fermentation

MDPI

Supplementary material—File S1

The detailed proofs of Theorem 1, Theorem 2 and Theorem 3 are presented at what follows.

Theorem 1 [Convergence of the weighted sum of the observer errors]. *Consider the model (1), (2) subject to assumptions 1 to 3 and the observer (5)–(7), with definitions (8)–(10) and observer error* $\bar{x}_2 = \hat{x}_2 - x_2$. *As a result of this observer:*

Ti) the function $z = \bar{x}_2 - \sigma\omega\bar{x}_1$, $\sigma = sign(b)$, *satisfies* $\frac{dz}{dt} = (-1)\omega|b|(z+\delta)$, *where* $\delta = \frac{1}{b}\left(\frac{\delta_2}{\sigma\omega} - \delta_1\right)$;

Tii) the function z *converges to* $\Omega_z = [z^l,\ z^u]$, *where* $z^l = -\delta_{max} \le 0$, $z^u = -\delta_{min} \ge 0$, *and* δ_{min}, δ_{max} *are unknown constants that satisfy* $\delta \ge \delta_{min}$, $\delta_{min} \in (-\infty, 0]$, $\delta \le \delta_{max}$, $\delta_{max} \in [0, \infty)$;

Tiii) the upper bound of the transient response of z is

$$|z| \le \left|\psi_{z|to}\right| e^{-\omega b_{min}(t-t_0)} + max\,\{-\delta_{min}, \delta_{max}\}.$$

Proof. Consider the general observer form :

$$\frac{d\hat{x}_1}{dt} = b\hat{x}_2 - bg_1 + h_1 \tag{S1}$$

$$\frac{d\hat{x}_2}{dt} = -bg_2 + h_2 \tag{S2}$$

where g_1 and g_2 are functions that will be defined later, whereas h_1, h_2, b are terms of model (1), (2). Substracting the x_1 dynamics (1) from expression (S1) and subtracting the x_2 dynamics (2) from expression (S2), yields

$$\frac{d\bar{x}_1}{dt} = b\bar{x}_2 - bg_1 - \delta_1 \tag{S3}$$

$$\frac{d\bar{x}_2}{dt} = -bg_2 - \delta_2 \tag{S4}$$

where $\bar{x}_1 = \hat{x}_1 - x_1$, $\bar{x}_2 = \hat{x}_2 - x_2$. Thus, the dynamics of the observation errors $\bar{x}_1$ and $\bar{x}_2$ are given by Eqs (S3), (S4), which can be arranged as:

$$\frac{d\bar{x}_2}{dt} = -bg_2 - \delta_2 \tag{S5}$$

$$\sigma\omega\frac{d\bar{x}_1}{dt} = \sigma\omega b\bar{x}_2 - \sigma\omega bg_1 - \sigma\omega\delta_1 \tag{S6}$$

where $\sigma = sign(b)$. Adding and arranging Eqs. (S5), (S6), yields

$$\frac{d}{dt}(\bar{x}_2 - \sigma\omega\bar{x}_1) = (-1)\omega\sigma b\left(\bar{x}_2 + \frac{1}{\sigma\omega}g_2 - g_1 + \frac{1}{b}\left(\frac{\delta_2}{\sigma\omega} - \delta_1\right)\right)$$

which can be rewritten as

$$\frac{dz}{dt} = (-1)\omega|b|\left(z + \sigma\omega\bar{x}_1 + \frac{1}{\sigma\omega}g_2 - g_1 + \delta\right) \tag{S7}$$

where

$$z = \bar{x}_2 - \sigma\omega\bar{x}_1 \tag{S8}$$

$$\delta = \frac{1}{b}\left(\frac{\delta_2}{\sigma\omega} - \delta_1\right) \tag{S9}$$

To counteract the effect of the term $\sigma\omega\bar{x}_1 + \frac{1}{\sigma\omega}g_2$ appearing in Eq (S7), the function g_1 is defined as

$$g_1 = \sigma\omega\bar{x}_1 + \frac{1}{\sigma\omega}g_2 \tag{S10}$$

So, it remains to define the function g_2. Substituting into Eq (S7), yields

$$\frac{dz}{dt} = (-1)\omega|b|(z+\delta) \tag{S11}$$

This completes the proof of Ti.

The term $z + \delta$ appearing in Eq (S11) satisfies:

$$z + \delta \ge z + \delta_{min} \ge 0\ for\ z \ge -\delta_{min} \ge 0 \tag{S12a}$$

$$z + \delta \le z + \delta_{max} \le 0\ for\ z \le -\delta_{max} \le 0 \tag{S12b}$$

And also

$$z + \delta \ge z + \delta_{min} > 0\ for\ z > -\delta_{min} \ge 0 \tag{S13a}$$

$$z + \delta \le z + \delta_{max} < 0\ for\ z < -\delta_{max} \le 0 \tag{S13b}$$

where δ_{min}, δ_{max} are unknown constants that satisfy

$$\delta \ge \delta_{min}\ ,\ \delta_{min} \in (-\infty, 0],\ \delta \le \delta_{max},\ \delta_{max} \in [0, \infty)$$

Therefore,

$$z+\delta>0\ for\ z>z^u,\ \ z^u=-\delta_{min}\geq 0$$
$$z+\delta<0\ for\ z<z^l,\ \ z^l=-\delta_{max}\leq 0$$

To prove the convergence of z, we consider the Lyapunov function

$$V_z=\frac{1}{2}\psi_z^2 \tag{S14}$$

$$\psi_z=\begin{cases} z+\delta_{min}\ \ for\ z\geq -\delta_{min}\geq 0 \\ 0\ for\ z\ \in(-\delta_{max},-\delta_{min}) \\ z+\delta_{max}\ \ for\ z\leq -\delta_{max}\leq 0 \end{cases} \tag{S15}$$

The main properties of ψ_z are:

$$\psi_z\geq 0\ \ for\ z\geq -\delta_{min}\geq 0$$
$$\psi_z\leq 0 for\ z\leq -\delta_{max}\leq 0$$
$$\psi_z=0\ \ for\ z\ \in[-\delta_{max},-\delta_{min}] \tag{S16}$$

Therefore, the main properties of V_z (S14) are:

$$V_z=0\ \ \ for\ z\in[z^l,\ z^u]$$
$$V_z>0\ \ \ for\ z\notin[z^l,\ z^u]$$
$$V_z\ is\ continuous\ with\ respect\ to\ z$$

Where $z^u=-\delta_{min}\geq 0,\ \ z^l=-\delta_{max}\leq 0$. Differentiating V_z (S14) with respect to time, yields

$$\frac{dV_z}{dt}=\psi_z\frac{dz}{dt}$$

Substituting the expression of dz/dt (S11) and arranging, yields

$$\frac{dV_z}{dt}=(-1)\omega|b|\psi_z(z+\delta) \tag{S17}$$

From the definition of ψ_z (S15) and properties (S13a), (S13b), (S16) it follows that

$$z+\delta\geq\psi_z>0\ \ for\ z>-\delta_{min}\geq 0$$
$$z+\delta\leq\psi_z<0\ \ \ for\ z<-\delta_{max}\leq 0$$
$$\psi_z(z+\delta)=0=\psi_z^2\ \ for\ z\ \in[-\delta_{max},-\delta_{min}] \tag{S18}$$

Therefore,

$$sgn(z+\delta)=sgn(\psi_z)\neq 0\ \ for\ z\ \notin[-\delta_{max},-\delta_{min}]$$
$$|z+\delta|\geq|\psi_z|\ \ for\ \ z\ \notin[-\delta_{max},-\delta_{min}]$$

From these properties it follows that

$$(z+\delta)\psi_z=|z+\delta|\ |\psi_z|\ \geq\psi_z^2\ \ for\ \ z\ \notin[-\delta_{max},-\delta_{min}] \tag{S19}$$

Combining properties (S18), (S19), yields $(z+\delta)\psi_z\geq\psi_z^2$. Hence $-\psi_z(z+\delta)\leq -\psi_z^2$. Using this property in Eq (S17), yields

$$\frac{dV_z}{dt}\leq(-1)\omega|b|\psi_z^2\leq 0$$

Using the definition of V_z (S14), we have

$$\frac{dV_z}{dt}\leq -2\omega|b|V_z\leq 0 \tag{S20}$$

Hence, $V_z\leq V_{z|to}e^{-2\omega\int_{to}^{t}|b|dt}$. Using the property (3), yields $V_z\leq V_{z|to}e^{-2\omega b_{min}(t-t_0)}$. Using the definition of V_z (S14) and applying square root, yields

$$|\psi_z|\leq\left|\psi_{z|to}\right|e^{-\omega b_{min}(t-t_0)} \tag{S21}$$

Therefore, $V_z\in\mathcal{L}_\infty$ and V_z converges to zero. From this result and the definition of V_z (S14), it follows that $\psi_z\in\mathcal{L}_\infty$ and ψ_z converges to zero. This and the definition of ψ_z (S15) imply that $z\in\mathcal{L}_\infty$ and z converges to $\Omega_z=[z^l,\ z^u],\ z^l=-\delta_{max}\leq 0,\ z^u=-\delta_{min}\geq 0$. This completes the proof of Tii.

From the definition of ψ_z (S15) it follows that $|z|\leq|\psi_z|+\max\{-\delta_{min},\delta_{max}\}$. Substituting the equation for the convergence of ψ_z (S21), yields

$$|z|\leq\left|\psi_{z|to}\right|e^{-\omega b_{min}(t-t_0)}+\max\{-\delta_{min},\delta_{max}\} \tag{S22}$$

This completes the proof. □

Remark 7. *The convergence region of z, that is Ω_z, depends on the bounds of δ (19), hence on the bounds of $\delta_2/(b\omega)$ and δ_1/b. Consequently, its width can be reduced to some extent by choosing a high ω value, but it cannot be made arbitrarily small.*

Remark 8. *The convergence rate of z is given by the dz/dt expression (21) and the definition of δ (19): i) large values of ω increase the convergence rate of z; ii) large values of ω decrease the effect of disturbance δ_2, but not the effect of disturbance term δ_1.*

Theorem 2. [Convergence of $\bar{x}_1$ and boundedness of the updated parameter]. *Consider the model (1), (2) subject to assumptions 1 to 3 and the observer (5)-(7), with definitions (8)-(10). As a result of this observer: Ti) the updated parameter $\hat{\theta}_\delta$ remains bounded; Tii) the observer error $\bar{x}_1 = \hat{x}_1 - x_1$ asymptotically converges to $\Omega_{x1} = [-\varepsilon,\ \varepsilon]$; Tiii) the bound of the transient response of $\bar{x}_1$ is: $|\bar{x}_1| \le \varepsilon + \sqrt{\psi_{x1|to}^2 + \psi_{z|to}^2 + \gamma^{-1}\tilde{\theta}_{to}^2}$, where $\psi_{x1|to}$, $\psi_{z|to}, \tilde{\theta}_{to}$ are the initial values of ψ_{x1}, ψ_z, $\tilde{\theta}_\delta$, respectively, ψ_{x1} is given by*

$$\psi_{x1} = \begin{cases} \bar{x}_1 - \varepsilon \ \ for\ \bar{x}_1 \ge \varepsilon \\ 0 \ \ for\ \bar{x}_1 \ \epsilon\ [-\varepsilon, \varepsilon] \\ \bar{x}_1 + \varepsilon \ \ for\ \bar{x}_1 \le -\varepsilon \end{cases},$$

$\tilde{\theta}_\delta$ is $\tilde{\theta}_\delta = \hat{\theta}_\delta - \theta_\delta$, and θ_δ is the upper bound of $(-\delta_{zt} - \delta_1/b)$.

Proof. Using the defined function g_1 (S10), the x_1 observer (S11) takes the form

$$\frac{d\hat{x}_1}{dt} = b\hat{x}_2 - b\left(\sigma\omega\bar{x}_1 + \frac{1}{\sigma\omega}g_2\right) + h_1$$

and the dynamics of $\bar{x}_1$ (S13) takes the form

$$\frac{d\bar{x}_1}{dt} = b\bar{x}_2 - b\left(\sigma\omega\bar{x}_1 + \frac{1}{\sigma\omega}g_2\right) - \delta_1 \tag{S23}$$

It remains to define the function g_2. The $\bar{x}_1$ dynamics (S23) can be rewritten as

$$\frac{d\bar{x}_1}{dt} = b\left(z - \frac{1}{\sigma\omega}g_2 - \frac{\delta_1}{b}\right) \tag{S24}$$

Where z is defined in Eq (S18). The subsystem Lyapunov function for $\bar{x}_1$ and z is selected:

$$V_{zx1} = V_{x1} + V_z \tag{S25}$$

where

$$V_{x1} = \frac{1}{2}\psi_{x1}^2 \tag{S26}$$

$$\psi_{x1} = \begin{cases} \bar{x}_1 - \varepsilon \ \ for\ \bar{x}_1 \ge \varepsilon \\ 0 \ \ for\ \bar{x}_1 \ \epsilon\ [-\varepsilon, \varepsilon] \\ \bar{x}_1 + \varepsilon \ \ for\ \bar{x}_1 \le -\varepsilon \end{cases} \tag{S27}$$

and V_z is defined in Eq (S14). The main properties of V_{x1} are:

$$V_{x1} = 0 \ \ for\ \bar{x}_1 \ \epsilon\ [-\varepsilon, \varepsilon]$$
$$V_{x1} > 0 \ \ for\ \bar{x}_1 \ \notin\ [-\varepsilon, \varepsilon]$$
$$V_{x1}\ is\ continuous\ with\ respect\ to\ \bar{x}_1$$

Differentiating V_{zx1} (S25) and V_{x1} (S26) with respect to time, yields

$$\frac{dV_{zx1}}{dt} = \frac{dV_z}{dt} + \frac{dV_{x1}}{dt}, \qquad \frac{dV_{x1}}{dt} = \psi_{x1}\frac{d\bar{x}_1}{dt}$$

Combining these expressions with the expressions for dV_z/dt (S20) and $d\bar{x}_1/dt$ (S24), yields:

$$\frac{dV_{zx1}}{dt} \le -2\omega|b|V_z + \psi_{x1}b\left(z - \frac{1}{\sigma\omega}g_2 - \frac{\delta_1}{b}\right) \tag{S28}$$

Let

$$\delta_{zt} = \psi_z - z \tag{S29}$$

Using the definition of ψ_z (S15), yields

$$\delta_{zt} = \begin{cases} \delta_{min}\ for\ \ z \ge -\delta_{min} \ge 0 \\ -z\ \ for\ \ z\ \in (-\delta_{max}, -\delta_{min}) \\ \delta_{max}\ for\ \ z \le -\delta_{max} \le 0 \end{cases}$$

Hence, $\delta_{zt} \in [\delta_{min}, \delta_{max}]$, and

$$|\delta_{zt}| \le \theta_{\delta zt} \tag{S30}$$
$$\theta_{\delta zt} = \max\{-\delta_{min}, \delta_{max}\}$$

From the definition of δ_{zt} (S29), it follows that $z = \psi_z - \delta_{zt}$. Substituting into Eq (S28) and arranging, yields:

$$\frac{dV_{zx1}}{dt} \le -\omega|b|\left(\psi_z^2 - \frac{1}{\sigma\omega}\psi_{x1}\psi_z\right) + b\psi_{x1}\left(-\frac{1}{\sigma\omega}g_2 - \delta_{zt} - \frac{\delta_1}{b}\right) \tag{S31}$$

The term $\psi_z^2 - (1/(\sigma\omega))\psi_{x1}\psi_z$ can be rewritten as

$$\psi_z^2 - \frac{1}{\sigma\omega}\psi_{x1}\psi_z = \left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2 - \frac{1}{4\omega^2}\psi_{x1}^2$$

Substituting into Eq. (S31) and arranging, yields

$$\begin{aligned}\frac{dV_{zx1}}{dt} &\le |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2 + \frac{1}{4\omega}\psi_{x1}^2\right)\\ &+|b|\frac{1}{\sigma}\psi_{x1}\left(k\sigma\psi_{x1} - \frac{1}{\sigma\omega}g_2\right) + b\psi_{x1}\left(-\delta_{zt} - \frac{\delta_1}{b}\right)\end{aligned} \tag{S32}$$

where $-\delta_{zt}, \delta_1, 1/b$ are bounded as indicated by Eqs (S30), (S4), (S2), so that $(-\delta_{zt} - \delta_1/b)$ is bounded and:

$$b\psi_{x1}\left(-\delta_{zt} - \frac{\delta_1}{b}\right) \le |b|\ |\psi_{x1}|\ \left|-\delta_{zt} - \frac{\delta_1}{b}\right| \le \theta_\delta|\psi_{x1}|\ |b| \tag{S33}$$

where θ_δ is the upper bound of $(-\delta_{zt} - \delta_1/b)$, and it is unknown, positive, and constant: $|-\delta_{zt} - \delta_1/b| \le \theta_\delta$. Let

$$\tilde{\theta}_\delta = \hat{\theta}_\delta - \theta_\delta \tag{S34}$$

Hence $\theta_\delta = \hat{\theta}_\delta - \tilde{\theta}_\delta$. Substituting into Eq (S33), yields

$$\begin{aligned}b\psi_{x1}\left(-\delta_{zt} - \frac{\delta_1}{b}\right) &\le |b|(\hat{\theta}_\delta - \tilde{\theta}_\delta)\,|\psi_{x1}| = |b|\ |\psi_{x1}|\hat{\theta}_\delta - |\psi_{x1}|\tilde{\theta}_\delta|b|\\ &= |b|\frac{1}{\sigma}\psi_{x1}\sigma sgn(\psi_{x1})\hat{\theta}_\delta - |\psi_{x1}|\tilde{\theta}_\delta|b|\end{aligned}$$

Substituting into Eq (S32) and arranging, yields

$$\begin{aligned}\frac{dV_{zx1}}{dt} &\le |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2 + \frac{1}{4\omega}\psi_{x1}^2\right)\\ &+|b|\left(\frac{1}{\sigma}\psi_{x1}\left(k\sigma\psi_{x1} - \frac{1}{\sigma\omega}g_2 + \sigma sgn(\psi_{x1})\hat{\theta}_\delta\right) - |\psi_{x1}|\tilde{\theta}_\delta\right)\end{aligned}$$

Arranging, yields

$$\begin{aligned}\frac{dV_{zx1}}{dt} &\le |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2\right)\\ &+|b|\left(\psi_{x1}\left(\frac{1}{4\omega}\psi_{x1} + k\psi_{x1} - \frac{1}{\omega}g_2\right) + |\psi_{x1}|\hat{\theta}_\delta - |\psi_{x1}|\tilde{\theta}_\delta\right)\end{aligned} \tag{S35}$$

Which is equivalent to

$$\begin{aligned}\frac{dV_{zx1}}{dt} &\le |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2\right)\\ &+|b|\left(\psi_{x1}\left(\frac{1}{4\omega}\psi_{x1} + k\psi_{x1} - \frac{1}{\omega}g_2 + sgn(\psi_{x1})\hat{\theta}_\delta\right) - |\psi_{x1}|\tilde{\theta}_\delta\right)\end{aligned} \tag{S36}$$

This equation would lead to a g_2 definition involving the discontinuous signal $sgn(\psi_{x1})$. To remedy this, we will use a saturation signal instead, but guaranteeing that the negativeness of dV_{zx1}/dt is not affected. From definition (S27), it follows that $|\psi_{x1}|$ satisfies:

$$|\psi_{x1}| = sat_{x1}\psi_{x1} \tag{S37}$$

$$where$$

$$sat_{x1} = \begin{cases} 1\ \ for\ \bar{x}_1 \ge \varepsilon \\ \frac{1}{\varepsilon}\bar{x}_1\ \ for\ \ \bar{x}_1 \in\ [-\varepsilon, \varepsilon] \\ -1\ \ for\ \bar{x}_1 \le -\varepsilon \end{cases}$$

Substituting (S37) into Eq (S35), yields

$$\begin{aligned}\frac{dV_{zx1}}{dt} &\le |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2\right)\\ &+|b|\left(\psi_{x1}\left(\frac{1}{4\omega}\psi_{x1} + k\psi_{x1} + sat_{x1}\hat{\theta}_\delta - \frac{1}{\omega}g_2\right) - |\psi_{x1}|\tilde{\theta}_\delta\right)\end{aligned} \tag{S38}$$

To counteract the effect of the term $\psi_{x1}\left(\frac{1}{4\omega}\psi_{x1} + k\psi_{x1} + sat_{x1}\hat{\theta}_\delta\right)$, we choose

$$g_2 = \omega\left(\left(k + \frac{1}{4\omega}\right)\psi_{x1} + sat_{x1}\hat{\theta}_\delta\right)$$

Substituting into Eq (S38), yields

$$\frac{dV_{zx1}}{dt} \leq |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2 - |\psi_{x1}|\tilde{\theta}_\delta\right) \tag{S39}$$

We choose the subsystem Lyapunov function for $\bar{x}_1$, z, $\tilde{\theta}_\delta$ to be:

$$V_{z\theta x1} = V_{zx1} + V_\theta, \tag{S40}$$

$$V_\theta = \frac{1}{2}\gamma^{-1}\tilde{\theta}_\delta^2 \tag{S41}$$

where V_{zx1} is defined in Eq (S25) and $\tilde{\theta}_\delta$ in Eq (S34). Differentiating (S40) with respect to time, yields

$$\frac{dV_{z\theta x1}}{dt} = \frac{dV_{zx1}}{dt} + \frac{dV_\theta}{dt} \tag{S42}$$

Differentiating V_θ (S41) with respect to time, using the definition of $\tilde{\theta}_\delta$ (S34), we get

$$\frac{dV_\theta}{dt} = \gamma^{-1}\tilde{\theta}_\delta\frac{d\hat{\theta}_\delta}{dt} \tag{S43}$$

Substituting the expression for dV_{zx1}/dt (S39) and dV_θ/dt (S43) into Eq (S42) and arranging, yields

$$\frac{dV_{z\theta x1}}{dt} \leq |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2\right) + \tilde{\theta}_\delta\left(-|b||\psi_{x1}| + \gamma^{-1}\frac{d\hat{\theta}_\delta}{dt}\right) \tag{S44}$$

To eliminate the effect of the term $\tilde{\theta}_\delta(-|b||\psi_{x1}|)$, we choose the update law

$$\frac{d\hat{\theta}_\delta}{dt} = \gamma|b||\psi_{x1}|$$

Substituting into Eq (S44), yields

$$\frac{dV_{z\theta x1}}{dt} \leq |b|\left(-k\psi_{x1}^2 - \omega\left(\psi_z - \frac{1}{2\sigma\omega}\psi_{x1}\right)^2\right) \tag{S45}$$

In summary, the observer equations are:

$$\frac{d\hat{x}_1}{dt} = b\hat{x}_2 - |b|\left(\omega\bar{x}_1 + \left(k + \frac{1}{4\omega}\right)\psi_{x1} + sat_{x1}\hat{\theta}_\delta\right) + h_1$$

$$\frac{d\hat{x}_2}{dt} = -b\omega\left(\left(k + \frac{1}{4\omega}\right)\psi_{x1} + sat_{x1}\hat{\theta}_\delta\right) + h_2$$

$$\frac{d\hat{\theta}_\delta}{dt} = \gamma|b||\psi_{x1}|$$

$$g_2 = \omega\left(\left(k + \frac{1}{4\omega}\right)\psi_{x1} + sat_{x1}\hat{\theta}_\delta\right)$$

$$sat_{x1} = \begin{cases} 1 \ for\ \bar{x}_1 \geq \varepsilon \\ \frac{1}{\varepsilon}\bar{x}_1 \ for\ \bar{x}_1 \in [-\varepsilon, \varepsilon] \\ -1 \ for\ \bar{x}_1 \leq -\varepsilon \end{cases}$$

$$\psi_{x1} = \begin{cases} \bar{x}_1 - \varepsilon \ for\ \bar{x}_1 \geq \varepsilon \\ 0 \ for\ \bar{x}_1 \in [-\varepsilon, \varepsilon] \\ \bar{x}_1 + \varepsilon \ for\ \bar{x}_1 \leq -\varepsilon \end{cases}$$

We prove that $\bar{x}_1$ converges to $\Omega_{x1} = [-\varepsilon, \varepsilon]$ at what follows. From Eq (S45) it follows that $\dot{V}_{z\theta x1} \leq -k|b|\psi_{x1}^2 \leq 0$. Since coefficient b satisfies property (3), we have

$$\frac{dV_{z\theta x1}}{dt} \leq -kb_{min}\psi_{x1}^2 \leq 0 \tag{S46}$$

Arranging and integrating, yields $V_{z\theta x1} + kb_{min}\int_{to}^{t}\psi_{x1}^2 dt \leq V_{z\theta x1|to}$ where $V_{z\theta x1|to}$ is the initial value of $V_{z\theta x1}$. Hence,

$$V_{z\theta x1} \leq V_{z\theta x1|to} \tag{S47}$$

and $kb_{min}\int_{to}^{t}\psi_{x1}^2 dt \leq V_{z\theta x1|to}$, so that $V_{z\theta x1} \in \mathcal{L}_\infty$, $\psi_{x1}^2 \in \mathcal{L}_1$. This implies:

i) $V_{zx1} \in \mathcal{L}_\infty$, $V_\theta \in \mathcal{L}_\infty$, $V_z \in \mathcal{L}_\infty$, $V_{x1} \in \mathcal{L}_\infty$, what follows from the definitions of $V_{z\theta x1}$(S40) and V_{zx1} (S25)

ii) $\psi_z \in \mathcal{L}_\infty$, $z \in \mathcal{L}_\infty$, what follows from the definitions of $V_{z\theta x1}$(S40), V_{zx1} (S25), V_z(S14), and ψ_z (S15)

iii) $\psi_{x1} \in \mathcal{L}_\infty$, $\bar{x}_1 \in \mathcal{L}_\infty$, $\psi_{x1}^2 \in \mathcal{L}_\infty$, what follows from the definitions of V_{x1} (S26) and ψ_{x1} (S27)

iv) $\tilde{\theta}_\delta \in \mathcal{L}_\infty$, $\hat{\theta}_\delta \in \mathcal{L}_\infty$, what follows from the definitions of V_θ (S41) and $\tilde{\theta}_\delta$ (S34),

This completes the proof of Ti.

To apply the Barbalat's lemma, the fulfilment of condition $d\psi_{x1}^2/dt \in \mathcal{L}_\infty$ is required, which requires the fulfillment of $d\bar{x}_1/dt \in \mathcal{L}_\infty$. The latter condition is verified based on Eq. (S24) and the boundedness of the signals involved therein. Applying the Barbalat's lemma ([28]), yields

$$\lim_{t\to\infty} \psi_{x1}^2 = 0 \tag{S48}$$

This result and the definition of ψ_{x1} (S27) imply that $\bar{x}_1$ converges asymptotically to $\Omega_{x1} = [-\varepsilon, \varepsilon]$. This completes the proof of Tii.

From the definition of ψ_{x1} (S27) it follows that

$$|\bar{x}_1| \le |\psi_{x1}| + \varepsilon \tag{S49}$$

Using the definitions of $V_{z\theta x1}$ (S40), V_{zx1} (S25), V_θ (S41), V_{x1} (S26), V_z (S14) in Eq (S47), yields:

$$\frac{1}{2}\psi_{x1}^2 + \frac{1}{2}\psi_z^2 + \frac{1}{2}\gamma^{-1}\tilde{\theta}^2 \le \frac{1}{2}\psi_{x1|to}^2 + \frac{1}{2}\psi_{z|to}^2 + \frac{1}{2}\gamma^{-1}\tilde{\theta}_{to}^2$$

where $\psi_{x1|to}$, $\psi_{z|to}$, $\tilde{\theta}_{to}$ are the initial values of ψ_{x1}, ψ_z, $\tilde{\theta}_b$. Therefore, $|\psi_{x1}| \le \left(\psi_{x1|to}^2 + \psi_{z|to}^2 + \gamma^{-1}\tilde{\theta}_{to}^2\right)^{1/2}$. Combining this result with Eq (S49), yields

$$|\bar{x}_1| \le \varepsilon + \sqrt{\psi_{x1|to}^2 + \psi_{z|to}^2 + \gamma^{-1}\tilde{\theta}_{to}^2} \tag{S50}$$

This completes the proof. □

Theorem 3. [Convergence of $\bar{x}_2$: upper bound of the transient response and convergence region]. *Consider the model (1), (2) subject to assumptions 1 to 3 and the observer (5)-(7), with definitions (8)-(10). As a result of this observer: Ti) the transient response of the observer error $\bar{x}_2 = \hat{x}_2 - x_2$ satisfies:*

$$|\bar{x}_2| \le |\psi_{z|to}| e^{-\omega b_{min}(t-t_0)} + max\{-\delta_{min}, \delta_{max}\} + \omega\left(\varepsilon + \sqrt{\psi_{x1|to}^2 + \psi_{z|to}^2 + \gamma^{-1}\tilde{\theta}_{to}^2}\right),$$

where $\delta = \frac{1}{b}\left(\frac{\delta_2}{\sigma\omega} - \delta_1\right)$; *Tii)* $|\bar{x}_2| \le |\psi_{z|to}| e^{-\omega b_{min}(t-t_0)} + max\{-\delta_{min}, \delta_{max}\} + \omega(\varepsilon + |\psi_{z|to}|)$ *holds true for* $\hat{x}_{1|to} \in [-\varepsilon + x_{1|to},\ \varepsilon + x_{1|to}]$ *and a high γ value leading to $\gamma^{-1}\tilde{\theta}_{to}^2 \approx 0$; Tiii) the observer error $\bar{x}_2$ asymptotically converges to*

$\Omega_{x2} = \{\bar{x}_2 : |\bar{x}_2| \le max\{-\delta_{min}, \delta_{max}\} + \omega\varepsilon\}$.

Proof. From the definition of z (S18), it follows that

$$|\bar{x}_2| = |z - \sigma\omega\bar{x}_1| \le |z| + |\omega\bar{x}_1| = |z| + \omega|\bar{x}_1| \tag{S51}$$

Recall that z can be expressed as function of ψ_z and $\bar{x}_1$ as function of ψ_{x1}. Substituting Eqs (S22), (S50) into Eq (S51), yields

$$|\bar{x}_2| \le |\psi_{z|to}| e^{-\omega b_{min}(t-t_0)} + \max\{-\delta_{min}, \delta_{max}\} + \omega\left(\varepsilon + \sqrt{\psi_{x1|to}^2 + \psi_{z|to}^2 + \gamma^{-1}\tilde{\theta}_{to}^2}\right) \tag{S52}$$

This completes the proof of Ti.

Low values of the terms $\psi_{x1|to}$ and $\gamma^{-1}\tilde{\theta}_{to}^2$ appearing in Eq (S52) can be obtained as follows:

i) $\psi_{x1|to} = 0$ if $\hat{x}_{1|to} \in [-\varepsilon + x_{1|to},\ \varepsilon + x_{1|to}]$, what implies $\bar{x}_{1|to} \in [-\varepsilon,\ \varepsilon]$
ii) a low value $\gamma^{-1}\tilde{\theta}_{to}^2 \approx 0$ is obtained by using a high γ value

Substituting these values into Eq (S52), yields

$$|\bar{x}_2| \le |\psi_{z|to}| e^{-\omega b_{min}(t-t_0)} + \max\{-\delta_{min}, \delta_{max}\} + \omega(\varepsilon + |\psi_{z|to}|)$$
$$\text{for } \hat{x}_{1|to} \in [-\varepsilon + x_{1|to},\ \varepsilon + x_{1|to}] \text{ and a high } \gamma \text{ value leading to } \gamma^{-1}\tilde{\theta}_{to}^2 \approx 0 \tag{S53}$$

This completes the proof of Tii.

Expression (S51), jointly with the convergence of z to Ω_z (See Theorem 1) and the convergence of $\bar{x}_1$ to $\Omega_{x1} = [-\varepsilon, \varepsilon]$ (Theorem 2) implies that $\bar{x}_2$ converges to

$$\Omega_{x2} = \{\bar{x}_2 : |\bar{x}_2| \leq \max\{-\delta_{min}, \delta_{max}\} + \omega\varepsilon\} \tag{S54}$$

This completes the proof of Tiii. □