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Review

A Table of Some Coherency Matrices, Coherency Matrix Factors, and Their Respective Mueller Matrices

Colin J. R. Sheppard [1,2,*], Artemi Bendandi [1,3,4], Aymeric Le Gratiet [1,5] and Alberto Diaspro [1,3]

1 Nanoscopy and NIC@IIT, Istituto Italiano di Tecnologia, Via Enrico Melen, 83 Edificio B, 16152 Genova, Italy; artemi.bendandi@gmail.com (A.B.); aymericlegratiet@gmail.com (A.L.G.); alberto.diaspro@iit.it (A.D.)
2 Molecular Horizons, School of Chemistry and Molecular Biosciences, University of Wollongong, Wollongong, NSW 2522, Australia
3 Dipartimento di Fisica, University of Genoa, 16146 Genova, Italy
4 Department of Biochemistry, University of Zurich, 8006 Zurich , Switzerland
5 Institut FOTON-UMR 6082, CNRS, Université de Rennes, 22305 Lannion, France
* Correspondence: colinjrsheppard@gmail.com

Abstract: Many books on polarization give tables of Mueller matrices. The coherency matrix has been found useful for interpretetion of the Mueller matrix. Here we give a table of Mueller matrices **M**, coherency matrices **C**, and coherency matrix factors **F** for different polarization components and systems. **F** is not given for some complicated nondeterministic cases. In many cases, though, **F** has a very simple form. In particular, we give expressions for **F** for the general case of an homogeneous elliptic diattenuating retarder. Different coordinate systems for describing diattenuating retarders are compared, on a generalized retardation sphere, analogous to the Poincaré sphere. For the general homogeneous deterministic case, expressions for the Mueller matrix have particularly simple forms for Cartesian or stereographic coordinates in generalized retardation space.

Keywords: polarization; polarization imaging; mueller matrix

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1. Introduction

Transformation of the polarization state of a partially polarized plane wave can be described by a Mueller matrix. Many books on polarization give tables of Mueller matrices, denoted **M**, a recent one being that by Gill and Ossikovski [1]. Older books are by Shurcliff [2] and Collett [3]. The coherency matrix **C**, introduced by Cloude [4], has proved useful in understanding and interpreting the Mueller matrix [4–12], so here we present a table of coherency matrices, which should be a useful reference resource. The coherency matrix is Hermitian, its eigenvalues are real and nonnegative for physical realizability, and for a deterministic Mueller matrix there is a single nonzero eigenvalue [4]. A deterministic system is sometimes called pure, and can be represented by a Jones matrix, so that a deterministic Mueller matrix is also called a Mueller-Jones matrix. A deterministic Mueller matrix is sometimes called nondepolarizing, but this terminology is confusing as nondepolarizing (deterministic) systems can depolarize and depolarizing (nondeterministic) systems can polarize [13].

It has been shown that the coherency matrix can be factorized into a product $\mathbf{C} = \mathbf{F}\mathbf{F}^{\dagger}$ [14] of the coherency matrix factor (CMF) and its conjugate transpose, denoted by †. For a deterministic system, the coherency matrix can be written as the product of two coherency vectors, $\mathbf{C} = \mathbf{z}\mathbf{z}^{\dagger} = \mathbf{z} \otimes \mathbf{z}^{*}$, where $\mathbf{z}$ is the coherency vector, introduced in detail by Chipman [15], and $\otimes$ represents a Kronecker product. The elements of the coherency vector are the fundamental components of the polarization transformation, expressed in the Pauli basis. For a nondeterministic Mueller matrix, the four columns of the coherency matrix factor **F** are the coherency vectors of the deterministic components [14]. Examples of nondeterministic systems are the ABCD matrix (Section 5.12), and media of scattering particles (Section 5.13). Each coherency vector is an eigenvector of the coherency matrix, multiplied by the square root of the corresponding eigenvalue, each of which are necessarily non-negative. The eigenvectors are mutually orthogonal. So the coherency matrix factor displays directly many of the important properties of the Mueller matrix in an illustrative format [14]. For example, the dominant c-vector and the number of non-zero eigenvalues can be seen immediately. A deterministic Mueller matrix can also be factorized into a product $\mathbf{M} = \mathbf{Z}\mathbf{Z}^*$ [16], where * means complex conjugate, and the matrix $\mathbf{Z}$, called the Z-matrix, the polarization coupling matrix, or the state-generating matrix [15,16], is

$$\mathbf{Z} = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 & z_0 & -iz_3 & iz_2 \\ z_2 & iz_3 & z_0 & -iz_1 \\ z_3 & -iz_2 & iz_1 & z_0 \end{pmatrix}, \tag{1}$$

in which the elements $z_j, j = 0,1,2,3$, are the elements of the coherency vector $\mathbf{z}$. This represents a straightforward approach to calculate a determinstic Mueller matrix from its coherency vector. z_0, Z_{00} and F_{00} are in general complex, but $C_{00} = |z_0|^2$ is real.

$\mathbf{M}, \mathbf{C}, \mathbf{Z}, \mathbf{F}, \mathbf{z}$ are all expressed in the Stokes basis, generated using the Pauli matrices, so their elements are all related to the fundamental components of the polarization transformation. In an analogous way, we have $\mathbf{Q}, \mathbf{H}, \mathbf{X}, \mathbf{E}, \mathbf{j}$ in the standard Cartesian basis, where $\mathbf{Q}$ is the Parke matrix [17,18], $\mathbf{H}$ is the (Hermitian) covariance matrix [19], $\mathbf{X}$ is the Parke matrix factor [11], $\mathbf{E}$ is the covariance matrix factor [11] and $\mathbf{j}$ is the Jones vector, also written in the matrix form $\mathbf{J}$ as the Jones matrix [20]. Then $\mathbf{H} = \mathbf{E}\mathbf{E}^\dagger$, and for a deterministic system $\mathbf{Q} = \mathbf{X}\mathbf{X}^\dagger$, $\mathbf{H} = \mathbf{j}\mathbf{j}^\dagger = \mathbf{j} \otimes \mathbf{j}^*$, $\mathbf{Q} = \mathbf{J} \otimes \mathbf{J}^*$. However, we do not list these matrices, as those in the Stokes basis are more illustrative of the polarization properties. Tables of Jones matrices are already available [1].

For the general case, the Mueller matrix can be calculated from the coherency matrix using [1,4,17]

$$\mathbf{M}(z) = \frac{1}{2}\begin{pmatrix} C_{00}+C_{11}+C_{22}+C_{33} & C_{01}+C_{10}-i(C_{23}-C_{32}) & C_{02}+C_{20}+i(C_{13}-C_{31}) & C_{03}+C_{30}-i(C_{12}-C_{21}) \\ C_{01}+C_{10}+i(C_{23}-C_{32}) & C_{00}+C_{11}-C_{22}-C_{33} & C_{12}+C_{21}+i(C_{03}-C_{30}) & C_{13}+C_{31}-i(C_{02}-C_{20}) \\ C_{02}+C_{20}-i(C_{13}-C_{31}) & C_{12}+C_{21}-i(C_{03}-C_{30}) & C_{00}-C_{11}+C_{22}-C_{33} & C_{23}+C_{32}+i(C_{01}-C_{10}) \\ C_{03}+C_{30}+i(C_{12}-C_{21}) & C_{13}+C_{31}+i(C_{02}-C_{20}) & C_{23}+C_{32}-i(C_{01}-C_{10}) & C_{00}-C_{11}-C_{22}+C_{33} \end{pmatrix}. \tag{2}$$

The trace of the coherency matrix, equal to the sum of its eigenvalues, is equal to $2M_{00}$. The Mueller matrix can alternatively be calculated from the coherency matrix by applying a (16×16) matrix transformation to the coherency matrix written in 16-element vector form, $\mathbf{c}$ [7,17].

In Section 3, we consider diattenuaters, including elliptic diattenuaters (Section 3.1), linear diattenuaters (Section 3.6), circular diattenuaters (Section 3.11), elliptic polarizers (Section 3.12), linear polarizers (Section 3.17) and circular polarizers (Section 3.22). We use the terminology as in Ref. [1]. The phase shift $\Delta = 0$, the ellipticity angle is χ, and the azimuthal angle is ϕ. The diattenuation D is

$$D = \cos\kappa = \frac{p_1^2 - p_2^2}{p_1^2 + p_2^2}, \tag{3}$$

where p_1^2, p_2^2 are the maximum and minimum intensity coefficients. $M_{00} = (p_1^2 + p_2^2)/2 = p_1^2/(1+\cos\kappa)$. Then

$$\sqrt{1-D^2} = \sin\kappa = \frac{2p_1p_2}{p_1^2 + p_2^2}; \tag{4}$$

and

$$
\begin{aligned}
1+D = 1+\cos\kappa = \frac{2p_1^2}{p_1^2+p_2^2}; \quad 1-D = 1-\cos\kappa = \frac{2p_2^2}{p_1^2+p_2^2};\\
1+\sqrt{1-D^2} = 1+\sin\kappa = \frac{(p_1+p_2)^2}{p_1^2+p_2^2}; \quad 1-\sqrt{1-D^2} = 1-\sin\kappa = \frac{(p_1-p_2)^2}{p_1^2+p_2^2}.
\end{aligned} \tag{5}
$$

Also,

$$
\sqrt{\frac{1+D}{2}} = \cos\left(\frac{\kappa}{2}\right) = \frac{p_1}{\sqrt{p_1^2+p_2^2}}; \quad \sqrt{\frac{1-D}{2}} = \sin\left(\frac{\kappa}{2}\right) = \frac{p_2}{\sqrt{p_1^2+p_2^2}}. \tag{6}
$$

In Section 4, we consider retarders, including elliptic retarders (Section 4.1), quarter-wave elliptic retarders (Section 4.6), half-wave elliptic retarders (Section 4.7), linear retarders (Section 4.8), quarter-wave linear retarders (Section 4.13), half-wave linear retarders (pseudorotators) (Section 4.18), and circular retarders (rotators) (Section 4.23).

In Section 5, we continue with elliptic homogeneous retarding diattenuaters in Section 5.1, circular retarding diattenuaters in Section 5.2, linear homogeneous retarding diattenuaters in Section 5.3, quarter wave elliptic homogeneous retarding diattenuaters (Section 5.8), and half wave elliptic homogeneous retarding diattenuaters (Section 5.9). For homogeneous retarding diattenuaters, the retardation is $R = \sin\Delta \sin\kappa$, where Δ is the retardance, so for an ideal retarder, $R = \sin\Delta$.

We then consider some other special cases, including scattering with different geometry, Mueller matrices of different symmetry, and canonical Mueller matrices. Section 6 deals with the general case of a determinstic Mueller matrix, using a parameterized approach [21]. Section 6.2 specializes to homogeneous Mueller matrices, and Section 6.3 to linear homogeneous Mueller matrices. Section 7 considers the case of a uniform polarizing medium of finite thickness [22–24]. In Section 8, we discuss the tables, including the relationship between the measures of diattenuation and retardance, κ and Δ, and the parametric representation of Mueller matrices, ψ, θ. We investigate the relationship on a generalized retardance sphere, analogous to the Poincaré sphere. We also discuss two other coordinate systems on the generalized retardance sphere, based on direction cosines, F, D, R, or the stereographic projection, u, v. These lead to simple expressions for $\mathbf{F}$ for the general case of an homogeneous elliptic retarding diattenuater.

2. Two Simple Special Cases

2.1. Free Space

For no change in polarization state, or propagation in free space:

$$
\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \sqrt{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

2.2. Perfect Depolarizer

For a perfect depolarizer:

$$
\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

3. Table for Diattenuaters

3.1. Elliptic Diattenuater

An elliptic diattenuater is also called a partial polarizer. Ref. [1], Equation (4.74). Phase shift, $\Delta = 0$. The ratio between the strengths of components of the electric field $= \tan\chi$, ellipticity angle $= \chi$. Oriented at an angle ϕ. Diattenuation $D = \cos\kappa$. $M_{00} = (p_1^2 + p_2^2)/2 = p_1^2/(1+\cos\kappa)$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & \cos\kappa\cos 2\chi\cos 2\phi & \cos\kappa\cos 2\chi\sin 2\phi & \cos\kappa\sin 2\chi \\ \cos\kappa\cos 2\chi\cos 2\phi & \begin{matrix}(1-\sin\kappa)\cos^2 2\chi\cos^2 2\phi \\ +\sin\kappa\end{matrix} & \begin{matrix}(1-\sin\kappa) \\ \times\cos^2 2\chi\sin 2\phi\cos 2\phi\end{matrix} & \begin{matrix}(1-\sin\kappa) \\ \times\sin 2\chi\cos 2\chi\cos 2\phi\end{matrix} \\ \cos\kappa\cos 2\chi\sin 2\phi & \begin{matrix}(1-\sin\kappa) \\ \times\cos^2 2\chi\sin 2\phi\cos 2\phi\end{matrix} & \begin{matrix}(1-\sin\kappa)\cos^2 2\chi\sin^2 2\phi \\ +\sin\kappa\end{matrix} & \begin{matrix}(1-\sin\kappa) \\ \times\sin 2\chi\cos 2\chi\sin 2\phi\end{matrix} \\ \cos\kappa\sin 2\chi & \begin{matrix}(1-\sin\kappa) \\ \times\sin 2\chi\cos 2\chi\cos 2\phi\end{matrix} & \begin{matrix}(1-\sin\kappa) \\ \times\sin 2\chi\cos 2\chi\sin 2\phi\end{matrix} & \sin^2 2\chi + \sin\kappa\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & \cos\kappa\cos 2\chi\cos 2\phi & \cos\kappa\cos 2\chi\sin 2\phi & \cos\kappa\sin 2\chi \\ \cos\kappa\cos 2\chi\cos 2\phi & (1-\sin\kappa)\cos^2 2\chi\cos^2 2\phi & \cos\kappa\cos^2 2\chi\sin 2\phi\cos 2\phi & \cos\kappa\sin 2\chi\cos 2\chi\cos 2\phi \\ \cos\kappa\cos 2\chi\sin 2\phi & \cos\kappa\cos^2 2\chi\sin 2\phi\cos 2\phi & (1-\sin\kappa)\cos^2 2\chi\sin^2 2\phi & \cos\kappa\sin 2\chi\cos 2\chi\sin 2\phi \\ \cos\kappa\sin 2\chi & \cos\kappa\sin 2\chi\cos 2\chi\cos 2\phi & \cos\kappa\sin 2\chi\cos 2\chi\sin 2\phi & (1-\sin\kappa)\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\cos 2\chi\cos 2\phi & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\cos 2\chi\sin 2\phi & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.2. Elliptic Diattenuater at $\phi = 0°$

$\phi = 0°$. Ref. [1], Equation (4.88).

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & \cos\kappa\cos 2\chi & 0 & \cos\kappa\sin 2\chi \\ \cos\kappa\cos 2\chi & \cos^2 2\chi + \sin\kappa\sin^2 2\chi & 0 & (1-\sin\kappa)\sin 2\chi\cos 2\chi \\ 0 & 0 & \sin\kappa & 0 \\ \cos\kappa\sin 2\chi & (1-\sin\kappa)\sin 2\chi\cos 2\chi & 0 & \sin^2 2\chi + \sin\kappa\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & \cos\kappa\cos 2\chi & 0 & \cos\kappa\sin 2\chi \\ \cos\kappa\cos 2\chi & (1-\sin\kappa)\cos^2 2\chi & 0 & \cos\kappa\sin 2\chi\cos 2\chi \\ 0 & 0 & 0 & 0 \\ \cos\kappa\sin 2\chi & \cos\kappa\sin 2\chi\cos 2\chi & 0 & (1-\sin\kappa)\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\cos 2\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.3. Elliptic Diattenuater at $\phi = 45°$

$\phi = 45°$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & \cos\kappa\cos 2\chi & \cos\kappa\sin 2\chi \\ 0 & \sin\kappa & 0 & 0 \\ \cos\kappa\cos 2\chi & 0 & \cos^2 2\chi + \sin\kappa\sin^2 2\chi & (1-\sin\kappa)\sin 2\chi\cos 2\chi \\ \cos\kappa\sin 2\chi & 0 & (1-\sin\kappa)\sin 2\chi\cos 2\chi & \sin^2 2\chi + \sin\kappa\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & 0 & \cos\kappa\cos 2\chi & \cos\kappa\sin 2\chi \\ 0 & 0 & 0 & 0 \\ \cos\kappa\cos 2\chi & 0 & (1-\sin\kappa)\cos^2 2\chi & \cos\kappa\sin 2\chi\cos 2\chi \\ \cos\kappa\sin 2\chi & 0 & \cos\kappa\sin 2\chi\cos 2\chi & (1-\sin\kappa)\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\cos 2\chi & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.4. Elliptic Diattenuater at $\phi = 90°$

$\phi = 90°$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & \cos\kappa\sin 2\chi & \\ -\cos\kappa\cos 2\chi & \cos^2 2\chi + \sin\kappa\sin^2 2\chi & 0 & -(1-\sin\kappa)\sin 2\chi\cos 2\chi \\ 0 & 0 & \sin\kappa & 0 \\ \cos\kappa\sin 2\chi & -(1-\sin\kappa)\sin 2\chi\cos 2\chi & 0 & \sin^2 2\chi + \sin\kappa\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & -\cos\kappa\cos 2\chi & 0 & \cos\kappa\sin 2\chi \\ -\cos\kappa\cos 2\chi & (1-\sin\kappa)\cos^2 2\chi & 0 & -\cos\kappa\sin 2\chi\cos 2\chi \\ 0 & 0 & 0 & 0 \\ \cos\kappa\sin 2\chi & -\cos\kappa\sin 2\chi\cos 2\chi & 0 & (1-\sin\kappa)\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ -\sqrt{1-\sin\kappa}\cos 2\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.5. Elliptic Diattenuater at $\phi = 135°$

$\phi = 135°$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & -\cos\kappa\cos 2\chi & \cos\kappa\sin 2\chi \\ 0 & \sin\kappa & 0 & 0 \\ -\cos\kappa\cos 2\chi & 0 & \cos^2 2\chi + \sin\kappa\sin^2 2\chi & -(1-\sin\kappa)\sin 2\chi\cos 2\chi \\ \cos\kappa\sin 2\chi & 0 & -(1-\sin\kappa)\sin 2\chi\cos 2\chi & \sin^2 2\chi + \sin\kappa\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & 0 & -\cos\kappa\cos 2\chi & \cos\kappa\sin 2\chi \\ 0 & 0 & 0 & 0 \\ -\cos\kappa\cos 2\chi & 0 & (1-\sin\kappa)\cos^2 2\chi & -\cos\kappa\sin 2\chi\cos 2\chi \\ \cos\kappa\sin 2\chi & 0 & -\cos\kappa\sin 2\chi\cos 2\chi & (1-\sin\kappa)\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{1-\sin\kappa}\cos 2\chi & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.6. Linear Diattenuater

Ref. [1], Equation (4.92). Phase shift, $\Delta = 0$. Ellipticity angle $= \chi = 0$. Oriented at an angle ϕ. $M_{00} = (p_1^2 + p_2^2)/2 = p_1^2/(1+\cos\kappa)$.

$$\mathbf{M} = \frac{p_1^2}{1+\cos\kappa}\begin{pmatrix} 1 & \cos\kappa\cos 2\phi & \cos\kappa\sin 2\phi & 0 \\ \cos\kappa\cos 2\phi & \cos^2 2\phi + \sin\kappa\sin^2 2\phi & (1-\sin\kappa)\sin 2\phi\cos 2\phi & 0 \\ \cos\kappa\sin 2\phi & (1-\sin\kappa)\sin 2\phi\cos 2\phi & \sin^2 2\phi + \sin\kappa\cos^2 2\phi & 0 \\ 0 & 0 & 0 & \sin\kappa \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{1+\cos\kappa}\begin{pmatrix} 1+\sin\kappa & \cos\kappa\cos 2\phi & \cos\kappa\sin 2\phi & 0 \\ \cos\kappa\cos 2\phi & (1-\sin\kappa)\cos^2 2\phi & (1-\sin\kappa)\sin 2\phi\cos 2\phi & 0 \\ \cos\kappa\sin 2\phi & (1-\sin\kappa)\sin 2\phi\cos 2\phi & (1-\sin\kappa)\sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\cos 2\phi & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa}\sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ (p_1-p_2)\cos 2\phi & 0 & 0 & 0 \\ (p_1-p_2)\sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.7. Linear Diattenuater at $\phi = 0°$

$\phi = 0°$. Ref. [1], Equation (4.79).

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & \cos\kappa & 0 & 0 \\ \cos\kappa & 1 & 0 & 0 \\ 0 & 0 & \sin\kappa & 0 \\ 0 & 0 & 0 & \sin\kappa \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p_1^2+p_2^2 & p_1^2-p_2^2 & 0 & 0 \\ p_1^2-p_2^2 & p_1^2+p_2^2 & 0 & 0 \\ 0 & 0 & 2p_1p_2 & 0 \\ 0 & 0 & 0 & 2p_1p_2 \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & \cos\kappa & 0 & 0 \\ \cos\kappa & 1-\sin\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (p_1+p_2)^2 & p_1^2-p_2^2 & 0 & 0 \\ p_1^2-p_2^2 & (p_1-p_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ p_1-p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.8. Linear Diattenuater at $\phi = 45°$

$\phi = 45°$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & \cos\kappa & 0 \\ 0 & \sin\kappa & 0 & 0 \\ \cos\kappa & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\kappa \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p_1^2+p_2^2 & 0 & p_1^2-p_2^2 & 0 \\ 0 & 2p_1p_2 & 0 & 0 \\ p_1^2-p_2^2 & 0 & p_1^2+p_2^2 & 0 \\ 0 & 0 & 0 & 2p_1p_2 \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & 0 & \cos\kappa & 0 \\ 0 & 0 & 0 & 0 \\ \cos\kappa & 0 & 1-\sin\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (p_1+p_2)^2 & 0 & p_1^2-p_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ p_1^2-p_2^2 & 0 & (p_1-p_2)^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_1-p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.9. Linear Diattenuater at $\phi = 90^\circ$

$\phi = 90^\circ$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & -\cos\kappa & 0 & 0 \\ -\cos\kappa & 1 & 0 & 0 \\ 0 & 0 & \sin\kappa & 0 \\ 0 & 0 & 0 & \sin\kappa \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p_1^2+p_2^2 & -(p_1^2-p_2^2) & 0 & 0 \\ -(p_1^2-p_2^2) & p_1^2+p_2^2 & 0 & 0 \\ 0 & 0 & 2p_1p_2 & 0 \\ 0 & 0 & 0 & 2p_1p_2 \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & -\cos\kappa & 0 & 0 \\ -\cos\kappa & 1-\sin\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (p_1+p_2)^2 & -(p_1^2-p_2^2) & 0 & 0 \\ -(p_1^2-p_2^2) & (p_1-p_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ -\sqrt{1-\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ -(p_1-p_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.10. Linear Diattenuater at $\phi = 135^\circ$

$\phi = 135^\circ$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & -\cos\kappa & 0 \\ 0 & \sin\kappa & 0 & 0 \\ -\cos\kappa & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\kappa \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p_1^2+p_2^2 & 0 & -(p_1^2-p_2^2) & 0 \\ 0 & 2p_1p_2 & 0 & 0 \\ -(p_1^2-p_2^2) & 0 & p_1^2+p_2^2 & 0 \\ 0 & 0 & 0 & 2p_1p_2 \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\sin\kappa & 0 & -\cos\kappa & 0 \\ 0 & 0 & 0 & 0 \\ -\cos\kappa & 0 & 1-\sin\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (p_1+p_2)^2 & 0 & -(p_1^2-p_2^2) & 0 \\ 0 & 0 & 0 & 0 \\ -(p_1^2-p_2^2) & 0 & (p_1-p_2)^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{1-\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(p_1-p_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.11. Circular Diattenuater

Ref. [1], Equation (4.90). $M_{00} = (p_1^2+p_2^2)/2 = p_1^2/(1+\cos\kappa)$.
Right, $\chi = 45^\circ$.

$$\mathbf{M} = \frac{p_1^2}{1+\cos\kappa}\begin{pmatrix} 1 & 0 & 0 & \cos\kappa \\ 0 & \sin\kappa & 0 & 0 \\ 0 & 0 & \sin\kappa & 0 \\ \cos\kappa & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{1+\cos\kappa}\begin{pmatrix} 1+\sin\kappa & 0 & 0 & \cos\kappa \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\kappa & 0 & 0 & 1-\sin\kappa \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\sin\kappa} & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_1-p_2 & 0 & 0 & 0 \end{pmatrix}.$$

Left, $\chi = -45°$.

$$\mathbf{M} = \frac{p_1^2}{1+\cos\kappa}\begin{pmatrix} 1 & 0 & 0 & -\cos\kappa \\ 0 & \sin\kappa & 0 & 0 \\ 0 & 0 & \sin\kappa & 0 \\ -\cos\kappa & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{1+\cos\kappa}\begin{pmatrix} 1+\sin\kappa & 0 & 0 & -\cos\kappa \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\cos\kappa & 0 & 0 & 1-\sin\kappa \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{1+\cos\kappa}}\begin{pmatrix} \sqrt{1+\sin\kappa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{1-\sin\kappa} & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_1+p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(p_1-p_2) & 0 & 0 & 0 \end{pmatrix}.$$

3.12. Elliptic Polarizer

A polarizer is the special case of a diattenuater when $D = 1$, $p_2 = 0$, $\kappa = 0$. Phase shift, $\Delta = 0$. The ratio between the strengths of components of the electric field $= \tan\chi$, ellipticity angle $= \chi$. Oriented at an angle ϕ. $M_{00} = p_1^2/2$.

$$\mathbf{M} = \frac{p_1^2}{2}\begin{pmatrix} 1 & \cos 2\chi\cos 2\phi & \cos 2\chi\sin 2\phi & \sin 2\chi \\ \cos 2\chi\cos 2\phi & \cos^2 2\chi\cos^2 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi & \sin 2\chi\cos 2\chi\cos 2\phi \\ \cos 2\chi\sin 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi & \cos^2 2\chi\sin^2 2\phi & \sin 2\chi\cos 2\chi\sin 2\phi \\ \sin 2\chi & \sin 2\chi\cos 2\chi\cos 2\phi & \sin 2\chi\cos 2\chi\sin 2\phi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{2}\begin{pmatrix} 1 & \cos 2\chi\cos 2\phi & \cos 2\chi\sin 2\phi & \sin 2\chi \\ \cos 2\chi\cos 2\phi & \cos^2 2\chi\cos^2 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi & \sin 2\chi\cos 2\chi\cos 2\phi \\ \cos 2\chi\sin 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi & \cos^2 2\chi\sin^2 2\phi & \sin 2\chi\cos 2\chi\sin 2\phi \\ \sin 2\chi & \sin 2\chi\cos 2\chi\cos 2\phi & \sin 2\chi\cos 2\chi\sin 2\phi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \cos 2\chi\cos 2\phi & 0 & 0 & 0 \\ \cos 2\chi\sin 2\phi & 0 & 0 & 0 \\ \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

For polarizers, $\mathbf{M} = \mathbf{C}$.

3.13. Elliptic Polarizer at $\phi = 0°$

$\phi = 0°$. Ref. [1], Equation (4.89).

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\chi & 0 & \sin 2\chi \\ \cos 2\chi & \cos^2 2\chi & 0 & \sin 2\chi\cos 2\chi \\ 0 & 0 & 0 & 0 \\ \sin 2\chi & \sin 2\chi\cos 2\chi & 0 & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{2}\begin{pmatrix} 1 & \cos 2\chi & 0 & \sin 2\chi \\ \cos 2\chi & \cos^2 2\chi & 0 & \sin 2\chi\cos 2\chi \\ 0 & 0 & 0 & 0 \\ \sin 2\chi & \sin 2\chi\cos 2\chi & 0 & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \cos 2\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.14. Elliptic Polarizer at $\phi = 45°$

$\phi = 45°$.

$$\mathbf{M} = \frac{p_1^2}{2}\begin{pmatrix} 1 & 0 & \cos 2\chi & \sin 2\chi \\ 0 & 0 & 0 & 0 \\ \cos 2\chi & 0 & \cos^2 2\chi & \sin 2\chi \cos 2\chi \\ \sin 2\chi & 0 & \sin 2\chi \cos 2\chi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{2}\begin{pmatrix} 1 & 0 & \cos 2\chi & \sin 2\chi \\ 0 & 0 & 0 & 0 \\ \cos 2\chi & 0 & \cos^2 2\chi & \sin 2\chi \cos 2\chi \\ \sin 2\chi & 0 & \sin 2\chi \cos 2\chi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos 2\chi & 0 & 0 & 0 \\ \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.15. Elliptic Polarizer at $\phi = 90°$

$\phi = 90°$.

$$\mathbf{M} = \frac{p_1^2}{2}\begin{pmatrix} 1 & -\cos 2\chi & 0 & \sin 2\chi \\ -\cos 2\chi & \cos^2 2\chi & 0 & -\sin 2\chi \cos 2\chi \\ 0 & 0 & 0 & 0 \\ \sin 2\chi & -\sin 2\chi \cos 2\chi & 0 & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{2}\begin{pmatrix} 1 & -\cos 2\chi & 0 & \sin 2\chi \\ -\cos 2\chi & \cos^2 2\chi & 0 & -\sin 2\chi \cos 2\chi \\ 0 & 0 & 0 & 0 \\ \sin 2\chi & -\sin 2\chi \cos 2\chi & 0 & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\cos 2\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.16. Elliptic Polarizer at $\phi = 135°$

$\phi = 135°$.

$$\mathbf{M} = \frac{p_1^2}{2}\begin{pmatrix} 1 & 0 & -\cos 2\chi & \sin 2\chi \\ 0 & 0 & 0 & 0 \\ -\cos 2\chi & 0 & \cos^2 2\chi & -\sin 2\chi \cos 2\chi \\ \sin 2\chi & 0 & -\sin 2\chi \cos 2\chi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \frac{p_1^2}{2}\begin{pmatrix} 1 & 0 & -\cos 2\chi & \sin 2\chi \\ 0 & 0 & 0 & 0 \\ -\cos 2\chi & 0 & \cos^2 2\chi & -\sin 2\chi \cos 2\chi \\ \sin 2\chi & 0 & -\sin 2\chi \cos 2\chi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \frac{p_1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\cos 2\chi & 0 & 0 & 0 \\ \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

3.17. Linear Polarizer

Ref. [1], Equation (4.93). $p_1 = 1, p_2 = 0$. Phase shift, $\Delta = 0$. Ellipticity angle $\chi = 0$. Oriented at an angle ϕ. $M_{00} = p_1^2/2$.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\phi & \sin 2\phi & 0 \\ \cos 2\phi & \cos^2 2\phi & \sin 2\phi \cos 2\phi & 0 \\ \sin 2\phi & \sin 2\phi \cos 2\phi & \sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\phi & \sin 2\phi & 0 \\ \cos 2\phi & \cos^2 2\phi & \sin 2\phi \cos 2\phi & 0 \\ \sin 2\phi & \sin 2\phi \cos 2\phi & \sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \cos 2\phi & 0 & 0 & 0 \\ \sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.18. Linear Polarizer at $\phi = 0°$

$\phi = 0°$. Ref. [1], Equation (4.94).

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.19. Linear Polarizer at $\phi = 45°$

$\phi = 45°$.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.20. Linear Polarizer at $\phi = 90°$

$\phi = 90°$.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.21. Linear Polarizer at $\phi = 135°$

$\phi = 135°$.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.22. Circular Polarizer

Ref. [1], Equation (4.91). $p_1 = 1, p_2 = 0, \kappa = 0°$.
Right, $\chi = 45°$.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Left, $\chi = -45°$.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

4. Table for Retarders

4.1. Elliptic Retarder

Phase shift Δ. Ellipticity angle $= \chi$. Oriented at an angle ϕ.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \begin{matrix}\cos\Delta \\ +(1-\cos\Delta)\cos^2 2\chi\cos^2 2\phi\end{matrix} & \begin{matrix}\sin^2\frac{\Delta}{2}\cos^2 2\chi\sin 4\phi \\ +\sin\Delta\sin 2\chi\end{matrix} & \begin{matrix}\sin^2\frac{\Delta}{2}\sin 4\chi\cos 2\phi \\ -\sin\Delta\cos 2\chi\sin 2\phi\end{matrix} \\ 0 & \begin{matrix}\sin^2\frac{\Delta}{2}\cos^2 2\chi\sin 4\phi \\ -\sin\Delta\sin 2\chi\end{matrix} & \begin{matrix}\cos\Delta \\ +(1-\cos\Delta)\cos^2 2\chi\sin^2 2\phi\end{matrix} & \begin{matrix}\sin^2\frac{\Delta}{2}\sin 4\chi\sin 2\phi \\ +\sin\Delta\cos 2\chi\cos 2\phi\end{matrix} \\ 0 & \begin{matrix}\sin^2\frac{\Delta}{2}\sin 4\chi\cos 2\phi \\ +\sin\Delta\cos 2\chi\sin 2\phi\end{matrix} & \begin{matrix}\sin^2\frac{\Delta}{2}\sin 4\chi\sin 2\phi \\ -\sin\Delta\cos 2\chi\cos 2\phi\end{matrix} & \begin{matrix}\sin^2 2\chi \\ +\cos\Delta\cos^2 2\chi\end{matrix} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & -i\sin\Delta\cos 2\chi\cos 2\phi & -i\sin\Delta\cos 2\chi\sin 2\phi & -i\sin\Delta\sin 2\chi \\ i\sin\Delta\cos 2\chi\cos 2\phi & 2\sin^2\frac{\Delta}{2}\cos^2 2\chi\cos^2 2\phi & \sin^2\frac{\Delta}{2}\cos^2 2\chi\sin 4\phi & \sin^2\frac{\Delta}{2}\sin 4\chi\cos 2\phi \\ i\sin\Delta\cos 2\chi\sin 2\phi & \sin^2\frac{\Delta}{2}\cos^2 2\chi\sin 4\phi & 2\sin^2\frac{\Delta}{2}\cos^2 2\chi\sin^2 2\phi & \sin^2\frac{\Delta}{2}\sin 4\chi\sin 2\phi \\ i\sin\Delta\sin 2\chi & \sin^2\frac{\Delta}{2}\sin 4\chi\cos 2\phi & \sin^2\frac{\Delta}{2}\sin 4\chi\sin 2\phi & 2\sin^2\frac{\Delta}{2}\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\cos 2\chi\cos 2\phi & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\cos 2\chi\sin 2\phi & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

The elements of **M** can be written in various different equivalent forms using the identities

$$\begin{aligned} \sin^2 2\chi + \cos\Delta\cos^2 2\chi &\equiv \cos^2\frac{\Delta}{2} - \sin^2\frac{\Delta}{2}\cos 4\chi, \\ \cos^2 2\chi + \cos\Delta\sin^2 2\chi &\equiv \cos^2\frac{\Delta}{2} + \sin^2\frac{\Delta}{2}\cos 4\chi, \\ \sin^2\frac{\Delta}{2}\sin 4\chi &\equiv (1-\cos\Delta)\sin 2\chi\cos 2\chi, \\ \sin^2\frac{\Delta}{2}\cos^2 2\chi \equiv \frac{1}{2}(1-\cos\Delta)\cos^2 2\chi &\equiv \frac{1}{2}\sin^2\frac{\Delta}{2}(1+\cos 4\chi) \equiv \frac{1}{4}(1-\cos\Delta)(1+\cos 4\chi). \end{aligned}$$

4.2. Elliptic Retarder at $\phi = 0^\circ$

Ref. [1], Equation (4.38).

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\chi + \cos\Delta\sin^2 2\chi & \sin\Delta\sin 2\chi & (1-\cos\Delta)\sin 2\chi\cos 2\chi \\ 0 & -\sin\Delta\sin 2\chi & \cos\Delta & \sin\Delta\cos 2\chi \\ 0 & (1-\cos\Delta)\sin 2\chi\cos 2\chi & -\sin\Delta\cos 2\chi & \sin^2 2\chi + \cos\Delta\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & -i\sin\Delta\cos 2\chi & 0 & -i\sin\Delta\sin 2\chi \\ i\sin\Delta\cos 2\chi & 2\sin^2\frac{\Delta}{2}\cos^2 2\chi & 0 & \sin^2\frac{\Delta}{2}\sin 4\chi \\ 0 & 0 & 0 & 0 \\ i\sin\Delta\sin 2\chi & \sin^2\frac{\Delta}{2}\sin 4\chi & 0 & 2\sin^2\frac{\Delta}{2}\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\cos 2\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

4.3. Elliptic Retarder at $\phi = 45^\circ$

$\phi = 45^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Delta & \sin\Delta\sin 2\chi & -\sin\Delta\cos 2\chi \\ 0 & -\sin\Delta\sin 2\chi & \cos^2 2\chi + \cos\Delta\sin^2 2\chi & \sin^2\frac{\Delta}{2}\sin 4\chi \\ 0 & \sin\Delta\cos 2\chi & \sin^2\frac{\Delta}{2}\sin 4\chi & \sin^2 2\chi + \cos\Delta\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & 0 & -i\sin\Delta\cos 2\chi & -i\sin\Delta\sin 2\chi \\ 0 & 0 & 0 & 0 \\ i\sin\Delta\cos 2\chi & 0 & 2\sin^2\frac{\Delta}{2}\cos^2 2\chi & \sin^2\frac{\Delta}{2}\sin 4\chi \\ i\sin\Delta\sin 2\chi & 0 & \sin^2\frac{\Delta}{2}\sin 4\chi & 2\sin^2\frac{\Delta}{2}\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\cos 2\chi & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

4.4. Elliptic Retarder at $\phi = 90^\circ$

$\phi = 90^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\chi + \cos\Delta\sin^2 2\chi & \sin\Delta\sin 2\chi & -\sin^2\frac{\Delta}{2}\sin 4\chi \\ 0 & -\sin\Delta\sin 2\chi & \cos\Delta & -\sin\Delta\cos 2\chi \\ 0 & -\sin^2\frac{\Delta}{2}\sin 4\chi & \sin\Delta\cos 2\chi & \sin^2 2\chi + \cos\Delta\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & i\sin\Delta\cos 2\chi & 0 & -i\sin\Delta\sin 2\chi \\ -i\sin\Delta\cos 2\chi & 2\sin^2\frac{\Delta}{2}\cos^2 2\chi & 0 & -\sin^2\frac{\Delta}{2}\sin 4\chi \\ 0 & 0 & 0 & 0 \\ i\sin\Delta\sin 2\chi & -\sin^2\frac{\Delta}{2}\sin 4\chi & 0 & 2\sin^2\frac{\Delta}{2}\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ -i\sin\frac{\Delta}{2}\cos 2\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

4.5. Elliptic Retarder at $\phi = 135°$

$\phi = 135°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Delta & \sin\Delta\sin 2\chi & \sin\Delta\cos 2\chi \\ 0 & -\sin\Delta\sin 2\chi & \cos^2 2\chi + \cos\Delta\sin^2 2\chi & -\sin^2\frac{\Delta}{2}\sin 4\chi \\ 0 & -\sin\Delta\cos 2\chi & -\sin^2\frac{\Delta}{2}\sin 4\chi & \sin^2 2\chi + \cos\Delta\cos^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & 0 & i\sin\Delta\cos 2\chi & -i\sin\Delta\sin 2\chi \\ 0 & 0 & 0 & 0 \\ -i\sin\Delta\cos 2\chi & 0 & 2\sin^2\frac{\Delta}{2}\cos^2 2\chi & -\sin^2\frac{\Delta}{2}\sin 4\chi \\ i\sin\Delta\sin 2\chi & 0 & -\sin^2\frac{\Delta}{2}\sin 4\chi & 2\sin^2\frac{\Delta}{2}\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\sin\frac{\Delta}{2}\cos 2\chi & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

4.6. Quarter-Wave Elliptic Retarder

$\Delta = 90°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\chi\cos^2 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi + \sin 2\chi & \cos 2\chi(\sin 2\chi\cos 2\phi - \sin 2\phi) \\ \cos^2 2\chi\sin^2 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi - \sin 2\chi & \cos^2 2\chi\sin^2 2\phi & \cos 2\chi(\sin 2\chi\sin 2\phi + \cos 2\phi) \\ 0 & \cos 2\chi(\sin 2\chi\cos 2\phi + \sin 2\phi) & \cos 2\chi(\sin 2\chi\sin 2\phi - \cos 2\phi) & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & -i\cos 2\chi\cos 2\phi & -i\cos 2\chi\sin 2\phi & -i\sin 2\chi \\ i\cos 2\chi\cos 2\phi & \cos^2 2\chi\cos^2 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi & \sin 2\chi\cos 2\chi\cos 2\phi \\ i\cos 2\chi\sin 2\phi & \cos^2 2\chi\sin 2\phi\cos 2\phi & \cos^2 2\chi\sin^2 2\phi & \sin 2\chi\cos 2\chi\sin 2\phi \\ i\sin 2\chi & \sin 2\chi\cos 2\chi\cos 2\phi & \sin 2\chi\cos 2\chi\sin 2\phi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\cos 2\chi\cos 2\phi & 0 & 0 & 0 \\ i\cos 2\chi\sin 2\phi & 0 & 0 & 0 \\ i\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

4.7. Half-Wave Elliptic Retarder

$\Delta = 180°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\chi\cos 4\phi - \sin^2 2\chi & \cos^2 2\chi\sin 4\phi & \sin 4\chi\cos 2\phi \\ 0 & \cos^2 2\chi\sin 4\phi & -\cos^2 2\chi\cos 4\phi - \sin^2 2\chi & \sin 4\chi\sin 2\phi \\ 0 & \sin 4\chi\cos 2\phi & \sin 4\chi\sin 2\phi & -\cos 4\chi \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\cos^2 2\chi\cos^2 2\phi & \cos^2 2\chi\sin 4\phi & \sin 4\chi\cos 2\phi \\ 0 & \cos^2 2\chi\sin 4\phi & 2\cos^2 2\chi\sin^2 2\phi & \sin 4\chi\sin 2\phi \\ 0 & \sin 4\chi\cos 2\phi & \sin 4\chi\sin 2\phi & 2\sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}i\begin{pmatrix} 0 & 0 & 0 & 0 \\ \cos 2\chi\cos 2\phi & 0 & 0 & 0 \\ \cos 2\chi\sin 2\phi & 0 & 0 & 0 \\ \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

4.8. Linear Retarder

Ref. [1], Equation (4.40). $\chi = 0^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\phi + \cos\Delta \sin^2 2\phi & \sin^2\frac{\Delta}{2}\sin 4\phi & -\sin\Delta\sin 2\phi \\ 0 & \sin^2\frac{\Delta}{2}\sin 4\phi & \sin^2 2\phi + \cos\Delta\cos^2 2\phi & \sin\Delta\cos 2\phi \\ 0 & \sin\Delta\sin 2\phi & -\sin\Delta\cos 2\phi & \cos\Delta \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & -i\sin\Delta\cos 2\phi & -i\sin\Delta\sin 2\phi & 0 \\ i\sin\Delta\cos 2\phi & 2\sin^2\frac{\Delta}{2}\cos^2 2\phi & \sin^2\frac{\Delta}{2}\sin 4\phi & 0 \\ i\sin\Delta\sin 2\phi & \sin^2\frac{\Delta}{2}\sin 4\phi & 2\sin^2\frac{\Delta}{2}\sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\cos 2\phi & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2}\sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.9. Linear Retarder at $\phi = 0^\circ$

$\phi = 0^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Delta & \sin\Delta \\ 0 & 0 & -\sin\Delta & \cos\Delta \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & -i\sin\Delta & 0 & 0 \\ i\sin\Delta & 2\sin^2\frac{\Delta}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.10. Linear Retarder at $\phi = 45^\circ$

$\phi = 45^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Delta & 0 & -\sin\Delta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\Delta & 0 & \cos\Delta \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & 0 & -i\sin\Delta & 0 \\ 0 & 0 & 0 & 0 \\ i\sin\Delta & 0 & 2\sin^2\frac{\Delta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\sin\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.11. Linear Retarder at $\phi = 90^\circ$

$\phi = 90^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Delta & -\sin\Delta \\ 0 & 0 & \sin\Delta & \cos\Delta \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & i\sin\Delta & 0 & 0 \\ -i\sin\Delta & 2\sin^2\frac{\Delta}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ -i\sin\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.12. Linear Retarder at $\phi = 135^\circ$

$\phi = 135^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Delta & 0 & \sin\Delta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\Delta & 0 & \cos\Delta \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & 0 & i\sin\Delta & 0 \\ 0 & 0 & 0 & 0 \\ -i\sin\Delta & 0 & 2\sin^2\frac{\Delta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\sin\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.13. Quarter-Wave Linear Retarder

$\Delta = 90^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\phi & \sin 2\phi\cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi\cos 2\phi & \sin^2 2\phi & \cos 2\phi \\ 0 & \sin 2\phi & -\cos 2\phi & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & -i\cos 2\phi & -i\sin 2\phi & 0 \\ i\cos 2\phi & \cos^2 2\phi & \sin 2\phi\cos 2\phi & 0 \\ i\sin 2\phi & \sin 2\phi\cos 2\phi & \sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\cos 2\phi & 0 & 0 & 0 \\ i\sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.14. Quarter-Wave Linear Retarder at $\phi = 0^\circ$

$\phi = 0^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.15. Quarter-Wave Linear Retarder at $\phi = 45^\circ$

$\phi = 45^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.16. Quarter-Wave Linear Retarder at $\phi = 90^\circ$

$\phi = 90^\circ$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.17. Quarter-Wave Linear Retarder at $\phi = 135°$

$\phi = 135°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.18. Half-Wave Linear Retarder (Pseudorotator)

Ref. [1], Equation (4.42). $\Delta = 180°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\phi & \sin 4\phi & 0 \\ 0 & \sin 4\phi & -\cos 4\phi & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\cos^2 2\phi & \sin 4\phi & 0 \\ 0 & \sin 4\phi & 2\sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}i \begin{pmatrix} 0 & 0 & 0 & 0 \\ \cos 2\phi & 0 & 0 & 0 \\ \sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.19. Half-Wave Linear Retarder at $\phi = 0°$

$\phi = 0°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{C} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \sqrt{2}i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.20. Half-Wave Linear Retarder at $\phi = 45°$

$\phi = 45°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{C} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = \sqrt{2}i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.21. Half-Wave Linear Retarder at $\phi = 90°$

M, **C** are the same as for $\phi = 0°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{C} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = -\sqrt{2}i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.22. Half-Wave Linear Retarder at $\phi = 135°$

M, **C** are the same as for $\phi = 45°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{F} = -\sqrt{2}i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.23. Circular Retarder (Rotator, with $2\theta = \Delta$)

Ref. [1], Equation (4.30). $\chi = \pm 45°$, for right or left.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Delta & \pm\sin\Delta & 0 \\ 0 & \mp\sin\Delta & \cos\Delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2\cos^2\frac{\Delta}{2} & 0 & 0 & \mp i\sin\Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm i\sin\Delta & 0 & 0 & 2\sin^2\frac{\Delta}{2} \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2}\begin{pmatrix} \cos\frac{\Delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm i\sin\frac{\Delta}{2} & 0 & 0 & 0 \end{pmatrix}.$$

4.24. Quarter-Wave Circular Retarder, Right

$\chi = 45°$, $\Delta = 90°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

4.25. Quarter-Wave Circular Retarder, Left

$\chi = -45°$, $\Delta = 90°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \sqrt{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

4.26. Half-Wave Circular Retarder

$\Delta = 180°, \theta = 90°$.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \pm i\sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

5. Table for Other Systems

5.1. Elliptic Homogeneous Retarding Diattenuater

If a device exhibits both retardance and diattenuation, it could be called either a retarding diattenuater or a diattenuating retarder. The special case of a homogeneous retarding diattenuater can result from an aligned series combination of a retarder and a diattenuater. If these two elements are not aligned, the overall device is inhomogeneous. See Section 8 for further discussion. As the expressions for **M**, **C** are lengthy, the elements are presented. There is a singularity if $\Delta = 180°, \kappa = 90°$, where the absolute phase of **F** is undefined.

$$\begin{aligned}
M_{01,10} =& M_{00}\cos\kappa\cos 2\chi\cos 2\phi,\\
M_{02,20} =& M_{00}\cos\kappa\cos 2\chi\sin 2\phi,\\
M_{03,30} =& M_{00}\cos\kappa\sin 2\chi,\\
M_{11} =& M_{00}\left[\cos\Delta\sin\kappa + (1-\cos\Delta\sin\kappa)\cos^2 2\chi\cos^2 2\phi\right],\\
M_{22} =& M_{00}\left[\cos\Delta\sin\kappa + (1-\cos\Delta\sin\kappa)\cos^2 2\chi\sin^2 2\phi\right],\\
M_{33} =& M_{00}(\sin^2 2\chi + \cos\Delta\sin\kappa\cos^2 2\chi),\\
M_{12,21} =& M_{00}\left[\pm\sin\Delta\sin\kappa\sin 2\chi + (1-\cos\Delta\sin\kappa)\cos^2 2\chi\sin 2\phi\cos 2\phi\right],\\
M_{13,31} =& M_{00}[\mp\sin\Delta\sin\kappa\cos 2\chi\sin 2\phi + (1-\cos\Delta\sin\kappa)\sin 2\chi\cos 2\chi\cos 2\phi],\\
M_{23,32} =& M_{00}[\pm\sin\Delta\sin\kappa\sin 2\chi\cos 2\phi + (1-\cos\Delta\sin\kappa)\sin 2\chi\cos 2\chi\sin 2\phi],
\end{aligned}$$

$$\begin{aligned}
C_{00} =& M_{00}(1+\cos\Delta\sin\kappa),\\
C_{01,10} =& M_{00}(\cos\kappa \mp i\sin\Delta\sin\kappa)\cos 2\chi\cos 2\phi,\\
C_{02,20} =& M_{00}(\cos\kappa \mp i\sin\Delta\sin\kappa)\cos 2\chi\sin 2\phi,\\
C_{03,30} =& M_{00}(\cos\kappa \mp i\sin\Delta\sin\kappa)\sin 2\chi,\\
C_{11} =& M_{00}(1-\cos\Delta\sin\kappa)\cos^2 2\chi\cos^2 2\phi,\\
C_{22} =& M_{00}(1-\cos\Delta\sin\kappa)\cos^2 2\chi\sin^2 2\phi,\\
C_{33} =& M_{00}(1-\cos\Delta\sin\kappa)\sin^2 2\chi,\\
C_{12,21} =& M_{00}(1-\cos\Delta\sin\kappa)\cos^2 2\chi\sin 2\phi\cos 2\phi,\\
C_{13,31} =& M_{00}(1-\cos\Delta\sin\kappa)\sin 2\chi\cos 2\chi\cos 2\phi,\\
C_{23,32} =& M_{00}(1-\cos\Delta\sin\kappa)\sin 2\chi\cos 2\chi\sin 2\phi,
\end{aligned}$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0\\ (\cos\kappa + i\sin\Delta\sin\kappa)\cos 2\chi\cos 2\phi & 0 & 0 & 0\\ (\cos\kappa + i\sin\Delta\sin\kappa)\cos 2\chi\sin 2\phi & 0 & 0 & 0\\ (\cos\kappa + i\sin\Delta\sin\kappa)\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

5.2. Circular Retarding Diattenuater

See Section 8. $\chi = 45^\circ$. There is a singularity if $\Delta = 180^\circ, \kappa = 90^\circ$, where the absolute phase of $\mathbf{F}$ is undefined.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & 0 & \cos\kappa\\ 0 & \cos\Delta\sin\kappa & \sin\Delta\sin\kappa & 0\\ 0 & -\sin\Delta\sin\kappa & \cos\Delta\sin\kappa & 0\\ \cos\kappa & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & \cos\kappa - i\sin\Delta\sin\kappa\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \cos\kappa + i\sin\Delta\sin\kappa & 0 & 0 & 1-\cos\Delta\sin\kappa \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \cos\kappa + i\sin\Delta\sin\kappa & 0 & 0 & 0 \end{pmatrix}.$$

$\chi = -45°$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 0 & 0 & 0 & -\cos\kappa \\ 0 & \cos\Delta\sin\kappa & -\sin\Delta\sin\kappa & 0 \\ 0 & \sin\Delta\sin\kappa & \cos\Delta\sin\kappa & 0 \\ -\cos\kappa & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} (1+\cos\Delta\sin\kappa) & 0 & 0 & -(\cos\kappa - i\sin\Delta\sin\kappa) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(\cos\kappa + i\sin\Delta\sin\kappa) & 0 & 0 & (1-\cos\Delta\sin\kappa) \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(\cos\kappa + i\sin\Delta\sin\kappa) & 0 & 0 & 0 \end{pmatrix}.$$

5.3. Linear Homogeneous Retarding Diattenuater

Terminology as in Ref. [1], Equation (4.98). Ellipticity angle, $\chi = 0°$. Phase shift, Δ. Oriented at an azimuthal angle ϕ. There is a singularity if $\Delta = 180°, \kappa = 90°$, where the absolute phase of **F** is undefined.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & \cos\kappa\cos 2\phi & \cos\kappa\sin 2\phi & 0 \\ \cos\kappa\cos 2\phi & \cos^2 2\phi + \cos\Delta\sin\kappa\sin^2 2\phi & (1-\cos\Delta\sin\kappa)\sin 2\phi\cos 2\phi & -\sin\Delta\sin\kappa\sin 2\phi \\ \cos\kappa\sin 2\phi & (1-\cos\Delta\sin\kappa)\sin 2\phi\cos 2\phi & \sin^2 2\phi + \cos\Delta\sin\kappa\cos^2 2\phi & \sin\Delta\sin\kappa\cos 2\phi \\ 0 & \sin\Delta\sin\kappa\sin 2\phi & -\sin\Delta\sin\kappa\cos 2\phi & \cos\Delta\sin\kappa \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\cos\Delta\sin\kappa & (\cos\kappa - i\sin\Delta\sin\kappa)\cos 2\phi & (\cos\kappa - i\sin\Delta\sin\kappa)\sin 2\phi & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa)\cos 2\phi & (1-\cos\Delta\sin\kappa)\cos^2 2\phi & (1-\cos\Delta\sin\kappa)\sin 2\phi\cos 2\phi & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa)\sin 2\phi & (1-\cos\Delta\sin\kappa)\sin 2\phi\cos 2\phi & (1-\cos\Delta\sin\kappa)\sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa)\cos 2\phi & 0 & 0 & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa)\sin 2\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.4. Linear Homogeneous Retarding Diattenuater at $\phi = 0°$ (Horizontal)

$\phi = 0°$. Ref. [1], Equation (4.99). There is a singularity if $\Delta = 180°, \kappa = 90°$, where the absolute phase of **F** is undefined.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & \cos\kappa & 0 & 0 \\ \cos\kappa & 1 & 0 & 0 \\ 0 & 0 & \cos\Delta\sin\kappa & \sin\Delta\sin\kappa \\ 0 & 0 & -\sin\Delta\sin\kappa & \cos\Delta\sin\kappa \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\cos\Delta\sin\kappa & (\cos\kappa - i\sin\Delta\sin\kappa) & 0 & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa) & (1-\cos\Delta\sin\kappa) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.5. Linear Homogeneous Retarding Diattenuater at $\phi = 45°$

$\phi = 45°$. There is a singularity if $\Delta = 180°, \kappa = 90°$, where the absolute phase of **F** is undefined.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & \cos\kappa & 0 \\ 0 & \cos\Delta\sin\kappa & 0 & -\sin\Delta\sin\kappa \\ \cos\kappa & 0 & 1 & 0 \\ 0 & \sin\Delta\sin\kappa & 0 & \cos\Delta\sin\kappa \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & (\cos\kappa - i\sin\Delta\sin\kappa) & 0 \\ 0 & 0 & 0 & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa) & 0 & (1-\cos\Delta\sin\kappa) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (\cos\kappa + i\sin\Delta\sin\kappa) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.6. Linear Homogeneous Retarding Diattenuater at $\phi = 90°$

$\phi = 90°$. There is a singularity if $\Delta = 180°, \kappa = 90°$, where the absolute phase of **F** is undefined.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & -\cos\kappa & 0 & 0 \\ -\cos\kappa & 1 & 0 & 0 \\ 0 & 0 & \cos\Delta\sin\kappa & -\sin\Delta\sin\kappa \\ 0 & 0 & \sin\Delta\sin\kappa & \cos\Delta\sin\kappa \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\cos\Delta\sin\kappa & -(\cos\kappa - i\sin\Delta\sin\kappa) & 0 & 0 \\ -(\cos\kappa + i\sin\Delta\sin\kappa) & (1-\cos\Delta\sin\kappa) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0 \\ -(\cos\kappa + i\sin\Delta\sin\kappa) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.7. Linear Homogeneous Retarding Diattenuater at $\phi = 135°$

$\phi = 135°$. There is a singularity if $\Delta = 180°, \kappa = 90°$, where the absolute phase of **F** is undefined.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & 0 & -\cos\kappa & 0 \\ 0 & \cos\Delta\sin\kappa & 0 & \sin\Delta\sin\kappa \\ -\cos\kappa & 0 & 1 & 0 \\ 0 & -\sin\Delta\sin\kappa & 0 & \cos\Delta\sin\kappa \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & -(\cos\kappa - i\sin\Delta\sin\kappa) & 0 \\ 0 & 0 & 0 & 0 \\ -(\cos\kappa + i\sin\Delta\sin\kappa) & 0 & (1-\cos\Delta\sin\kappa) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{\sqrt{M_{00}}}{\sqrt{1+\cos\Delta\sin\kappa}}\begin{pmatrix} 1+\cos\Delta\sin\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(\cos\kappa + i\sin\Delta\sin\kappa) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.8. Quarter-Wave Elliptic Homogeneous Retarding Diattenuater

$\Delta = 90°$.

$$\begin{aligned}
M_{01,10} =& M_{00} \cos\kappa \cos 2\chi \cos 2\phi, \\
M_{02,20} =& M_{00} \cos\kappa \cos 2\chi \sin 2\phi, \\
M_{03,30} =& M_{00} \cos\kappa \sin 2\chi, \\
M_{11} =& M_{00} \cos^2 2\chi \cos^2 2\phi, \\
M_{22} =& M_{00} \cos^2 2\chi \sin^2 2\phi, \\
M_{33} =& M_{00} \sin^2 2\chi, \\
M_{12,21} =& M_{00}(\pm \sin\kappa \sin 2\chi + \cos^2 2\chi \sin 2\phi \cos 2\phi), \\
M_{13,31} =& M_{00} \cos 2\chi(\mp \sin\kappa \sin 2\phi + \sin 2\chi \cos 2\phi), \\
M_{23,32} =& M_{00} \sin 2\chi(\pm \sin\kappa \cos 2\phi + \cos 2\chi \sin 2\phi),
\end{aligned}$$

$$\mathbf{C} = M_{00} \begin{pmatrix} 1 & e^{-i\kappa} \cos 2\chi \cos 2\phi & e^{-i\kappa} \cos 2\chi \sin 2\phi & e^{-i\kappa} \sin 2\chi \\ e^{i\kappa} \cos 2\chi \cos 2\phi & \cos^2 2\chi \cos^2 2\phi & \cos^2 2\chi \sin 2\phi \cos 2\phi & \sin 2\chi \cos 2\chi \cos 2\phi \\ e^{i\kappa} \cos 2\chi \sin 2\phi & \cos^2 2\chi \sin 2\phi \cos 2\phi & \cos^2 2\chi \sin^2 2\phi & \sin 2\chi \cos 2\chi \sin 2\phi \\ e^{i\kappa} \sin 2\chi & \sin 2\chi \cos 2\chi \cos 2\phi & \sin 2\chi \cos 2\chi \sin 2\phi & \sin^2 2\chi \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{i\kappa} \cos 2\chi \cos 2\phi & 0 & 0 & 0 \\ e^{i\kappa} \cos 2\chi \sin 2\phi & 0 & 0 & 0 \\ e^{i\kappa} \sin 2\chi & 0 & 0 & 0 \end{pmatrix}.$$

5.9. Half-Wave Elliptic Homogeneous Retarding Diattenuater

$\Delta = 180°$. There is a singularity if $\kappa = 90°$, so the absolute phase of $\mathbf{F}$ is undefined and does not agree with that given for a pure retarder in Sections 4.7 and 4.18.

$$\begin{aligned}
M_{01,10} =& M_{00} \cos\kappa \cos 2\chi \cos 2\phi, \\
M_{02,20} =& M_{00} \cos\kappa \cos 2\chi \sin 2\phi, \\
M_{03,30} =& M_{00} \cos\kappa \sin 2\chi, \\
M_{11} =& M_{00}\Big[(1 + \sin\kappa) \cos^2 2\chi \cos^2 2\phi - \sin\kappa\Big], \\
M_{22} =& M_{00}\Big[(1 + \sin\kappa) \cos^2 2\chi \sin^2 2\phi - \sin\kappa\Big], \\
M_{33} =& M_{00}\Big[(1 + \sin\kappa) \sin^2 2\chi - \sin\kappa\Big], \\
M_{12,21} =& M_{00}\Big[(1 + \sin\kappa) \cos^2 2\chi \sin 2\phi \cos 2\phi\Big], \\
M_{13,31} =& M_{00}[(1 + \sin\kappa) \sin 2\chi \cos 2\chi \cos 2\phi], \\
M_{23,32} =& M_{00}[(1 + \sin\kappa) \sin 2\chi \cos 2\chi \sin 2\phi],
\end{aligned}$$

$$
\begin{aligned}
C_{00} =& M_{00}(1-\sin\kappa),\\
C_{01,10} =& M_{00}\cos\kappa\cos 2\chi\cos 2\phi,\\
C_{02,20} =& M_{00}\cos\kappa\cos 2\chi\sin 2\phi,\\
C_{03,30} =& M_{00}\cos\kappa\sin 2\chi,\\
C_{11} =& M_{00}(1+\sin\kappa)\cos^2 2\chi\cos^2 2\phi,\\
C_{22} =& M_{00}(1+\sin\kappa)\cos^2 2\chi\sin^2 2\phi,\\
C_{33} =& M_{00}(1+\sin\kappa)\sin^2 2\chi,\\
C_{12,21} =& M_{00}(1+\sin\kappa)\cos^2 2\chi\sin 2\phi\cos 2\phi,\\
C_{13,31} =& M_{00}(1+\sin\kappa)\sin 2\chi\cos 2\chi\cos 2\phi,\\
C_{23,32} =& M_{00}(1+\sin\kappa)\sin 2\chi\cos 2\chi\sin 2\phi,
\end{aligned}
$$

$$
\mathbf{F} = \sqrt{M_{00}}\begin{pmatrix} \sqrt{1-\sin\kappa} & 0 & 0 & 0\\ \sqrt{1+\sin\kappa}\cos 2\chi\cos 2\phi & 0 & 0 & 0\\ \sqrt{1+\sin\kappa}\cos 2\chi\sin 2\phi & 0 & 0 & 0\\ \sqrt{1+\sin\kappa}\sin 2\chi & 0 & 0 & 0 \end{pmatrix}.
$$

5.10. Inhomogeneous Right Circular Polarizer

An inhomogeneous circular polarizer constructed from a horizontal linear polarizer, followed by a quarter-wave linear retarder at 45° [2].

$$
\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 \end{pmatrix}, \mathbf{C} = \frac{1}{4}\begin{pmatrix} 1 & 1 & -i & 1\\ 1 & 1 & -i & 1\\ i & i & 1 & i\\ 1 & 1 & -i & 1 \end{pmatrix}, \mathbf{F} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ i & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

It is seen that the phase of the components z_1, z_2, z_3 are not all equal, so the system is inhomogeneous.

5.11. Normal Reflection from a Dielectric

Apart from multiplying factors, $\mathbf{M}, \mathbf{C}, \mathbf{F}$ are the same as for a half-wave linear retarder at $\phi = 0°, 90°$, as in Sections 4.19 and 4.21. Refractive index n. The negative sign in $\mathbf{F}$ accounts for a phase change on reflection, and cancels on calculating $\mathbf{C}$ [25].

$$
\mathbf{M} = \left(\frac{n-1}{n+1}\right)^2\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{C} = 2\left(\frac{n-1}{n+1}\right)^2\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\mathbf{F} = -\sqrt{2}\left(\frac{n-1}{n+1}\right)\begin{pmatrix} 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

5.12. Aligned Linear Retarding Diattenuater: ABCD Matrix

Ref. [14]

$$\mathbf{M} = \begin{pmatrix} A & B & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -D & C \end{pmatrix}, \mathbf{C} = \begin{pmatrix} A+C & B-iD & 0 & 0 \\ B+iD & A-C & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{\sqrt{2R}} \begin{pmatrix} (R+C)\sqrt{(R-C)(A+R)} & -(R-C)\sqrt{(R+C)(A-R)} & 0 & 0 \\ \sqrt{(A+R)/(R+C)}(B+iD) & \sqrt{(A-R)/(R-C)}(B-iD) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$A \geq R, R = \sqrt{B^2+C^2+D^2}$. There are two homogeneous deterministic components in general, reducing to one if $A = R$.

5.13. Medium of Scattering Particles Obeying Reciprocity and with a Plane of Symmetry

This is the most common form used in the literature on scattering by nonspherical particles. Refs. [6,26].

$$\mathbf{M} = \begin{pmatrix} A_0+B_0 & C & 0 & 0 \\ C & A+B & 0 & 0 \\ 0 & 0 & A-B & D \\ 0 & 0 & -D & A_0-B_0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} A_0+A & C-iD & 0 & 0 \\ C+iD & B_0+B & 0 & 0 \\ 0 & 0 & B_0-B & 0 \\ 0 & 0 & 0 & A_0-A \end{pmatrix},$$

$A_0 \geq A \geq 0, B_0 \geq B \geq 0$. There are, in general, four deterministic components.

5.14. Rotationally Symmetric Medium of Asymmetric Scatterers in the Direct Forward Scattering Direction

Refs. [6,26].

$$\mathbf{M} = \begin{pmatrix} A_0+B_0 & 0 & 0 & F+I \\ 0 & A & J & 0 \\ 0 & -J & A & 0 \\ I-F & 0 & 0 & A_0-B_0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} A_0+A & 0 & 0 & I-iJ \\ 0 & B_0 & iF & 0 \\ 0 & -iF & B_0 & 0 \\ I+iJ & 0 & 0 & A_0-A \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{(R+A)\sqrt{(R-A)(A_0+R)}}{R} & -\frac{(R-A)\sqrt{(R+A)(A_0-R)}}{R} & 0 & 0 \\ 0 & 0 & \sqrt{B_0+F} & \sqrt{B_0-F} \\ 0 & 0 & -i\sqrt{B_0+F} & i\sqrt{B_0-F} \\ \sqrt{\frac{A_0+R}{R(R+A)}}(I+iJ) & \sqrt{\frac{A_0-R}{R(R-A)}}(I+iJ) & 0 & 0 \end{pmatrix},$$

$A_0 \geq R \geq 0, B_0 \geq F, R = \sqrt{A^2+I^2+J^2}$.

5.15. Medium of Asymmetric Scatterers in the Exact Backscattering Direction

Refs. [26–28].

$$\mathbf{M} = \begin{pmatrix} A_0+B & C & L & I \\ C & A+B & J & K \\ -L & -J & A-B & D \\ I & K & -D & A_0-B \end{pmatrix}, \mathbf{C} = \begin{pmatrix} A_0+A & C-iD & 0 & I-iJ \\ C+iD & 2B & 0 & K-iL \\ 0 & 0 & 0 & 0 \\ I+iJ & K+iL & 0 & A_0-A \end{pmatrix},$$

$A_0 \geq A \geq 0, B \geq 0$. Maximum of three deterministic components.

5.16. Medium of Asymmetric Scatterers in the Non-Exact Backscattering Direction, with Rotational Symmetry of the Medium

Refs. [6,26].

$$\mathbf{M} = \begin{pmatrix} A_1+B & 0 & 0 & I \\ 0 & B & 0 & 0 \\ 0 & 0 & -B & 0 \\ I & 0 & 0 & A_2-B \end{pmatrix},$$

$$\mathbf{C} = \frac{1}{2}\begin{pmatrix} A_1+A_2 & 0 & 0 & 2I \\ 0 & 4B+A_1-A_2 & 0 & 0 \\ 0 & 0 & A_1-A_2 & 0 \\ 2I & 0 & 0 & A_1+A_2 \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{2}\begin{pmatrix} \sqrt{A_1+A_2+2I} & \sqrt{A_1+A_2-2I} & 0 & 0 \\ 0 & 0 & -\sqrt{2(4B+A_1-A_2)} & 0 \\ 0 & 0 & 0 & \sqrt{2(A_1-A_2)} \\ \sqrt{A_1+A_2+2I} & -\sqrt{A_1+A_2-2I} & 0 & 0 \end{pmatrix},$$

$A_1 \geq A_2 \geq 0, A_1+A_2 \geq |2I|, B \geq 4(A_2-A_1)$.

5.17. Medium of Asymmetric Scatterers in the Exact Backscattering Direction, with Rotational Symmetry of the Medium

Refs. [6,14,25–29]. $A_1 = A_2 = A_0$.

$$\mathbf{M} = \begin{pmatrix} A_0+B & 0 & 0 & I \\ 0 & B & 0 & 0 \\ 0 & 0 & -B & 0 \\ I & 0 & 0 & A_0-B \end{pmatrix}, \mathbf{C} = \begin{pmatrix} A_0 & 0 & 0 & I \\ 0 & 2B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & A_0 \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{A_0+I} & \sqrt{A_0-I} & 0 & 0 \\ 0 & 0 & -2\sqrt{B} & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{A_0+I} & -\sqrt{A_0-I} & 0 & 0 \end{pmatrix},$$

$A_0 \geq 0, A_0 \geq |I|, B \geq 0$. Maximum of three deterministic components.

5.18. G-Antisymmetric Mueller Matrix

With equal diagonal elements. Ref. [30].

$$\mathbf{M} = \begin{pmatrix} d & p_1 & p_2 & p_3 \\ p_1 & d & p_6 & -p_5 \\ p_2 & -p_6 & d & p_4 \\ p_3 & p_5 & -p_4 & d \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2d & p_1-ip_4 & p2-ip_5 & p_3-ip_6 \\ p_1+ip_4 & 0 & 0 & 0 \\ p2+ip_5 & 0 & 0 & 0 \\ p_3+ip_6 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \begin{pmatrix} (d+r)^{3/2} & (d+r)^{3/2} & 0 & 0 \\ \sqrt{d+r}(p_1+ip_4) & \sqrt{d-r}(p_1+ip_4) & 0 & 0 \\ \sqrt{d+r}(p_2+ip_5) & \sqrt{d-r}(p_2+ip_5) & 0 & 0 \\ \sqrt{d+r}(p_3+ip_6) & \sqrt{d-r}(p_3+ip_6) & 0 & 0 \end{pmatrix},$$

$r = \sqrt{d^2+p_1^2+p_2^2+p_3^2+p_4^2+p_5^2+p_6^2}, d \leq r$.

5.19. G-Symmetric Mueller Matrix

Diagonal elements may be different.

$$
\mathbf{M} = \begin{pmatrix} d_0 & p_1 & p_2 & p_3 \\ -p_1 & d_1 & p_6 & p_5 \\ -p_2 & p_6 & d_2 & p_4 \\ -p_3 & p_5 & p_4 & d_3 \end{pmatrix},
$$

$$
\mathbf{C} = \frac{1}{2} \begin{pmatrix} d_0+d_1+d_2+d_3 & 0 & 0 & 0 \\ 0 & d_1+d_2-d_3-d_4 & 2(p_6+ip_3) & 2(p_5-ip_3) \\ 0 & 2(p_6-ip_3) & d_0-d_1+d_2-d_3 & 2(p_4+ip_1) \\ 0 & 2(p_5+ip_2) & 2(p_4-ip_1) & d_0-d_1-d_2+d_3 \end{pmatrix}.
$$

5.20. Symmetric Mueller Matrix

Diagonal elements may be different.

$$
\mathbf{M} = \begin{pmatrix} d_0 & p_1 & p_2 & p_3 \\ p_1 & d_1 & p_6 & p_5 \\ p_2 & p_6 & d_2 & p_4 \\ p_3 & p_5 & p_4 & d_3 \end{pmatrix},
$$

$$
\mathbf{C} = \frac{1}{2} \begin{pmatrix} d_0+d_1+d_2+d_3 & p_1 & p_2 & p_3 \\ p_1 & d_0+d_1-d_2-d_3 & p_6 & p_5 \\ p_2 & p_6 & d_0-d_1+d_2-d_3 & p_4 \\ p_3 & p_5 & p_4 & d_0-d_1-d_2+d_3 \end{pmatrix}.
$$

5.21. Canonical Mueller Matrix, Type-I

Ref. [31].

$$
\mathbf{M} = \begin{pmatrix} d_0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_3 \end{pmatrix},
$$

$$
\mathbf{C} = \frac{1}{2} \begin{pmatrix} d_0+d_1+d_2+d_3 & 0 & 0 & 0 \\ 0 & d_0+d_1-d_2-d_3 & 0 & 0 \\ 0 & 0 & d_0-d_1+d_2-d_3 & 0 \\ 0 & 0 & 0 & d_0-d_1-d_2+d_3 \end{pmatrix},
$$

$$
\mathbf{F} = \sqrt{2} \begin{pmatrix} \sqrt{d_0+d_1+d_2+d_3} & 0 & 0 & 0 \\ 0 & \sqrt{d_0+d_1-d_2-d_3} & 0 & 0 \\ 0 & 0 & \sqrt{d_0-d_1+d_2-d_3} & 0 \\ 0 & 0 & 0 & \sqrt{d_0-d_1-d_2+d_3} \end{pmatrix}.
$$

5.22. Canonical Mueller Matrix, Type-II (Bolshakov)

Refs. [32–34].

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 2d_0 & d_0 - d_1 & 0 & 0 \\ -(d_0 - d_1) & 2d_1 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix},$$

$$\mathbf{C} = \frac{1}{2}\begin{pmatrix} d_0 + d_1 + d_2 & 0 & 0 & 0 \\ 0 & d_0 + d_1 - d_2 & 0 & 0 \\ 0 & 0 & d_0 - d_1 & i(d_0 - d_1) \\ 0 & 0 & -i(d_0 - d_1) & d_0 - d_1 \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{d_0 + d_1 + d_2} & 0 & 0 & 0 \\ 0 & \sqrt{d_0 + d_1 - d_2} & 0 & 0 \\ 0 & 0 & i\sqrt{d_0 - d_1} & 0 \\ 0 & 0 & \sqrt{d_0 - d_1} & 0 \end{pmatrix}.$$

5.23. Canonical Mueller Matrix, Type-II (Ossikovski 1)

Ref. [35].

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 2d_0 & d_0 & 0 & 0 \\ d_0 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}, \mathbf{C} = \frac{1}{2}\begin{pmatrix} d_0 + d_2 & 0 & 0 & 0 \\ 0 & d_0 - d_2 & 0 & 0 \\ 0 & 0 & d_0 & -id_0 \\ 0 & 0 & id_0 & d_0 \end{pmatrix},$$

$$\mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{d_0 + d_2} & 0 & 0 & 0 \\ 0 & \sqrt{d_0 - d_2} & 0 & 0 \\ 0 & 0 & -i\sqrt{d_0} & 0 \\ 0 & 0 & \sqrt{d_0} & 0 \end{pmatrix}.$$

5.24. Canonical Mueller matrix, Type-II (Ossikovski 2)

Ref. [36].

$$\mathbf{M} = \begin{pmatrix} d_0 & -d_1 & 0 & 0 \\ d_1 & d_0 - 2d_1 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} d_0 - d_1 + d_2 & 0 & 0 & 0 \\ 0 & d_0 - d_1 - d_2 & 0 & 0 \\ 0 & 0 & d_1 & -id_1 \\ 0 & 0 & id_1 & d_1 \end{pmatrix},$$

$$\mathbf{F} = \begin{pmatrix} \sqrt{d_0 - d_1 + d_2} & 0 & 0 & 0 \\ 0 & \sqrt{d_0 - d_1 - d_2} & 0 & 0 \\ 0 & 0 & -i\sqrt{d_1} & 0 \\ 0 & 0 & \sqrt{d_1} & 0 \end{pmatrix}.$$

6. Parameterized Deterministic Mueller Matrix

6.1. Parameterized Deterministic Mueller Matrix

Following Ref. [21], l, m, n are direction cosines in Poincaré space; $l = \cos 2\chi \cos 2\phi$, $m = \cos 2\chi \sin 2\phi$, $n = \sin 2\chi$; $l^2 + m^2 + n^2 = 1$. $\sin\psi$ determines the strength of the polarization transformation. θ_i determine the balance between retardance and diattenuation. $\Delta\theta_i = (\theta_{i-1} - \theta_{i+1})$ in cyclic order, so that $\Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3 = 0$.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & \begin{matrix} l\sin\psi\cos\theta_1 \\ +mn(1-\cos\psi)\sin\Delta\theta_1 \end{matrix} & \begin{matrix} m\sin\psi\cos\theta_2 \\ +nl(1-\cos\psi)\sin\Delta\theta_2 \end{matrix} & \begin{matrix} n\sin\psi\cos\theta_3 \\ +lm(1-\cos\psi)\sin\Delta\theta_3 \end{matrix} \\ \begin{matrix} l\sin\psi\cos\theta_1 \\ -mn(1-\cos\psi)\sin\Delta\theta_1 \end{matrix} & \cos\psi + l^2(1-\cos\psi) & \begin{matrix} n\sin\psi\sin\theta_3 \\ +lm(1-\cos\psi) \end{matrix} & \begin{matrix} -m\sin\psi\sin\theta_2 \\ +nl(1-\cos\psi) \end{matrix} \\ \begin{matrix} m\sin\psi\cos\theta_2 \\ -nl(1-\cos\psi)\sin\Delta\theta_2 \end{matrix} & \begin{matrix} -n\sin\psi\sin\theta_3 \\ +lm(1-\cos\psi) \end{matrix} & \cos\psi + m^2(1-\cos\psi) & \begin{matrix} l\sin\psi\sin\theta_1 \\ +mn(1-\cos\psi) \end{matrix} \\ \begin{matrix} n\sin\psi\cos\theta_3 \\ -lm(1-\cos\psi)\sin\Delta\theta_3 \end{matrix} & \begin{matrix} m\sin\psi\sin\theta_2 \\ +nl(1-\cos\psi) \end{matrix} & \begin{matrix} -l\sin\psi\sin\theta_1 \\ +mn(1-\cos\psi) \end{matrix} & \cos\psi + n^2(1-\cos\psi) \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} (1+\cos\psi) & l\sin\psi e^{-i\theta_1} & m\sin\psi e^{-i\theta_2} & n\sin\psi e^{-i\theta_3} \\ l\sin\psi e^{i\theta_1} & l^2(1-\cos\psi) & lm(1-\cos\psi)e^{-i\Delta\theta_3} & nl(1-\cos\psi)e^{i\Delta\theta_2} \\ m\sin\psi e^{i\theta_2} & lm(1-\cos\psi)e^{i\Delta\theta_3} & m^2(1-\cos\psi) & mn(1-\cos\psi)e^{-i\Delta\theta_1} \\ n\sin\psi e^{i\theta_3} & nl(1-\cos\psi)e^{-i\Delta\theta_2} & mn(1-\cos\psi)e^{i\Delta\theta_1} & n^2(1-\cos\psi) \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2M_{00}}\begin{pmatrix} \cos\frac{\psi}{2} & 0 & 0 & 0 \\ l\sin\frac{\psi}{2}e^{i\theta_1} & 0 & 0 & 0 \\ m\sin\frac{\psi}{2}e^{i\theta_2} & 0 & 0 & 0 \\ n\sin\frac{\psi}{2}e^{i\theta_3} & 0 & 0 & 0 \end{pmatrix}.$$

6.2. Parameterized Homogeneous Deterministic Mueller Matrix

$l^2 + m^2 + n^2 = 1$. $\theta_i = \theta$, $\Delta\theta_i = 0$. The expressions are much simpler than in Section 5.1.

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & l\sin\psi\cos\theta & m\sin\psi\cos\theta & n\sin\psi\cos\theta \\ l\sin\psi\cos\theta & \cos\psi + l^2(1-\cos\psi) & n\sin\psi\sin\theta + lm(1-\cos\psi) & -m\sin\psi\sin\theta + nl(1-\cos\psi) \\ m\sin\psi\cos\theta & -n\sin\psi\sin\theta + lm(1-\cos\psi) & \cos\psi + m^2(1-\cos\psi) & l\sin\psi\sin\theta + mn(1-\cos\psi) \\ n\sin\psi\cos\theta & m\sin\psi\sin\theta + nl(1-\cos\psi) & -l\sin\psi\sin\theta + mn(1-\cos\psi) & \cos\psi + n^2(1-\cos\psi) \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} (1+\cos\psi) & l\sin\psi e^{-i\theta} & m\sin\psi e^{-i\theta} & n\sin\psi e^{-i\theta} \\ l\sin\psi e^{i\theta} & l^2(1-\cos\psi) & lm(1-\cos\psi) & nl(1-\cos\psi) \\ m\sin\psi e^{i\theta} & lm(1-\cos\psi) & m^2(1-\cos\psi) & mn(1-\cos\psi) \\ n\sin\psi e^{i\theta} & nl(1-\cos\psi) & mn(1-\cos\psi) & n^2(1-\cos\psi) \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2M_{00}}\begin{pmatrix} \cos\frac{\psi}{2} & 0 & 0 & 0 \\ l\sin\frac{\psi}{2}e^{i\theta} & 0 & 0 & 0 \\ m\sin\frac{\psi}{2}e^{i\theta} & 0 & 0 & 0 \\ n\sin\frac{\psi}{2}e^{i\theta} & 0 & 0 & 0 \end{pmatrix}.$$

Putting $F = \cos\psi$, $D = \sin\psi\cos\theta$, $R = \sin\psi\sin\theta$, where $D^2 + R^2 = 1 - F^2$, we then obtain even simpler expressions,

$$\mathbf{M} = M_{00}\begin{pmatrix} 1 & lD & mD & nD \\ lD & l^2 + (m^2+n^2)F & nR + lm(1-F) & -mR + nl(1-F) \\ mD & -nR + lm(1-F) & m^2 + (n^2+l^2)F & lR + mn(1-F) \\ nD & mR + nl(1-F) & -lR + mn(1-F) & n^2 + (l^2+m^2)F \end{pmatrix},$$

$$\mathbf{C} = M_{00}\begin{pmatrix} (1+F) & l(D-iR) & m(D-iR) & n(D-iR) \\ l(D+iR) & l^2(1-F) & lm(1-F) & nl(1-F) \\ m(D+iR) & lm(1-F) & m^2(1-F) & mn(1-F) \\ n(D+iR) & nl(1-F) & mn(1-F) & n^2(1-F) \end{pmatrix},$$

$$\mathbf{F} = \sqrt{M_{00}} \begin{pmatrix} \sqrt{1+F} & 0 & 0 & 0 \\ l\sqrt{1-F}e^{i\theta} & 0 & 0 & 0 \\ m\sqrt{1-F}e^{i\theta} & 0 & 0 & 0 \\ n\sqrt{1-F}e^{i\theta} & 0 & 0 & 0 \end{pmatrix}.$$

l, m, n are direction cosines in Poincaré space, and F, D, R are also direction cosines in generalized retardation space (see Section 8).

6.3. Parameterized Linear Homogeneous Deterministic Mueller Matrix

$n = 0, l = \cos 2\phi, m = \sin 2\phi; l^2 + m^2 = 1. \theta_i = \theta, \Delta\theta_i = 0.$

$$\mathbf{M} = M_{00} \begin{pmatrix} 1 & \sin\psi\cos\theta\cos 2\phi & \sin\psi\cos\theta\sin 2\phi & 0 \\ \sin\psi\cos\theta\cos 2\phi & \cos^2 2\phi + \cos\psi\sin^2 2\phi & (1-\cos\psi)\sin 2\phi\cos 2\phi & -\sin\psi\sin\theta\sin 2\phi \\ \sin\psi\cos\theta\sin 2\phi & (1-\cos\psi)\sin 2\phi\cos 2\phi & \sin^2 2\phi + \cos\psi\cos^2 2\phi & \sin\psi\sin\theta\cos 2\phi \\ 0 & \sin\psi\sin\theta\sin 2\phi & -\sin\psi\sin\theta\cos 2\phi & \cos\psi \end{pmatrix},$$

$$\mathbf{M} = M_{00} \begin{pmatrix} 1 & lD & mD & 0 \\ lD & l^2 + m^2 F & lm(1-F) & -mR \\ mD & lm(1-F) & m^2 + l^2 F & lR \\ 0 & mR & -lR & (l^2+m^2)F \end{pmatrix},$$

$$\mathbf{C} = M_{00} \begin{pmatrix} (1+\cos\psi) & \sin\psi\cos 2\phi e^{-i\theta} & \sin\psi\sin 2\phi e^{-i\theta} & 0 \\ \sin\psi\cos 2\phi e^{i\theta} & (1-\cos\psi)\cos^2 2\phi & (1-\cos\psi)\sin 2\phi\cos 2\phi & 0 \\ \sin\psi\sin 2\phi e^{i\theta} & (1-\cos\psi)\sin 2\phi\cos 2\phi & (1-\cos\psi)\sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{F} = \sqrt{2M_{00}} \begin{pmatrix} \cos\frac{\psi}{2} & 0 & 0 & 0 \\ \sin\frac{\psi}{2}\cos 2\phi e^{i\theta} & 0 & 0 & 0 \\ \sin\frac{\psi}{2}\sin 2\phi e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

7. Uniform Deterministic Medium

By 'uniform medium' we mean one where the polarization properties are constant throughout space, in both the transverse directions and the direction of propagation. Some works call such a medium 'homogeneous', but we prefer to call it 'uniform', reserving 'homogeneous' to describe a medium where the Mueller matrix is homogeneous [37], so that $\theta_i = \theta$, sometimes called a 'normal' medium. A homogeneous deterministic medium or system has a coherency vector in which the phase of the elements z_1, z_2, z_3 are equal, so that the diattenuation and polarizance vectors are equal. On the other hand, for an inhomogeneous system, the values of θ_i are not equal.

We follow the notation of Refs. [22,23]. $L = LB - iLD$, $L' = lB' - iLD'$ and $C = CB - iCD$ are the elementary generalized retardances of the medium, where the linear diattenuation in the x-y and 45° directions are LD and LD', respectively, and the circular diattenuation is CD; the corresponding retardances are LB, LB', CB, all real $[\mathrm{m}^{-1}]$; the amplitude absorption parameter is $(\alpha/2)[\mathrm{m}^{-1}]$. Then the total generalized retardance vector is $\mathbf{T} = (L, L', -C)^T$, where ^{T} means transpose, and $\mathbf{T} \bullet \mathbf{T}^* = |L|^2 + |L'|^2 + |C|^2 = t^2$ (real nonnegative scalar), $T = (\mathbf{T} \bullet \mathbf{T})^{1/2} = \left(L^2 + L'^2 + C^2\right)^{1/2} = TB - iTD$ (complex scalar).

The solution for a medium of thickness $\bar{z}$ is then that of Section 6.1, with an effective ψ equal to $T\bar{z}$. In general the component z_0 of $\mathbf{z}$ is complex. For the inverse problem for a given Mueller matrix, we introduce a complex parameter p,

$$p^2 = -\left(l^2 e^{2i\theta_1} + m^2 e^{2i\theta_2} + n^2 e^{2i\theta_3}\right). \tag{7}$$

$|p|$ is the homogeneity, defined for the thickness $\bar{z}$, so for an homogeneous medium, when $\theta_1 = \theta_2 = \theta_3$, $|p| = 1$. Then for general p [38,39],

$$M_{00}\left(\cos^2\frac{\psi}{2} + p^2\sin^2\frac{\psi}{2}\right) = e^{-\alpha\bar{z}}e^{-2i\bar{\kappa}}\left(\cos^2\frac{1}{2}T\bar{z} + \sin^2\frac{1}{2}T\bar{z}\right) = e^{-\alpha\bar{z}}e^{-2i\bar{\kappa}}, \tag{8}$$

where $\bar{\kappa}$ is the phase of $\cos(T\bar{z}/2)$, so that

$$\bar{\kappa} = -\tfrac{1}{2}\arg\left(1 + p^2\tan^2\frac{\psi}{2}\right), \quad e^{-\alpha\bar{z}} = M_{00}\frac{|1 + p^2\tan^2\frac{\psi}{2}|}{1+\tan^2\frac{\psi}{2}}, T\bar{z} = 2\arctan\left(p\tan\frac{\psi}{2}\right). \tag{9}$$

Putting [11]

$$t_z = \frac{\tan\frac{1}{2}Tz}{\frac{1}{2}Tz}, \tag{10}$$

so that for $z = \bar{z}$,

$$t_{\bar{z}} = \frac{\tan\frac{1}{2}T\bar{z}}{\frac{1}{2}T\bar{z}} = \frac{p\tan\frac{\psi}{2}}{\arctan\left(p\tan\frac{\psi}{2}\right)}, \tag{11}$$

the elementary generalized retardances, in terms of the parameters of the Mueller matrix, are [11]

$$L\bar{z} = \frac{2il}{t_{\bar{z}}}\tan\frac{\psi}{2}e^{i\theta_1};\ L'\bar{z} = \frac{2im}{t_{\bar{z}}}\tan\frac{\psi}{2}e^{i\theta_2};\ C\bar{z} = -\frac{2in}{t_{\bar{z}}}\tan\frac{\psi}{2}e^{i\theta_3}, \tag{12}$$

where $t_{\bar{z}}$ is approximately unity for small values of ψ. Then for a medium of general thickness z,

$$\begin{aligned}
M_{00}(z) &= \frac{1+\frac{1}{4}t^2z^2|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{01,10}(z) &= \frac{-\Im(Lt_z) \pm \frac{1}{2}\Im(L'C^*)|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{02,20}(z) &= \frac{-\Im(L't_z) \mp \frac{1}{2}\Im(LC^*)|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{03,30}(z) &= \frac{\Im(Ct_z) \pm \frac{1}{2}\Im(L'L^*)|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{11}(z) &= \frac{1+\frac{1}{4}(|L|^2-|L'|^2-|C|^2)z^2|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{22}(z) &= \frac{1+\frac{1}{4}(|L'|^2-|L|^2-|C|^2)z^2|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{33}(z) &= \frac{1+\frac{1}{4}(|C|^2-|L|^2-|L'|^2)z^2|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},
\end{aligned}$$

$$\begin{aligned}
M_{12,21}(z) &= \frac{\pm\Re(Ct_z) + \frac{1}{2}\Re(L'L^*)|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{13,31}(z) &= \frac{\pm\Re(L't_z) - \frac{1}{2}\Re(LC^*)|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},\\
M_{23,32}(z) &= \frac{\mp\Re(Lt_z) - \frac{1}{2}\Re(L'C^*)|t_z|^2}{|1+\frac{1}{4}T^2z^2t_z^2|}e^{-\alpha z},
\end{aligned}$$

$$
\mathbf{C}(z) = \frac{2e^{-\alpha z}}{\left|1+\frac{1}{4}T^2z^2t_z^2\right|}\begin{pmatrix} 1 & iL^*zt_z^*/2 & iL'^*zt_z^*/2 & -iC^*zt_z^*/2 \\ -iLzt_z/2 & |L|^2z^2|t_z|^2/4 & LL'^*z^2|t_z|^2/4 & -LC^*z^2|t_z|^2/4 \\ -iL'zt_z/2 & L'L^*z^2|t_z|^2/4 & |L'|^2z^2|t_z|^2/4 & -L'C^*z^2|t_z|^2/4 \\ iCzt_z/2 & -CL^*z^2|t_z|^2/4 & -CL'^*z^2|t_z|^2/4 & |C|^2z^2|t_z|^2/4 \end{pmatrix},
$$

$$
\mathbf{F}(z) = \frac{\sqrt{2}e^{-\alpha z/2}}{\left(1+\frac{1}{4}T^2z^2t_z^2\right)^{1/2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -iLzt_z/2 & 0 & 0 & 0 \\ -iL'zt_z/2 & 0 & 0 & 0 \\ iCzt_z/2 & 0 & 0 & 0 \end{pmatrix}.
$$

The expression for $\mathbf{M}(z)$ is lengthy, and has been presented before [22,23]. The phase of the premultiplying factor of $\mathbf{F}$ cancels out on calculating $\mathbf{C}$, but the generalized retardances are multiplied by the complex quantity t_z, thus altering the relative values of their real and imaginary parts, and so also the effective retardances and diattentuations [11].

8. Discussion

For diattenuaters, $\mathbf{F}$ is real, $\mathbf{C}$ is real and symmetric, and $\mathbf{M}$ is also symmetric. Thus the diattenuation vector and the polarizance vector are equal, so that the polarizance $P = (M_{10}^2 + M_{20}^2 + M_{30}^2)^{1/2}$ is equal to the diattenuation $D = (M_{01}^2 + M_{02}^2 + M_{03}^2)^{1/2}$. For retarders, $\mathbf{F}$ has a real top left element, the remaining elements being imaginary, $\mathbf{C}$ is G-symmetric and Hermitian, and $M_{00} = 1$, the other elements of the leading row and column of $\mathbf{M}$ being zero. The half-wave retarder is a special case, as $\mathbf{F}$ is then imaginary, with a zero in the top left element, $\mathbf{C}$ is real and symmetric with zeros in the leading row and column, and $\mathbf{M}$ is symmetric.

The homogeneous linear retarding diattenuater in Section 5.3, can be considered to be made up of (i.e. decomposed into) a series combination of a retarder, with retardance Δ, and a diattenuater, with diattenuation $\cos\kappa$. The diattenuation and polarizance vectors are not in general equal, but the diattenuation and polarizance are equal, as they are even for the inhomogeneous case. The Mueller matrices for the homogeneous case commute, so the overall matrix can also be decomposed into a diattenuator and a retarder, in this order. Even for this comparatively simple case (which is both homogeneous and linearly polarized), where the two Mueller matrices for the components commute, we observe that the retardance terms of the Mueller matrix $M_{13}, M_{31}, M_{23}, M_{32}$ are of magnitude $\sin\kappa \sin\Delta$, i.e., they are scaled by a function of the diattenuation. In addition, there are artifact terms M_{12}, M_{21} of magnitude $(1 - \cos\Delta \sin\kappa)$, corresponding to circular retardance, caused by cross-products between the linear retardance and linear diattenuation. However, the coherency vector $\mathbf{z}$, given by the first column of the coherency matrix factor $\mathbf{F}$, is of comparatively simple form compared with the Mueller matrix.

A general Mueller matrix is much more complicated, even for the deterministic case of Section 6.1, which has artifact cross-product diattenuation and retardance terms. The deterministic Mueller matrix contains first-order terms that are G-antisymmetric, and second-order cross-product terms that are G-symmetric. In the coherency matrix, the first-order terms are confined to the leading row and column, while the second-order terms are in other locations. For the homogeneous deterministic case, as in Section 6.2, the artifact diattenuation terms disappear, but the artifact retardance terms remain. Note that these latter terms are independent of the value of θ. In Section 6.3, we consider a parameterized linear homogeneous Mueller matrix. This can then be compared with the homogeneous linear retarding diattenuater in Section 5.3. Equating the elements of the coherency matrix factor, we find that

$$
\begin{aligned}
F &= \cos\psi = \cos\Delta \sin\kappa, \\
D &= \sin\psi \cos\theta = \cos\kappa, \\
R &= \sin\psi \sin\theta = \sin\Delta \sin\kappa,
\end{aligned} \tag{13}
$$

where we have put the three equations equal to F, D, R, respectively, as the second equation gives the strength of the diattenuation, and the third equation gives the strength of the retardation with the right-hand side again scaled by a function of the diattenuation. For $\Delta = 0$, $\theta = 0$, corresponding to pure diattenuation, $\psi + \kappa = \pi/2$. For pure retardance, $\theta = \pi/2$, and $\kappa = \pi/2$, and so $\psi = \Delta$.

It is interesting to note that these equations are analogous to those connecting the angles in the Stokes parameters, I, U, V, Q, for a fully polarized beam:

$$\begin{aligned} U/I &= \cos 2\chi \sin 2\phi = \sin 2\alpha \cos \delta, \\ V/I &= \sin 2\chi = \sin 2\alpha \sin \delta, \\ Q/I &= \cos 2\chi \cos 2\phi = \cos 2\alpha, \end{aligned} \tag{14}$$

where χ is the ellipticity angle, ϕ is the orientation angle, α is the auxiliary angle ($\tan \alpha = E_{0y}/E_{0x}$) and δ is the phase delay of the y component. In this case, these angles represent two systems of spherical polar coordinates on the Poincaré sphere, with different polar axes, along V and Q, respectively. Collett calls the $2\alpha, \delta$ sphere the 'observable polarization sphere' to distinguish it from the Poincaré sphere $2\chi, 2\psi$ [3], but we prefer to think of only a single sphere with two different spherical polar coordinate systems. Further, we consider a polarization transformation to be represented by a path on the sphere, rather than by a rotation of the sphere.

We conclude that for a linear Mueller matrix, the pairs of coordinates ψ, θ and κ, Δ can also be represented on an analogy of the Poincaré sphere, of radius unity, as shown in Figure 1, which we call the generalized retardation sphere. Note that a point on the generalized retardation sphere represents a polarization transformation, whereas a point on the Poincaré sphere represents a polarization state, rather than a change of state. As, by convention, $0 \le \Delta \le \pi$ and $0 \le \kappa \le \pi/2$, then $0 \le D \le 1$, $0 \le R \le 1$, $-1 \le F \le 1$, so we require a quarter of the complete sphere, whereas only an octant is shown in Figure 1. Then the artifact circular retardance term is of the form $(1 - \sin\kappa \cos\Delta) = 1 - F$. Note that this is maximized for a half-wave retarder, $\Delta = \pi$, when $F = -1$.

For a pure elliptic retarder, as in Section 4.1, we see that this expression is consistent with the parametric form in Section 6.2 if $\theta = \pi/2$ and $\Delta = \psi$, i.e., the point P lies in the plane $D = 0$, so $\kappa = \pi/2$. The plane $D = 0$ includes the point $F = -1$ for $\Delta = \psi = \pi$, corresponding to a pure half-wave retarder. For the plane $R = 0$, $\sin \Delta = 0$, corresponding to the case of a pure diattenuater, and taking $0 \le \psi < \pi$, then $\theta = 0$ and $D = \cos \kappa = \sin \psi$. Then $\Delta = 0$, $F = \cos \psi = \sin \kappa$, and $\psi = \pi/2 - \kappa$; or $\Delta = \pi$, $F = \cos \psi = -\sin \kappa$, and $\psi = \pi/2 + \kappa$. The point $F = -1$ is excluded from this plane. A quarter-wave retarding diattenuater has $\Delta = \pi/2$, so P is in the plane $F = 0$. Then $\psi = \pi/2$ and $\theta = \kappa$. As introducing the coordinates F, D, R, simplifies the expressions significantly, we have also presented this form for **M**, **C** and **F** in Section 6.2 for the parameterized homogeneous deterministic case.

The case of a linear homogeneous retarding diattenuater can be generalized to the elliptic case, using the identities

$$\sqrt{2} \sin \frac{\psi}{2} e^{i\theta} \equiv \frac{\sin \psi}{\sqrt{1 + \cos \psi}} e^{i\theta} \equiv \sqrt{1 - F} e^{i\theta} \equiv \frac{\cos \kappa + i \sin \Delta \sin \kappa}{\sqrt{1 + \cos \Delta \sin \kappa}}; \quad \psi \neq \pi, \tag{15}$$

giving **M**, **C**, **F**, in terms of Δ, κ and χ, as shown in Section 5.1. The special case of a circular retarding diattenuater is presented in Section 5.2. We stress that Section 6.1 applies for any deterministic Mueller matrix, even for the inhomogeneous case.

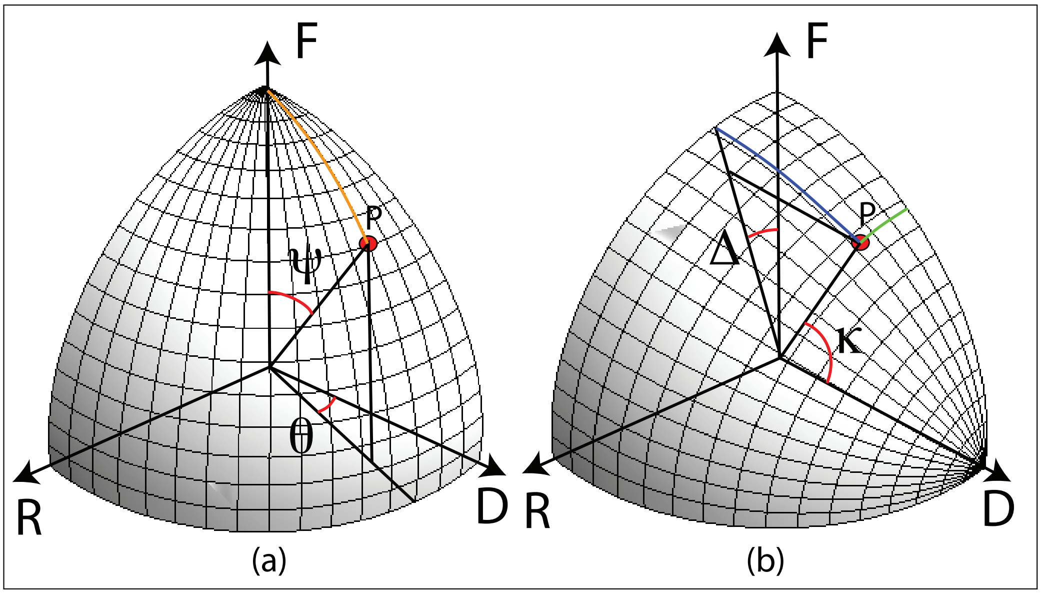

Figure 1. The representation of the transformation given by an homogeneous retarding diattenuater on the generalized retardation sphere, a portion of an analogy to the Poincaré sphere. (**a**) Polar coordinates ψ, θ aligned on the F-axis; (**b**) polar coordinates κ, Δ aligned on the D- axis The Cartesian coordinates $D = \cos\kappa$ and $R = \sin\Delta\sin\kappa$ are the diattenuation and the retardation, $0 \leq (D, R) \leq 1$. The coordinate $F = \cos\Delta\sin\kappa$, $-1 \leq F \leq 1$, is a measure of the strength of the polarization transformation, with the point $F = 1$ corresponding to no change, and the circle $F = 0, D^2 + R^2 = 1$ to a quarter-wave retarding diattenuater. The point $F = -1, D = R = 0$ (not shown) corresponds to a half-wave retarder.

In addition to giving simpler expressions for some general cases, the spherical polar coordinates ψ, θ have the advantage over the κ, Δ coordinates that they correctly determine the behaviour at the half-wave condition, $F = -1$, where there is a singularity that coincides with the pole of the coordinate system at $\psi = \pi$. However, the physical system only exhibits a single singularity, whereas the coordinate system ψ, θ has two singular points. There is some advantage, therefore, in introducing a set of coordinates based on a stereographic projection, related to the Riemann sphere, which exhibits a single singular point. We introduce the coordinates u, v, defined as

$$u = \frac{D}{1+F} = \frac{\cos\kappa}{1+\cos\Delta\sin\kappa}, v = \frac{R}{1+F} = \frac{\sin\Delta\sin\kappa}{1+\cos\Delta\sin\kappa}; \; u^2 + v^2 = \frac{1-F}{1+F}, \tag{16}$$

so that lines of constant u, v are orthogonal sets of circles on the generalized retardation sphere, passing through the point $F = -1$, and inclined at angles $\arctan u$, $\arctan v$, respectively, to the F axis, as illustrated in Figure 2. We have

$$F = \frac{1-u^2-v^2}{1+u^2+v^2}, \quad D = \frac{2u}{1+u^2+v^2}, \quad R = \frac{2v}{1+u^2+v^2}. \tag{17}$$

The relationships between the different coordinates is illustrated in Figure 3.

Then

$$\tan\frac{\psi}{2}e^{i\theta} = \frac{D+iR}{1+F} = u + iv, \tag{18}$$

and for a deterministic elliptical homogeneous retarding diattenuater, we obtain the simple expressions

$$\mathbf{M} = \frac{M_{00}}{1+u^2+v^2}\begin{pmatrix} 1+u^2+v^2 & 2lu & 2mu & 2nu \\ 2lu & 1+(2l^2-1)(u^2+v^2) & 2[nv+lm(u^2+v^2)] & 2[-mv+nl(u^2+v^2)] \\ 2mu & 2[-nv+lm(u^2+v^2)] & 1+(2m^2-1)(u^2+v^2) & 2[lv+mn(u^2+v^2)] \\ 2nu & 2[mv+nl(u^2+v^2)] & 2[-lv+mn(u^2+v^2)] & 1+(2n^2-1)(u^2+v^2) \end{pmatrix},$$

$$\mathbf{C} = \frac{2M_{00}}{1+u^2+v^2}\begin{pmatrix} 1 & l(u-iv) & m(u-iv) & n(u-iv) \\ l(u+iv) & l^2(u^2+v^2) & lm(u^2+v^2) & nl(u^2+v^2) \\ m(u+iv) & lm(u^2+v^2) & m^2(u^2+v^2) & mn(u^2+v^2) \\ n(u+iv) & nl(u^2+v^2) & mn(u^2+v^2) & n^2(u^2+v^2) \end{pmatrix},$$
$$\mathbf{F} = \sqrt{\frac{2M_{00}}{1+u^2+v^2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ l(u+iv) & 0 & 0 & 0 \\ m(u+iv) & 0 & 0 & 0 \\ n(u+iv) & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

The point $F = 1, u = v = 0$ corresponds to free space, $F = 0, u^2 + v^2 = 1$ to a quarter-wave elliptic retarding diattenuater, and $F = -1, u^2 + v^2 \to \infty$, to a half-wave elliptic retarder.

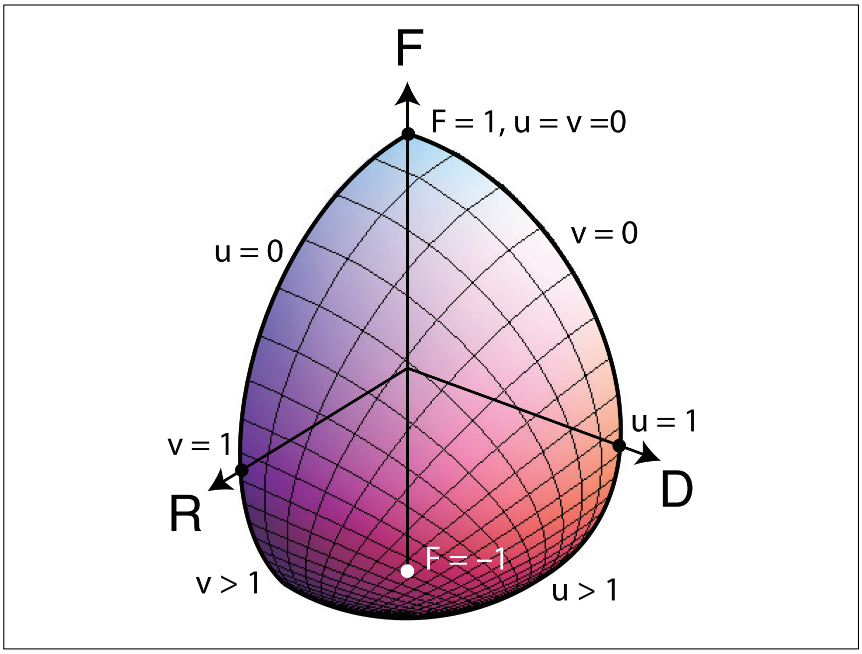

Figure 2. Lines of constant value of u, v on the surface of the generalized retardation sphere, two orthogonal sets of circles all passing through the point $F = -1$ (hidden from view).

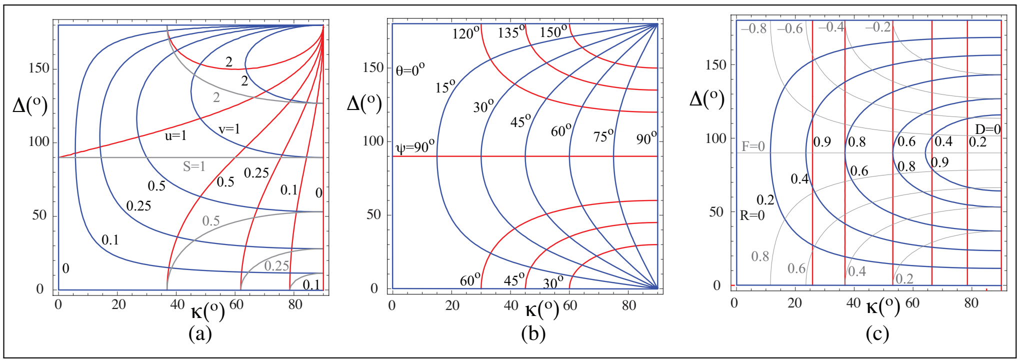

Figure 3. The variation of the parameters (**a**) u, v and S, (**b**) ψ and θ, and (**c**) D, R and F, with κ and Δ.

Summing up, an homogeneous deterministic system can be specified by five parameters (neglecting absolute phase), a scalar (the gain, M_{00}), and two unit vectors each with two independent parameters. The unit vectors can be taken as an orientation in Stokes space, described by angles χ, ϕ or direction cosines l, m, n; and an orientation in generalized retardation space, described by angles Δ, κ or ψ, θ, direction cosines F, D, R, or stereographic coordinates u, v. For an inhomogeneous deterministic system, there are seven parameters, again a scalar M_{00} and a unit vector in Stokes space, together with a further four parameters, which can conveniently be taken as $S = (u^2 + v^2)^{1/2} = [(1-F)/(1+F)]^{1/2} = \tan(\psi/2)$, specifying the (amplitude) strength of the polarization interaction, together with $\theta_1, \theta_2, \theta_3$, corresponding to a rotation in a 4D space. Then, for the homogeneous case, S represents a radius in the stereographic projection of the generalized retardance sphere, in which u, v are Cartesian coordinates, with the circle $S = 1$ corresponding to quarter-wave retarding diattenuaters.

Finally, the expressions in Sections 6.1, 6.2 and 7, and others, can be generalized to nondeterministic systems by including additional columns in the corresponding coherency matrix factor. However, for this to give a valid **F**, the columns must satisfy orthogonality conditions. For the general nondeterministic case, there are 16 parameters, with 7, 5, 3 and 1 parameters, respectively, for the component deterministic coherency vectors. There is also the special case of a nondeterministic system, with up to four homogeneous deterministic components.

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