac{(-)^m f^{(m)}(t)}{m!} \tau^m \right] d\tau. \quad (35)$$

that we will call regularized Liouville derivative.

Reconsidering property (24) we can write

$$D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha + N)} \int_0^\infty \tau^{N-\alpha-1} f^{(N)}(t-\tau) d\tau. \quad (36)$$

This is called Liouville-Caputo derivative [7].

Consider the property of the convolution

$$D_f^N [f(t) * g(t)] = f(t) * D_f^N [g(t)]. \quad (37)$$

It seems clear that *if the right hand side exists, then it also occurs for the left hand side* due to the following reasons:

- $D_f^N [g(t)]$  has a worst analytic behaviour than  $g(t)$ ; eventually it can be discontinuous.
- The convolution has a smoothing effect. Therefore, in the left side above we are computing the derivative of a function with “better behaviour”.

These considerations lead us to write the expression

$$D_f^\alpha f(t) = D_f^N \left[ \frac{1}{\Gamma(-\alpha + N)} \int_0^\infty \tau^{N-\alpha-1} f(t-\tau) d\tau \right], \quad (38)$$

that constitutes a derivative of the Riemann-Liouville type.

In conclusion, from the impulse response of the differintegrator, three different integral formulations were obtained from where current formulations can be derived, (35), (36) and (38). This topic will be discussed further in the sequel. A fair comparison of the 3 derivatives lead us to conclude that

- If  $f(t)$  has Laplace transform with a nondegenerate region of convergence, the three derivatives give the same result,
- The Liouville-Caputo derivative demands too much from analytical point of view, since it needs the unnecessary existence of the  $N^{\text{th}}$  order derivative,
- If  $f(t) = 1$ ,  $t \in \mathbf{R}$  the Riemann-Liouville derivative does not exist, since the integral is divergent.

### 3.5. The TF of the Differintegrator

Consider the exponential function  $f(t) = e^{st}$ ,  $t \in \mathbf{R}, s \in \mathbf{C}$ . From (22), the FD of this function is given by:

$$D_f^\alpha e^{st} = e^{st} \lim_{h \rightarrow 0^+} \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} e^{-nsh}}{h^\alpha}.$$

But

$$\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} e^{-nsh} = (1 - e^{-sh})^\alpha,$$

provided that  $\operatorname{Re}(s) > 0$ . Since  $(1 - e^{-sh})^\alpha$  is a multi-valued expression, we can assume that we define the left half real line as branch-cut. The calculation of the limit leads to

$$D_f^\alpha e^{st} = s^\alpha e^{st}, \quad (39)$$

for  $\operatorname{Re}(s) > 0$ , which is again the generalization of the classic result. We conclude that the causal differintegrator has a TF given by  $H(s) = s^\alpha$ ,  $\operatorname{Re}(s) > 0$ .

With  $s = i\omega$  we can give rise to Bode diagrams, close to the classic procedure. This leads to the following results:

$$\begin{cases} D^\alpha \cos(\omega t) = \omega^\alpha \cos(\omega t + \alpha \frac{\pi}{2}) \\ D^\alpha \sin(\omega t) = \omega^\alpha \sin(\omega t + \alpha \frac{\pi}{2}) \end{cases}, \quad t \in \mathbf{R}, \omega \in \mathbf{R}^+, \quad (40)$$

showing that the “differintegrators” produce a phase shift of  $\alpha \frac{\pi}{2}$ .

The differintegrators are realized by ideal fractional coils and capacitors.

### 3.6. Classic Riemann-Liouville and Caputo Derivatives

The commutative property of the convolution, allows us to write from (38) and (36):

$$D_f^\alpha f(t) = D_f^N \left[ \frac{1}{\Gamma(-\alpha + N)} \int_{-\infty}^t (t - \tau)^{N-\alpha-1} f(\tau) d\tau \right] \quad (41)$$

and

$$D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha + N)} \int_{-\infty}^t (t - \tau)^{N-\alpha-1} f^{(N)}(\tau) d\tau. \quad (42)$$

These expressions are the general formulations of Riemann-Liouville and Liouville-Caputo derivatives. The usual formulations of RL and C derivatives are obtained from the above relations assuming that the function is defined in a given interval  $[a, b]$ . Therefore, for  $t \in [a, b]$  the RL and C derivatives are given by

$${}^{RL}D_f^\alpha f(t) = D_f^N \left[ \frac{1}{\Gamma(-\alpha + N)} \int_a^t (t - \tau)^{N-\alpha-1} f(\tau) d\tau \right] \quad (43)$$

and

$${}^CD_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha + N)} \int_a^t (t - \tau)^{N-\alpha-1} f^{(N)}(\tau) d\tau. \quad (44)$$

These derivatives do not enjoy most of the important properties of the classic derivative previously presented, namely the results referring the derivatives of sinusoids that following (43) and (44) are no longer sinusoids [10]. Another example is given by the following differential equation  ${}^CD_f^\alpha f(t) + af(t) = {}^CD_f^\beta u(t)$ . If the initial conditions are null the output is also null.

## 4. The Linear Differential Equations

### 4.1. The Transfer Function and the Impulse Response

In practical applications fractional linear systems are described by an equation of the type:

$$\sum_{k=0}^N a_k D^{\alpha_k} y(t) = \sum_{k=0}^M b_k D^{\beta_k} x(t), \quad (45)$$

where  $a_k, b_k \in \mathbf{R}$ ,  $t \in \mathbf{R}$ . The parameters  $\alpha_k$  and  $\beta_k$  are the derivative orders that we assume to form strictly increasing sequences of positive numbers. The so-called commensurate systems correspond to the case  $\alpha_k = \beta_k = k\alpha$ . In real-world applications we consider that  $\beta_M \leq \alpha_N$  for stability reasons.

According to (39) the exponential is the eigenfunction of the system defined by (45). The corresponding eigenvalue,  $H(s)$ , is the transfer function (TF) of

$$H(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^M b_k s^{\beta_k}}{\sum_{k=0}^N a_k s^{\alpha_k}}. \quad (46)$$

As in the classical case we will name by “poles” the roots of the *characteristic pseudo-polynomial*,  $A(s)$ , in the denominator of the TF.

The impulse response may not be easy to compute in the general formulation stated above. Anyway the general expression of the impulse response is given by

$$h(t) = \sum_{k=1}^{K_0} A_k e^{p^{1/\gamma_k} t} u(t) + \frac{1}{2\pi i} \int_0^{\infty} [H(e^{-i\pi} u) - H(e^{i\pi} u)] e^{-\sigma t} du \cdot u(t), \quad (47)$$

where the constants  $A_k$ ,  $k = 1, 2, \dots, K_0$ , are the residues of (46) at  $p_k^{1/\gamma_k}$ ,  $k = 1, 2, \dots, K_0$ . Computing the LT of both sides in (47) we conclude that the transfer function can be decomposed in two components:

$$H(s) = H_i(s) + H_f(s), \quad (48)$$

where the integer order part is

$$H_i(s) = \sum_{k=1}^{K_0} \frac{A_k}{s - p^{1/\gamma_k}}, \quad \operatorname{Re}(s) > \max(\operatorname{Re}(p^{1/\gamma_k})), \quad (49)$$

and the fractional part is

$$H_f(s) = \frac{1}{2\pi i} \int_0^{\infty} [H(e^{-i\pi} u) - H(e^{i\pi} u)] \frac{1}{s + u} du, \quad (50)$$

that is valid for  $\operatorname{Re}(s) > 0$ . It is interesting to remark that the first component may or not exist, while the second exists always if at least one of the derivative orders is noninteger. The stability conditions are only established in terms of the first, meaning that the fractional component is always stable.

#### 4.2. The Initial Condition Problem

The free term of the output depends merely on the initial conditions (IC) that are the values assumed at the reference instant  $t = t_0$  by the system variables associated with energy storage.

It is the structure of the system that should impose the IC and not the method of calculating the derivatives. The IC influence the output when we excite the system by means of a fresh input at instant  $t = t_0$ .

The IC problem in fractional linear systems is not well solved with the one-sided Laplace transform and a suitable solution was proposed in [30]. Our point of view is the following:

1. Equation (45) is defined for any  $t \in \mathbf{R}$ ,
2. Our observation window is the unit step  $u(t - t_0)$ ,
3. The IC depend on the structure of the system and are independent of the tools that we adopt for the analysis,
4. The IC are the values assumed by the variables at the instant of opening the observation window.

These reasons point towards a simple procedure to rework the differential equation (45) to include the IC. The algorithm is based on the fractional jump formula [30]:

$$\begin{aligned} \sum_{k=0}^N a_k [y(t) \cdot u(t)]^{(\alpha_k)} &= \sum_{k=0}^M b_k [x(t) \cdot u(t)]^{(\beta_k)} \\ + \sum_{k=1}^N a_k \sum_{m=0}^{k-1} y^{(\alpha_m)}(t_0) \delta^{(\alpha_k - \alpha_m)}(t - t_0) &- \sum_{k=1}^M b_k \sum_{m=0}^{k-1} x^{(\beta_m)}(t_0) \delta^{(\beta_k - \beta_m)}(t - t_0). \end{aligned} \quad (51)$$

Considering  $\alpha_k = \beta_k = k$  and applying the two-sided LT to (51) we obtain an equation similar to the one obtained classically with the one-sided LT, provided that  $t = 0$ .

With this general formulation we obtained a fractional version of the current linear systems. In particular we obtain for the commensurate case

$$\begin{aligned} \sum_{k=0}^N a_k [y(t) \cdot u(t)]^{(\alpha_k)} &= \sum_{k=0}^M b_k [x(t) \cdot u(t)]^{(\alpha_k)} \\ + \sum_{k=1}^N a_k \sum_{m=0}^{k-1} y^{(\alpha_m)}(t_0) \delta^{(\alpha_k - \alpha_m)}(t - t_0) &- \sum_{k=1}^M b_k \sum_{m=0}^{k-1} x^{(\alpha_m)}(t_0) \delta^{(\alpha_k - \alpha_m)}(t - t_0). \end{aligned} \quad (52)$$

We note that the initial instant does not need to be  $t = 0$ . We conclude that the above formulation is fully compatible with classic results.

## 5. Conclusions

In this paper we discussed a sequence of operations that contains the classic derivatives as particular cases in order to develop operational tools for Signals and Systems and Applied Sciences. With a Signal Processing approach we can work more easily with FC tools in domains where the integer calculus is applied. Therefore, Signal Processing concepts provide a coherent formulation of FC in agreement with classic tools. Grünwald-Letnikov and regularized Liouville derivatives are the suitable derivatives for such objective.

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## Abbreviations

The following abbreviations are used in this manuscript:

C	Caputo
FT	Fourier transform
FD	Fractional derivative
FI	Fractional integral
GL	Grünwald-Letnikov
IC	Initial conditions
L	Liouville
LT	Laplace transform
RL	Riemann-Liouville
TF	Transfer function

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