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UPPER BOUNDS FOR THE SPECTRAL AND ℓ_p NORMS OF CAUCHY- TOEPLITZ AND CAUCHY-HANKEL MATRICES

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Abstract- In this study we have given some upper bounds for the spectral and ℓ_p norms of Cauchy-Toeplitz and Cauchy-Hankel matrices of the forms $T=[1/(1/2+|i-j|)]_{n\times n}$ and $H=[1/(1/2+(i+j))]_{n\times n}$ respectively.

Keywords- Cauchy-Toeplitz matrix, Cauchy-Hankel matrix, matrix norm, upper bound, Hadamard product.

1. INTRODUCTION

It's well known that the ℓ_p norm of a matrix A is

$$\|A\|_p=\left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right]^{1/p} \tag{1.1}$$

and also the spectral norm of a matrix A is

$$\|A\|_2=\sqrt{\max_{1\le i\le n}\lambda_i\left(A^H A\right)} \tag{1.2}$$

where A is $m\times n$ and A^H is the conjugate transpose of the matrix A. Define the maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of any matrix A by

$$c_1(A)=\max_j\sqrt{\sum_i|a_{ij}|^2} \tag{1.3}$$

and

$$r_1(A)=\max_i\sqrt{\sum_j|a_{ij}|^2} \tag{1.4}$$

respectively [1,2,3]. Let A, B and C be $m\times n$ matrices. If $A=B\circ C$ then

$$\|A\|_2\le r_1(B)c_1(C)\quad [2,3]. \tag{1.5}$$

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be $n\times n$ matrices. The Hadamard product of A and B is defined by $A\circ B=[a_{ij}b_{ij}]$. If $\|.\|$ is any matrix norm on $n\times m$ matrices, then

$$\|A\circ B\|\le\|A\|\|B\|\quad [1,2]. \tag{1.6}$$

A function Ψ is called a psi (or digamma) function if

$$\Psi(x)=\frac{d}{dx}\{\log[\Gamma(x)]\},$$

where

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt.$$

The n th derivative of a psi function is called a polygamma function, i.e.

$$\Psi(n,x)=\frac{d}{dx^n}Psi(x)=\frac{d}{dx^n}\left[\frac{d}{dx}\ln[\Gamma(x)]\right].$$

If n=0 then $\Psi(0,x)=Psi(x)=\frac{d}{dx}\{\ln[\Gamma(x)]\}$. On the other hand, if a>0, b is any number, and n is a positive integer, then

$$\lim_{n\to\infty}\Psi(a,n+b)=0 \text{ [4].} \tag{1.7}$$

Let $C=[1/(x_i-y_j)]_{i,j=1}^n$ $(x_i\neq y_j)$ be a Cauchy matrix and $T=[t_{j-i}]_{i,j=0}^n$ be a Toeplitz matrix. A Cauchy-Toeplitz matrix is defined as

$$T=\left[\frac{1}{g+(i-j)h}\right]_{i,j=1}^n \tag{1.8}$$

where $h\neq 0$, g and h are any numbers and g/h is not an integer.

On the other hand, let $H=[h_{i+j}]_{i,j=0}^n$ be a Hankel matrix. Every $n\times n$ Cauchy-Hankel matrix is of the form

$$H=\left[\frac{1}{g+(i+j)h}\right]_{i,j=1}^n \tag{1.9}$$

where $h\neq 0$, g and h are any numbers and g/h is not an integer. Hankel matrices are symmetric.

In this study we define the matrices T and H of the forms

$$T=\left[\frac{1}{1/2+|i-j|}\right]_{n\times n} \tag{1.10}$$

and

$$H=\left[\frac{1}{1/2+(i+j)}\right]_{n\times n} \tag{1.11}$$

and give some upper bounds connected with norms of the matrices T, H and $T\circ H$.

2. THE SPECTRAL NORMS OF CAUCHY-TOEPLITZ AND CAUCHY-HANKEL MATRICES

Theorem 2.1. i) Let T be as in (1.10). Then $\frac{1}{\sqrt{n}}\|T\|_2 \le \sqrt{\pi^2-4}$.

ii) Let H be as in (1.11). Then $\frac{1}{\sqrt{n}}\|H\|_2 \le \sqrt{\frac{\pi^2}{2}-\frac{40}{9}}$.

Proof.

i) If we take $A=[2]_{n\times n}$ and $B=\left[\frac{1}{1+2|i-j|}\right]_{n\times n}$ then $T=A\circ B$. From (1.3) and (1.4),

$$r_1(A)=\max_i\sqrt{\sum_j |a_{ij}|^2}=\sqrt{4n}$$

and

$$c_1(B)=\max_j\sqrt{\sum_i |b_{ij}|^2}=\begin{cases}\sqrt{2\sum\limits_{s=1}^{(n-1)/2}\frac{1}{(2s+1)^2}+1} & ,\ if\ n\ odd\\ \sqrt{\sum\limits_{s=1}^{n/2+1}\frac{1}{(2s-1)^2}+\sum\limits_{s=1}^{n/2-1}\frac{1}{(2s+1)^2}} & ,\ if\ n\ even\end{cases}\tag{2.1}$$

If we evaluate the right hand side of the inequality (1.3), we have

$$\sum_{s=1}^{(n-1)/2}\frac{1}{(2s+1)^2}=\frac{\pi^2}{8}-1-\frac{1}{4}\Psi(1,n/2+1),\tag{2.2}$$

$$\sum_{s=1}^{n/2+1}\frac{1}{(2s-1)^2}=\frac{\pi^2}{8}-\frac{1}{4}\Psi(1,(n+3)/2),\tag{2.3}$$

and

$$\sum_{s=1}^{n/2-1}\frac{1}{(2s+1)^2}=\frac{\pi^2}{8}-1-\frac{1}{4}\Psi(1,(n+1)/2).\tag{2.4}$$

Hence we get

$$c_1(B)=\begin{cases}\sqrt{2\left[\frac{\pi^2}{8}-1-\frac{1}{4}\Psi(1,n/2+1)\right]+1} & ,\ if\ n\ odd\\ \sqrt{\frac{\pi^2}{4}-1-\frac{1}{4}\left[\Psi(1,(n+3)/2)+\Psi(1,(n+1)/2)\right]} & ,\ if\ n\ even\end{cases}.$$

By the inequality (1.5) we have $\|T\|_2 \le r_1(A)c_1(B)$. Thus, we obtain

$$\|T\|_2 \le r_1(A)c_1(B) = \begin{cases} \sqrt{4n\left(2\left[\frac{\pi^2}{8}-1-\frac{1}{4}\Psi(1,n/2+1)\right]+1\right)} & , \ n \ odd \\ \sqrt{4n\left(\frac{\pi^2}{4}-1-\frac{1}{4}[\Psi(1,(n+3)/2)+\Psi(1,(n+1)/2)]\right)} & , \ n \ \text{even} \end{cases} . \tag{2.5}$$

If we divide by $\sqrt{n}$ both hand sides of the inequality (2.5), then

$$\frac{1}{\sqrt{n}}\|T\|_2 \le \begin{cases} \sqrt{4\left(2\left[\frac{\pi^2}{8}-1-\frac{1}{4}\Psi(1,n/2+1)\right]+1\right)} & , \ n \ odd \\ \sqrt{4\left(\frac{\pi^2}{4}-1-\frac{1}{4}[\Psi(1,(n+3)/2)+\Psi(1,(n+1)/2)]\right)} & , \ n \ \text{even} \end{cases} . \tag{2.6}$$

If we take the limit as $n \to \infty$ of the expressions in the right hand side of the inequality (2.6), from (1.7)

$$\frac{1}{\sqrt{n}}\|T\|_2 \le \sqrt{\pi^2-4}\,. \tag{2.7}$$

ii) If we take $A=[2]_{n\times n}$ and $B=\left[\frac{1}{1+2(i+j)}\right]_{n\times n}$ then $H=A\circ B$. From (1.3) and (1.4),

$$r_1(A)=\max_i \sqrt{\sum_j |a_{ij}|^2} = \sqrt{4n}$$

and

$$c_1(B)=\max_j \sqrt{\sum_i |b_{ij}|^2} = \sqrt{\sum_{s=1}^{n}\frac{1}{(2s+3)^2}} = \sqrt{\frac{\pi^2}{8}-\frac{10}{9}-\frac{1}{4}\Psi(1,n+5/2)}\,. \tag{2.8}$$

Since $\|H\|_2 \le r_1(A)c_1(B)$,

$$\|H\|_2 \le r_1(A)c_1(B) = \sqrt{4n\left[\frac{\pi^2}{8}-\frac{10}{9}-\frac{1}{4}\Psi(1,n+5/2)\right]}\,. \tag{2.9}$$

If we divide by $\sqrt{n}$ both hand sides of the inequality (2.9), then

$$\frac{1}{\sqrt{n}}\|H\|_2 \le \sqrt{4\left[\frac{\pi^2}{8}-\frac{10}{9}-\frac{1}{4}\Psi(1,n+5/2)\right]}\,. \tag{2.10}$$

If we take the limit as $n \to \infty$ of the expressions in the right hand side of the inequality (2.10), from (1.7)

$$\frac{1}{\sqrt{n}}\|H\|_2 \le \sqrt{\frac{\pi^2}{2}-\frac{40}{9}}. \tag{2.11}$$

Thus from (2.7) and (2.11) the proof is completed.

Corollary 2.1. Let T and H be as in (1.10) and (1.11). Then

$$\|T \circ H\|_2 \le \sqrt{(\pi^2-4)\left(\frac{\pi^2}{2}-\frac{40}{9}\right)}.$$

Proof. By define of the maximum row length norm ,

$$r_1(T) = \max_i \sqrt{\sum_j |t_{ij}|^2} = \begin{cases} \sqrt{4\left[2\sum_{s=1}^{(n-1)/2}\frac{1}{(2s+1)^2}+1\right]} & , n \textit{ odd} \\ \sqrt{4\left[\sum_{s=1}^{n/2+1}\frac{1}{(2s-1)^2}+\sum_{s=1}^{n/2-1}\frac{1}{(2s+1)^2}\right]} & , n \text{ even} \end{cases}$$

and from (2.2), (2.3) and (2.4) we get

$$r_1(T) = \begin{cases} \sqrt{4\left[\frac{\pi^2}{4}-1-\frac{1}{4}\Psi(1,n/2+1)\right]} & , n \textit{ odd} \\ \sqrt{4\left(\frac{\pi^2}{4}-1-\frac{1}{4}[\Psi(1,(n+3)/2)+\Psi(1,(n+1)/2)]\right)} & , n \text{ even} \end{cases}. \tag{2.12}$$

By define of the maximum column length norm and (2.8)

$$c_1(H) = \max_j \sqrt{\sum_i |h_{ij}|^2} = \sqrt{\sum_{s=1}^{n}\frac{2^2}{(2s+3)^2}} = \sqrt{4\left[\frac{\pi^2}{8}-\frac{10}{9}-\frac{1}{4}\Psi(1,n+5/2)\right]}. \tag{2.13}$$

Since $\|T \circ H\|_2 \le r_1(T)c_1(H)$, from (2.12) and (2.13) we have

$$\|T \circ H\|_2 \le \begin{cases} \sqrt{4\left[\frac{\pi^2}{4}-1-\frac{1}{4}\Psi(1,n/2+1)\right]4\left[\frac{\pi^2}{8}-\frac{10}{9}-\frac{1}{4}\Psi(1,n+5/2)\right]} & , n \textit{ odd} \\ \sqrt{4\left(\frac{\pi^2}{4}-1-\frac{1}{4}[\Psi(1,(n+3)/2)+\Psi(1,(n+1)/2)]\right)4\left[\frac{\pi^2}{8}-\frac{10}{9}-\frac{1}{4}\Psi(1,n+5/2)\right]} & , n \text{ even} \end{cases}.$$

If we take the limit as $n \to \infty$ of the expressions in the right hand side of the above inequality, then we get

$$\|T\circ H\|_2 \le \sqrt{(\pi^2-4)\left(\frac{\pi^2}{2}-\frac{40}{9}\right)}.$$

3. ℓ_p NORMS OF CAUCHY-TOEPLITZ AND CAUCHY-HANKEL MATRICES

Theorem 3.1. i) Let T be as in (1.10). Then

$$n^{-1/p}\|T\|_p \le \left[2\zeta(p)(2^p-1)-2^p\right]^{1/p}$$

where $2\le p<\infty$.

ii) Let H be as in (1.11). Then

$$\|H\|_p \le \left[2^p-\frac{3(2^p-1)}{2}\zeta(p)+(2^{p-1}-1)\zeta(p-1)\right]^{1/p}$$

where $3\le p<\infty$.

Proof.

i) From (1.1),

$$\begin{aligned}\|T\|_p^p &= 2\sum_{s=1}^{n-1}\left(\frac{2}{2s+1}\right)^p(n-s)+n2^p\\ &= 2^p\left[2n\sum_{s=1}^{n-1}\frac{1}{(2s+1)^p}-2\sum_{s=1}^{n-1}\frac{s}{(2s+1)^p}+n\right]\end{aligned} \tag{3.1}$$

If we divide by n both hand sides of the inequality (3.1), then

$$\frac{1}{n}\|T\|_p^p = 2^p\left[2\sum_{s=1}^{n-1}\frac{1}{(2s+1)^p}-\frac{2}{n}\sum_{s=1}^{n-1}\frac{s}{(2s+1)^p}+1\right]$$

where

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=1}^{n-1}\frac{s}{(2s+1)^p}\to 0$$

and

$$\lim_{n\to\infty}\sum_{s=1}^{n-1}\frac{1}{(2s+1)^p}=\lim_{n\to\infty}\left(\sum_{s=1}^{2n-1}\frac{1}{s^p}-\frac{1}{2^p}\sum_{s=1}^{n-1}\frac{1}{s^p}-1\right)=\left[\zeta(p)\left(1-\frac{1}{2^p}\right)-1\right].$$

Then

$$\frac{1}{n}\|T\|_p^p \le 2^p\left[2\left[\zeta(p)\left(1-\frac{1}{2^p}\right)-1\right]+1\right]\Rightarrow n^{-1/p}\|T\|_p^p \le \left[2\zeta(p)\left(2^p-1\right)-2^p\right]^{1/p}.$$

ii) From (1.1), we get

$$\|H\|_p^p = 2^p \left[\sum_{s=1}^{n} \frac{s}{(2s+3)^p} + \sum_{s=1}^{n-1} \frac{n-s}{(2s+2n+3)^p} \right]$$

where

$$\lim_{n\to\infty} \sum_{s=1}^{n} \frac{s}{(2s+3)^p} = 1 - \frac{3(2^p-1)}{2^{p+1}} \zeta(p) + \frac{2^{p-1}-1}{2^p} \zeta(p-1)$$

and

$$\lim_{n\to\infty} \sum_{s=1}^{n-1} \frac{n-s}{(2s+2n+3)^p} \to 0 .$$

Then

$$\|H\|_p^p \le \left[2^p - \frac{3(2^p-1)}{2} \zeta(p) + (2^{p-1}-1)\zeta(p-1) \right].$$

Thus the proof is completed.

Corollary 3.1. For Hadamard product of the matrices T and H which is given by (1.10) and (1.11), the inequality

$$n^{-1/p} \|T \circ H\|_p \le \left[\left(2^p - \frac{3(2^p-1)}{2} \zeta(p) + (2^{p-1}-1)\zeta(p-1) \right) \left(2\zeta(p)(2^p-1) - 2^p \right) \right]^{1/p}$$

is valid where $3 \le p < \infty$.

Proof. Since $\|A \circ B\| \le \|A\| \|B\|$, we can write

$$n^{-1/p} \|T \circ H\|_p \le n^{-1/p} \|T\|_p \|H\|_p .$$

Hence from Theorem 3.1 (i-ii), the proof is evident.

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