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Communication

Thermodynamic Reversibility in Polarimetry

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Abstract: The action of linear media on incident polarized electromagnetic waves can produce two kinds of thermodynamic irreversible effects, namely, loss of intensity, in general anisotropic, and reduction of the degree of polarization. Even though both phenomena can be described through specific properties, the overall degree of reversibility of polarimetric interactions can be characterized by means of a single parameter whose minimum and maximum values are achieved by fully irreversible and reversible polarimetric transformations, respectively. Furthermore, the sources of irreversibility associated to the entire family of Mueller matrices proportional to a given one are identified, leading to the definition of the specific reversibility as the square average of the degree of polarimetric purity and the polarimetric dimension index. The feasible values of the degree of reversibility with respect to the mean intensity coefficient and the degree of polarimetric purity are analyzed graphically, and the iso-reversibility branches are identified and analyzed. Furthermore, the behavior of the specific reversibility with respect to the achievable values of the polarimetric dimension index and the degree of polarizance is described by means of the purity figure, and it is compared to the iso-purity elliptical branches in such figure.

Keywords: Mueller matrices; polarimetry; diattenuation; polarizance; depolarization

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1. Introduction

The concept of thermodynamic irreversibility was formerly analyzed by Jones [1] in the context of optical phenomena, including attenuation of the intensity and reduction of the degree of polarization as two effects that are directly related to polarization.

In fact, the second-order description of the state of polarization of a plane wave is given by the corresponding Stokes vector, which combines information on the intensity, average polarization ellipse, and degree of polarization. Ideal processes where both intensity and degree of polarization are preserved and only changes of the shape and orientation of the average polarization ellipse occur can be reversed through the action of a transparent retarder. All other kinds of polarimetric transformations involve necessarily changes on either the intensity or the degree of polarization for certain incident totally polarized states and therefore they can be considered irreversible. In this communication, the concepts of polarimetric reversibility and irreversibility are revisited in the light of the recent advances in polarization theory and Stokes–Mueller algebra.

Linear polarimetric interactions are characterized by means of the corresponding Mueller matrices, which encompass all the measurable information regarding the changes of the Stokes parameters of the polarized light probe for each given interaction conditions (angle of incidence, spectral profile of light, angle of observation, spot-size of the sample, measurement time, etc.) As indicated above, two kinds of irreversible effects may occur (either combined or in an independent manner), namely, the loss of intensity of the interacting light and the reduction of the degree of polarization.

In general, polarimetric interactions in nature are passive, that is, a part of the intensity of the incident light beam is lost in the interaction, so that it cannot be recovered through a subsequent natural linear interaction [2–7]. Thus, media exhibiting attenuation (isotropic) or diattenuation (anisotropic) produce this kind of irreversible effects linked to the loss of intensity. An example of diattenuating interaction is the reflection of a plane wave on a plane surface, for which the Fresnel equations provide the angle-dependent reflectance and transmittance for both *s* and *p* components of the incident beam. This phenomenon preserves the degree of polarization of totally polarized incident light, while producing anisotropic loss of intensity, so that the process is reversible from the point of view of the polarimetric purity and irreversible from the point of view of intensity.

On the other hand, at least from a theoretical point of view, certain polarimetric interactions preserve the intensity while reducing the degree of polarization. An example is a parallel composition of transparent retarders, whose Mueller matrix is given by the convex sum of the Mueller matrices of the components and exhibits zero attenuation (and consequently zero diattenuation), whereas it reduces the degree of polarization of the incident light [8].

The above examples, together with other interesting physical situations, will be analyzed in further sections devoted to the main aim of this work, which consists of characterizing the reversibility of polarimetric transformations and introducing appropriate and physically significant descriptors.

The contents of this communication are organized as follows. Section 2 contains a summary of the concepts and notations that are necessary for the required developments in later sections. Section 3 is devoted to revisiting the concept of polarimetric reversibility, which leads to the definition, in Section 4, of the degree of polarimetric reversibility, whose analysis and graphical representation are also considered in the same section. Section 5 deals with the introduction of the concept of specific reversibility and the study of its behavior as a function of the so-called components of purity [8,9]. Sections 6 and 7 are devoted to the discussion and conclusions, respectively.

Since the Stokes–Mueller algebra applies to any kind of linear polarimetric transformations of electromagnetic waves, the results presented are not limited to the optical range but apply to the entire electromagnetic spectrum.

2. Theoretical Background

Let us consider the transformation of polarized light by the action of a linear media (under fixed interaction conditions). It can always be formulated as $\mathbf{s'} = \mathbf{M}\mathbf{s}$ where $\mathbf{s}$ and $\mathbf{s'}$ are the Stokes vectors that represent the states of polarization of the incident and emerging light beams, respectively, while $\mathbf{M}$ is the Mueller matrix associated with this kind of interaction and which can always be expressed as [10–12]

$$\begin{gathered}
\mathbf{M} = m_{00}\,\hat{\mathbf{M}}, \quad \hat{\mathbf{M}} \equiv \begin{pmatrix} 1 & \mathbf{D}^T \\ \mathbf{P} & \mathbf{m} \end{pmatrix}, \\
\mathbf{m} \equiv \frac{1}{m_{00}} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \\
\mathbf{D} \equiv \frac{(m_{01}, m_{02}, m_{03})^T}{m_{00}}, \ \mathbf{P} \equiv \frac{(m_{10}, m_{20}, m_{30})^T}{m_{00}},
\end{gathered} \tag{1}$$

where $m_{ij}\ (i, j = 0, 1, 2, 3)$ are the elements of $\mathbf{M}$; the superscript T indicates transpose; m_{00} is the mean intensity coefficient (MIC), i.e., the ratio between the intensity of the emerging light and the intensity of incident unpolarized light; $\mathbf{D}$ and $\mathbf{P}$ are the diattenuation and polarizance vectors, with absolute values D (diattenuation) and P (polarizance); and $\mathbf{m}$ is the normalized 3×3 submatrix associated with $\mathbf{M}$.

The passivity constraints (natural linear polarimetric interactions do not amplify the intensity of light) are completely characterized by the inequality $m_{00}(1 + Q) \leq 1$ [5,13], where $Q \equiv \max(D, P)$. Thus, $m_{00} \leq 1/(1 + Q)$, which implies that media exhibiting nonzero polarizance or diattenuation necessarily feature $m_{00} < 1$, while the limit $m_{00} = 1$ corresponds to *transparent* Mueller matrices, which have the general form

$$\mathbf{M} \equiv \begin{pmatrix} 1 & 0^T \\ 0 & \mathbf{m} \end{pmatrix}. \quad (2)$$

Some additional concepts and descriptors that will be useful for further developments and discussions are briefly reviewed below.

Leaving aside systems exhibiting magneto-optic effects, given a Mueller matrix $\mathbf{M}$, the Mueller matrix that represents the same linear interaction as $\mathbf{M}$, but with the incident and emergent directions of propagation of the electromagnetic wave interchanged, is given by [14–16]

$$\mathbf{M}^r = \mathrm{diag}(1,1,-1,1)\ \mathbf{M}^T\, \mathrm{diag}(1,1,-1,1), \quad (3)$$

and consequently, the diattenuation (polarizance) of $\mathbf{M}^r$ coincides with the polarizance (diattenuation) of $\mathbf{M}$, showing that D and P share a common essential nature related to the ability of the medium to enpolarize (increase the degree of polarization) unpolarized light incoming in either forward or reverse directions [16]. Since magneto-optic effects only affect to the sign of certain elements of $\mathbf{M}$, this does not alter D, P and other quantities considered below (when applied to $\mathbf{M}^r$), which are defined from square averages of some Mueller matrix elements.

A useful combined measure of diattenuation–polarizance, is given by the degree of polarizance P_P, or *enpolarizance*, defined as [8]

$$P_P \equiv \sqrt{\frac{D^2 + P^2}{2}}, \quad (4)$$

Regarding the ability of $\mathbf{M}$ to preserve the degree of polarization (DOP) of totally polarized incident light, a proper measure is given by the *degree of polarimetric purity* of $\mathbf{M}$ (also called *depolarization index*) [17], P_Δ, which can be expressed as

$$P_\Delta = \sqrt{\frac{D^2 + P^2 + 3P_S^2}{3}} = \sqrt{\frac{2P_P^2}{3} + P_S^2}, \quad (5)$$

where P_S is the *polarimetric dimension index* (also called the *degree of spherical purity*), defined as [9,16]

$$P_S \equiv \frac{\|\mathbf{m}\|_2}{\sqrt{3}} \qquad \left[\|\mathbf{m}\|_2 \equiv \frac{1}{m_{00}} \sqrt{\sum_{k,l=1}^{3} m_{kl}^2} \right], \quad (6)$$

$\|\mathbf{m}\|_2$ being the Frobenius norm of $\mathbf{m}$.

Because of the fundamental role that P_S plays in subsequent analyses, it is worth considering it in more detail by noting that the name degree of spherical purity originally coined for P_S [9,16] comes from the fact that both the I-image and the characteristic ellipsoid [16,18] associated with $\mathbf{M}$ are spherical if and only if $P_S = 1$. On the other hand, from a statistical point of view, P_S can be expressed as [16,19]

$$P_S = \frac{\sqrt{\sum\limits_{\substack{k,l=0 \\ k<l}}^{3} \left(\sigma_k^2 - \sigma_l^2\right)^2}}{\sqrt{3} \sum\limits_{k=0}^{3} \sigma_k^2} \qquad \left(\sum_{k=0}^{3} \sigma_k^2 = m_{00} \right), \quad (7)$$

where σ_k^2 represent the variances of the coherency matrix (denoted by $\mathbf{C}$) associated with $\mathbf{M}$, so that P_S formally adopts the form of the Mueller-algebra version of the dimensionality index defined originally for 3D polarization matrices [20] and then generalized for n-

dimensional density matrices [21]. In contrast to the generic dimensionality index of a density matrix ρ defined from the variances of the symmetric matrix constituting the real part of ρ, P_S is defined from the variances of **C**, which determine the diagonal elements of **M** by means of simple linear relations [19]

$$\begin{aligned} m_{00} &= \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2, \\ m_{11} &= \sigma_0^2 + \sigma_1^2 - \sigma_2^2 - \sigma_3^2, \\ m_{22} &= \sigma_0^2 - \sigma_1^2 + \sigma_2^2 - \sigma_3^2, \\ m_{33} &= \sigma_0^2 - \sigma_1^2 - \sigma_2^2 + \sigma_3^2. \end{aligned} \tag{8}$$

The *depolarizance* is defined as

$$D_\Delta = \sqrt{1 - P_\Delta^2}, \tag{9}$$

giving a measure of the overall ability of the medium to depolarize light, which constitutes the natural quadratic counterpart of P_Δ. The entropy-like version of D_Δ is the polarization entropy S_Δ, which increases monotonically as D_Δ increases. The detailed description of S_Δ and its relation to P_Δ via the indices of polarimetric purity [22] can be found in [16].

The parameters m_{00}, D, P, P_P, P_Δ, D_Δ, and P_S take their achievable values in the interval $[0,1]$. In particular, $m_{00} = 1$ for transparent media and $m_{00} = 0$ for opaque media (zero Mueller matrix); $P_P = 1$ corresponds to perfect polarizers $(D = P = 1)$, while the minimal value $P_P = 0$ is taken by nonenpolarizing media $(D = P = 0)$. Note that, from the abovementioned passivity condition, $m_{00} = 1$ implies $P_P = 0$, while $P_P = 1$ implies $m_{00} \leq 1/2$. The maximal degree of polarimetric purity, $P_\Delta = 1$, is exhibited uniquely by *nondepolarizing* (or *pure*) media (i.e., media that do not decrease de degree of polarization of totally polarized incident light), while $P_\Delta = 0$ is characteristic of perfect depolarizers, which are represented by Mueller matrices of the form $\mathbf{M}_{\Delta 0} = m_{00}\,\mathrm{diag}\,(1,0,0,0)$. Moreover, $P_S = 1$ corresponds uniquely to retarders (regardless of the value of m_{00}, i.e., regardless of whether they are transparent or exhibit certain amount of isotropic attenuation), and $P_S = 0$ corresponds to media exhibiting $\mathbf{m} = 0$.

3. The Concept of Polarimetric Reversibility

We will say that a linear polarimetric interaction, represented by a given Mueller matrix **M**, is reversible when there are two transformations $\mathbf{M}'$ and $\mathbf{M}''$ such that $\mathbf{M}\,\mathbf{M}' = \mathbf{I}$, and $\mathbf{M}''\mathbf{M} = \mathbf{I}$, where **I** is the identity matrix; that is, the corresponding serial combinations of the medium with those represented by $\mathbf{M}'$ and $\mathbf{M}''$ produce a completely neutral effect. From Mueller–Stokes algebra, it is well known that the only media that behave as reversible are the transparent retarders, whose Mueller matrices, hereafter denoted by $\mathbf{M}_R$, are proper orthogonal in the sense that they satisfy $\mathbf{M}_R^T = \mathbf{M}_R^{-1}$, with $\det\mathbf{M}_R = +1$ [16]. Thus, the general form of the Mueller matrix of a transparent retarder is

$$\mathbf{M}_R = \begin{pmatrix} 1 & 0^T \\ 0 & \mathbf{m}_R \end{pmatrix}, \quad \mathbf{m}_R^T = \mathbf{m}_R^{-1}, \quad \det\mathbf{m}_R = +1. \tag{10}$$

Consequently, the term *reversible* will be applied exclusively when **M** represents a transparent retarder, viz. $\mathbf{M} = \mathbf{M}_R$. Retarders exhibiting certain isotropic attenuation have the general form $m_{00}\mathbf{M}_R$, with $m_{00} < 1$.

It should be noted that the present work is focused on the thermodynamic reversibility of polarimetric transformations and not to retroreflection properties [23] nor the interchange of incident and emergent directions of the light probe, where, as seen above, the term *reverse* is commonly used to refer to such change [14–17,23–26].

As mentioned in the introduction, the thermodynamic polarimetric reversibility–irreversibility can be analyzed from two complementary points of view, namely, intensity and degree of polarization. Thus, for the sake of clarity, we will distinguish different kinds of transformations.

Transformations corresponding to pure Mueller matrices $(P_\Delta = 1)$, denoted as $\mathbf{M}_J$, will be said to be *DOP*-reversible, because they preserve the degree of polarization of totally polarized incident light (regardless of whether they present any kind of intensity attenuation or not).

Transparent transformations, characterized by $m_{00} = 1$ (which implies $P_P = 0$), will be said to be *intensity-reversible* (denoted as *I*-reversible, when appropriate, for brevity) because they do not alter the intensity of the light probe (regardless of whether they decrease the degree of polarization or not). The intensity-irreversible polarimetric transformations include the subclass of *intensity-normalizable-reversible* (denoted as *IN*-reversible), which are represented by Mueller matrices satisfying $P_P = 0$ and $m_{00} < 1$.

Since any given linear polarimetric transformation is fully characterized by its associated Mueller matrix, the above names are also applied to the Mueller matrices themselves. Table 1 summarizes the definitions and links among the different classes of polarimetric reversibility.

Table 1. Classification of Mueller matrices in terms of their reversibility-irreversibility properties.

<table>
<tr><td colspan="2"></td><td>DOP-reversible
$P_\Delta = 1$</td><td>DOP-irreversible
$P_\Delta < 1$</td></tr>
<tr><td colspan="2">I-reversible
$m_{00} = 1$
$\mathbf{M} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{m} \end{pmatrix}$</td><td>(Reversible)
$m_{00} = 1, P_S = P_\Delta = 1$
$(\mathbf{M} = \mathbf{M}_R)$</td><td>$m_{00} = 1, P_S = P_\Delta < 1$</td></tr>
<tr><td rowspan="2">I-irreversible
$m_{00} < 1$</td><td>IN-reversible
$m_{00} < 1, P_S = P_\Delta$
$\mathbf{M} = m_{00}\begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{m} \end{pmatrix}$</td><td>$m_{00} < 1, P_S = P_\Delta = 1$
$(\hat{\mathbf{M}} = \mathbf{M}_R)$</td><td>$m_{00} < 1, P_S = P_\Delta < 1$</td></tr>
<tr><td>IN-irreversible
$m_{00} < 1, P_S < P_\Delta$
$\mathbf{M} = m_{00}\begin{pmatrix} 1 & \mathbf{D}^T \\ \mathbf{P} & \mathbf{m} \end{pmatrix}$</td><td>$m_{00} < 1, P_S < P_\Delta = 1$
$(\mathbf{M} = \mathbf{M}_J,\ P_P > 0)$</td><td>$m_{00} < 1, P_S < P_\Delta < 1$</td></tr>
</table>

Even though certain serial combinations of *DOP*-irreversible transformations and enpolarizing elements result to be *DOP*-reversible (consider for instance a perfect depolarizer sandwiched by perfect polarizers), this polarimetric behavior involves necessarily anisotropic loss of intensity, showing that, in general, irreversibility arises from an intricate combination of loss of intensity and changes of the degree of polarization. Consequently, the classification in Table 1 has an academic character oriented to the interpretation of different physical situations.

The specific physical behaviors represented by the respective cells of Table 1 are briefly described below.

As indicated above, reversibility (shadowed cell in Table 1) implies both *I*-reversibility and *DOP*-reversibility $(m_{00} = 1, P_\Delta = 1)$. From passivity condition and Equation (5), it follows that reversibility implies $P_S = P_\Delta = 1$, and therefore, the conditions for reversibility can also be expressed as $m_{00} = 1, P_S = 1$. Obviously, since any physical linear transformation of polarized electromagnetic waves involves unavoidable loss of intensity (because of absorption, reflection, etc.), polarimetric reversibility constitutes an ideal limiting situation.

I-reversibility $(m_{00} = 1)$ may also be exhibited by *DOP*-irreversible processes $(P_\Delta < 1)$, in which case, $P_\Delta = P_S < 1$ (because of the property $m_{00} = 1 \Rightarrow P_P = 0$).

I-irreversible transformations $(m_{00} < 1)$ can be split into *IN*-reversible (when $P_P = 0$) and *IN*-irreversible (when $P_P > 0$). As shown in Table 1, *IN*-reversible behavior corresponds to nonenpolarizing media, which can be either *DOP*-reversible or not depending on whether $P_S = 1$ or $P_S < 1$, respectively. Moreover, *IN*-irreversible behavior corresponds to enpolarizing media, which can be either *DOP*-reversible or not depending on whether $P_\Delta = 1$ or $P_\Delta < 1$, respectively.

DOP-reversible polarimetric processes $(P_\Delta = 1)$ correspond to pure Mueller matrices, denoted as $\mathbf{M}_J$, and necessarily belong to one of the three following different categories

- Transparent retarders, denoted as $\mathbf{M}_R$, when $m_{00} = 1$, (reversible transformations).
- Attenuating retarders, whose Mueller matrix has the form $m_{00}\mathbf{M}_R$ (with $m_{00} < 1$), when $P_P = 0$ (i.e., $P_\Delta = P_S$) and $m_{00} < 1$ (*IN*-reversible transformations).
- Enpolarizing pure media, when $P_P > 0$ (*IN*-irreversible transformations).

DOP-irreversible polarimetric processes $(P_\Delta < 1)$ correspond to depolarizing Mueller matrices, which correspond to one of the following cases

- Depolarizing transparent media, when $m_{00} = 1$.
- Depolarizing-nonenpolarizing media ($P_P = 0$ with $m_{00} < 1$).
- Depolarizing-enpolarizing media, when $P_P > 0$.

4. The Degree of Polarimetric Reversibility

As shown in Table 1, the different types of polarimetric irreversibility can be described completely through the nondimensional descriptors m_{00}, P_P, and P_Δ. Nevertheless, the question arises whether it is possible to find a single parameter that provides a measure of the degree of reversibility of a polarimetric transformation. The present section is devoted to finding a positive answer to that question.

From the analyses preformed in the previous section, it turns out that the degree of *DOP*-reversibility is well characterized by P_Δ. Moreover, Equation (5) can be expressed as

$$P_\Delta = \sqrt{\frac{2}{3}P_P^2 + P_S^2}, \tag{11}$$

while the MIC, m_{00}, which in virtue of the passivity condition is critically limited by the enpolarizance of the medium, provides an overall measure of the degree of *I*-reversibility. Therefore, leaving aside the trivial and limiting case of opaque media $(m_{00} = 0)$, an overall measure of the degree of polarimetric reversibility is given by

$$P_R \equiv \frac{m_{00} + P_\Delta}{2} \quad (m_{00} > 0). \tag{12}$$

Thus, $P_R = 1$ is satisfied, uniquely, by transparent retarders $(m_{00} = 1, P_\Delta = 1)$, for which necessarily $P_P = 0$ and $P_S = 1$. Moreover $P_R \to 0$ when $m_{00} \to 0$ with $P_\Delta = 0$, as expected for a natural and well-defined degree of reversibility. Intermediate values are achieved depending on the values of m_{00} and P_Δ, as for instance, transparent perfect depolarizers, $\mathbf{M} = \operatorname{diag}(1,0,0,0)$ (which can be physically synthesized through appropriate parallel compositions of transparent retarders) for which $P_R = 1/2$.

Figure 1 shows the feasible region for the achievable values of the degree of reversibility with respect to the axes m_{00} and P_Δ. Iso-reversibility lines are given by the oblique segments (dashed) determined by fixed values of the function $(m_{00} + P_\Delta)/2$. Edge OG corresponds to the limiting case $m_{00} = 0$, associated exclusively to the zero Mueller matrix, for which P_Δ is undetermined. Thus, edge OG is excluded from the feasible region OEFGO. Point F $(P_R = 1)$ is associated biunivocally to transparent retarders. Line EG (vertex G excluded) corresponds to $P_R = 1/2$, which includes the central point of the squared figure, $1/2 = m_{00} = P_\Delta$, while vertex E is associated with transparent perfect depolarizers, $\mathbf{M} = \operatorname{diag}(1,0,0,0)$, (which can be physically synthesized through appropriate parallel compositions of transparent retarders [27]).

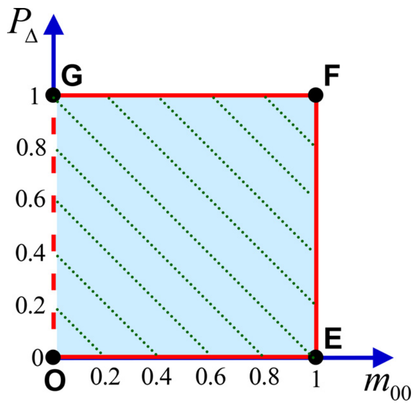

Figure 1. The feasible region for the degree of reversibility, P_R, is determined by the regular square OEFG, edge OG excluded. Edges OE (point O excluded), EF, and FG correspond, respectively, to perfect depolarizers, transparent media, and nondepolarizing media. Point F corresponds to transparent retarders, while point E corresponds to transparent perfect depolarizers. Iso-reversibility lines are represented by the dashed segments.

Since each point of Figure 1 is determined by the coordinates m_{00} and P_Δ, and a Mueller matrix depends up to sixteen independent elements, a given single point corresponds to many different Mueller matrices for which the values of m_{00} and P_Δ coincide. As shown in Figure 1, the length of the iso-reversibility lines is maximum when $P_R = 1/2$ and decreases as P_R decreases or increases from such an intermediate value.

5. The Specific Reversibility

To go deeper in the characterization of the reversibility properties associated with a given Mueller matrix $\mathbf{M}$, observe that the fact that the MIC, m_{00}, plays the role of a scale coefficient in the expression $\mathbf{M} = m_{00}\hat{\mathbf{M}}$, described in Equation (1), makes it common to use descriptors that, like D, P, P_P, P_S, and P_Δ, are insensitive to the effective value of m_{00}.

Thus, even though P_R characterizes the degree of reversibility of a linear polarimetric transformation, it is interesting to explore the way of defining a measure of the reversibility features of $\hat{\mathbf{M}}$, that is, with independence of the specific value of m_{00} and thus being valid for the entire family of Mueller matrices proportional to $\mathbf{M}$.

To do so, let us observe that P_P establishes critical limits for the degree of *I*-reversibility (recall that *IN*-reversibility implies $P_P = 0$). Furthermore, the bigger P_S is, the closer $\hat{\mathbf{M}}$ is to a transparent retarder. These considerations allow us to introduce the following definition of the *specific reversibility*, R_S, whose appropriateness will be illustrated below through some examples and by means of its features in the purity figure [8,9]

$$R_S \equiv \sqrt{\frac{P_S^2 + P_\Delta^2}{2}}. \tag{13}$$

Equivalently, from Equation (11)

$$R_S = \sqrt{P_\Delta^2 - \frac{P_P^2}{3}} = \sqrt{P_S^2 + \frac{P_P^2}{3}}. \tag{14}$$

This new parameter is limited by $0 \leq R_S \leq 1$. Transformations featuring $R_S = 1$ correspond uniquely to retarders (transparent or not), while the minimum, $R_S = 0$, is associated uniquely to perfect depolarizers. In the case of a pure Mueller matrix $(P_\Delta = 1)$

exhibiting certain amount of diattenuation, i.e., $D > 0$ (recall that pure Mueller matrices necessarily satisfy $P = D = P_P$ [28]), then

$$R_S^2 = 1 - \frac{D^2}{3}. \tag{15}$$

In particular, perfect polarizers $(D = P = 1)$ feature $R_S^2 = 2/3$, showing that the irreversibility produced by diattenuation of pure media is regulated by the effective value of D in a simple manner and with a natural correspondence with the partitioned structure of $\mathbf{M}$ shown in Equation (1); that is, for a perfect polarizer, $\mathbf{D}$ and $\mathbf{P}$ contribute, respectively, $1/3$ and $1/3$ to the value of R_S^2, while the value $\hat{P}_S^2 = 1/3$ is fixed directly from the condition $D = P = 1$.

Moreover, nonenpolarizing media $(P_P = 0)$ are fully characterized by the property $P_\Delta = P_S$ (see Equation (11)) and consequently satisfy, uniquely, $R_S = P_S = P_\Delta$.

Obviously, parameters P_R and R_S hold information of different nature. Regarding P_R, it cannot be increased by the successive application of linear polarimetric transformations, that is $P_R(\mathbf{M}_1\mathbf{M}_2) \leq P_R(\mathbf{M}_1)P_R(\mathbf{M}_2)$, which agrees with its character of an overall measure of reversibility. Nevertheless, R_S constitutes an index of the lack of reversibility derived from both enpolarizing and depolarizing properties of the medium.

The representation of linear of polarimetric transformations by means of the space defined by the axes P_S and P_P constitutes the so-called purity figure [8], for which different branches for values of P_Δ and R_S are shown in Figure 2. The feasible region for Mueller matrices is limited by the positive branches of the coordinate axes themselves, together with the elliptical edge $1 = 2P_P^2/3 + P_S^2$ (AC) and the hyperbolic edge $2P_P^2 = 1 + 3P_S^2$ (DC) [9].

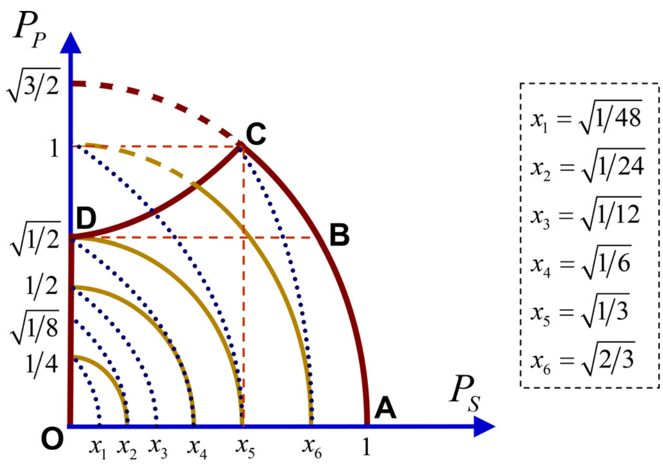

Figure 2. Purity figure, where the solid elliptical branches represent iso-purity Mueller matrices (P_Δ constant), and the dashed elliptical branches represent iso-specific-reversibility (R_S constant) Mueller matrices. Descriptors P_S, P_Δ, and R_S coincide along axis P_S, while the difference $R_S - P_\Delta$ increases as P_P increases. The feasible region is limited by points OACD. Points O, A, and C represent, respectively, perfect depolarizers, retarders, and perfect polarizers. The maximal achievable value of P_P for media with $P_S = 0$ corresponds to point D, while points over the line BD necessarily exhibit $P_S > 0$.

The iso-specific-reversibility (R_S constant) Mueller matrices describe the elliptical branches represented by the dashed lines in Figure 2, while the iso-purity ones (P_Δ constant) correspond to the elliptical branches represented by solid lines. The feasible region for Mueller matrices is given by the area OABCDO [8]. Point O is achieved, uniquely, by perfect depolarizers, which are the only media with zero specific reversibility. The elliptical segment AC corresponds, in an exclusive manner, to nondepolarizing media whose limiting points A and C represent retarders and perfect polarizers, respectively. Media exhibiting a zero polarimetric dimension $P_S = 0$ are represented by segment OD. Thus, descriptors P_S,

P_Δ, and R_S coincide along axis P_S, while the difference $R_S - P_\Delta$ increases as P_P increases. Media with $R_S = 1$ (retarders, transparent or not) are represented by point A; perfect polarizers $(P_P = 1)$ exhibit $R_S = \sqrt{2/3}$ and $P_\Delta = 1$ (point C); point D corresponds to Mueller matrices with $R_S = \sqrt{1/6}$ and $P_\Delta = \sqrt{1/3}$. Branches for other values of R_S and P_Δ are also represented in Figure 1 in order to illustrate the different behavior of them as P_P increases from $P_P = 0$ to its achievable maximum values.

Note that, while nondepolarizing Mueller matrices cover the entire elliptical segment AC, specific-reversible ones correspond exclusively to point A.

6. Discussion

Among the possible definitions for the degree of reversibility, the one introduced in Equation (12), P_R, is given simply by the arithmetic mean of the values of mean intensity coefficient (MIC), m_{00}, and the degree of polarimetric purity, P_Δ. P_R is applicable to any nonzero Mueller matrix, and its representation in the space determined by the coordinate axes m_{00} and P_Δ shows that P_R satisfies the natural conditions to be hold by a well-defined degree of polarimetric reversibility.

The partitioned structure of Mueller matrices, for which m_{00} plays the mathematical role of a scale parameter (whose feasible values are submitted to the passivity condition), makes it particularly useful for handling descriptors that are independent of m_{00}, as occurs with the degree of polarimetric purity, the depolarizance, the indices of polarimetric purity [22], the polarizance, the diattenuation, the enpolarizance, and the polarimetric dimension index. This supports the idea of defining a specific reversibility parameter, R_S, that takes a fixed value for all physical Mueller matrices that are proportional to a given $\mathbf{M}$.

The definition introduced in Equation (13) for R_S is formulated as a square average of P_Δ and P_S, which are mutually independent and are linked via P_P. From Equations (13) and (14), given a fixed value of P_S, the greater P_P is, the greater P_Δ is, while for a given value of P_Δ, the greater P_P is, the lower P_S is. That is, whereas P_P contributes positively to the purity (P_P is a measure of the ability to increase the degree of polarization), it reduces the specific reversibility (P_P can also be considered a measure of the overall diattenuation power of the medium, and therefore, it is linked to the decrease of the *I*-reversibility).

It should be noted that alternative definitions for the specific reversibility could be formulated as

$$R_S \equiv \sqrt{\cos^2\,\alpha\; P_S^2 + \sin^2\,\alpha\; P_\Delta^2}, \quad (0 \le \alpha \le \pi/2), \tag{16}$$

Or, equivalently,

$$R_S \equiv \sqrt{P_S^2 + \frac{2}{3}\sin^2\,\alpha P_P^2}\,, \quad (0 \le \alpha \le \pi/2), \tag{17}$$

all of them satisfying the natural exigency that $R_S = 0$ if and only if conditions $P_P = 0$ and $P_S = 0$ are simultaneously hold. The analyses performed in the previous sections support the simplest choice $\alpha = \pi/4$ taken for the definition proposed in Equation (13). In fact, the iso-specific-reversibility branches in the purity figure (see Figure 2) become particularly simple and admit a straightforward interpretation and comparison to the iso-purity branches.

7. Conclusions

A proper measure for the degree of thermodynamic reversibility of linear polarimetric transformations (which are represented by respective Mueller matrices) has been introduced in Equation (12) which accounts for the reversibility associated with the lack of attenuation of the intensity of the light probe, together with the reversibility associated with the preservation of the degree of polarization of incident totally polarized light. The feasible values for the degree of reversibility, P_R, have been discussed with the help of their representation with respect to the coordinates determined by the mean intensity coefficient (m_{00}) and the degree of polarimetric purity (P_Δ).

Moreover, since the analysis of specific properties of Mueller matrices that do not depend on the particular value of m_{00} has proven to be very useful for the study and classification of the possible physical behaviors represented by respective types of Mueller matrices, a new quantity, the specific reversibility, denoted as R_S, which is associated to the entire family of Mueller matrices that are proportional to a given $\mathbf{M}$, has been introduced in Equation (13) as a square average of the degree of polarimetric purity, P_Δ, and the polarimetric dimension index, P_S.

It has been shown that the enpolarizance, P_P, plays a critical and meaningful role because it can be combined quadratically with either the degree of polarimetric purity or the polarimetric dimension index to determine the value of R_S. Some interesting features of the specific reversibility have been analyzed through its representation, by means of elliptical branches, in the purity figure built from the coordinate axes P_S and P_S.

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