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# A COMPARISON ON THE COMMUTATIVE NEUTRIX PRODUCTS OF DISTRIBUTIONS

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**Abstract** - In this work, we define a new commutative neutrix product of distribution and than we make a comparison with present commutative product of distributions by providing a counterexample that two commutative neutrix products of the distributions differ.
**Keywords:** Distribution, delta-function, neutrix, neutrix product.

## 1. INTRODUCTION

Although there is no problem in defining the product of a distribution with an infinitely differentiable function, Schwartz [13] proved that, in general it is impossible to give definition for product of distributions. Despite this, products such as H(x) $\delta(x)$, $x^{-s}\delta^{(r)}(x)$ and $\delta^{(r)}(x)\delta^{(s)}(x)$ are of special interest to physicists and engineers. Therefore it has attracted attention at once after the creation of distributions and opened up a new area of mathematical research, with many attempts to try and give a satisfactory definition for the product of two distributions.

In the following, we let N be the neutrix, see van der Corput [1], having domain N' = $\{1,2,\dots,n,\dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$n^{\lambda}\ln^{r-1}n,\ \ln^{r}n : n \qquad \lambda > 0,\ r = 1, 2, \dots$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let p(x) be any infinitely differentiable function having the following properties:

(i) $p(x) = 0$ for $|x| \geq 1$,
(ii) $p(x) \geq 0$,
(iii) $p(x) = p(-x)$,
(iv) $\int_{-1}^{1} p(x)dx = 1$.

Putting $\delta_n(x) = n\,p(nx)$ for n = 1, 2, …then it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. Then if $f$ is an arbitrary distribution in D', we define

$f_n(x) = (f*\delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$

for n = 1, 2, .... It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution f(x).

A first extension of the product of a distribution and an infinitely differentiable function is the following definition, see for example [2].

DEFINITION 1. *Let f and g be distributions in D' for which on the interval (a, b), f is the k-th, derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product fg = gf of two distributions f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}\binom{k}{i}(-1)^i\left[Fg^{(i)}\right]^{(k-i)}$$

The following definition for the neutrix product of distributions was given in [3] and generalizes definition 1.

DEFINITION 2. *Let f and g be distribution in D' and let $g_n(x) = (g*\delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval*
*(a, b) if*

$$N-\lim_{n\to\infty}\langle f(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

*for all function $\phi$ in D with support contained in the interval (a, b).*

This definition 2 of the product is not symmetric, hence in general $f \circ g \neq g \circ f$.

## 2. A NEW COMMUTATIVE NEUTRIX PRODUCT

In the following we give a new commutative product of distributions.

DEFINITION 3. *Let f and g be distribution in D' and let $f_n(x) = (f*\delta_n)(x)$, $g_n(x) = (g*\delta_n)(x)$. We say that the neutrix product $f \Diamond g$ of f and g exists and is equal to the distribution h on the interval (a, b) if*

$$N-\lim_{n\to\infty}\frac{1}{2}\{\langle f(x)g_n(x)+f_n(x)g(x),\phi(x)\rangle\} = \langle h(x),\phi(x)\rangle$$

*for all function $\phi$ in D' with support contained in the interval (a, b).*
*Note that if*

$$\lim_{n\to\infty}\frac{1}{2}\{\langle f(x)g_n(x)+f_n(x)g(x),\phi(x)\rangle\} = \langle h(x),\phi(x)\rangle,$$

we simply say that the product $f\cdot g$ exists and equal $h$.

It is obvious that if the product $f \cdot g$ exists then the neutrix products $f\, o\, g$ and $g\, o f$ exist, so $f \Diamond g$ exists and $f \cdot g = f \Diamond g$. Note also that although the product defined in definition 1 is always commutative, neutrix product defined in definition 2 is non-commutative, but definition 3 is clearly commutative. We now prove the following theorem.

THEOREM 1. *Let f and g be distributions in D' and suppose that the neutrix products* $f \Diamond g^{(i)}$ *exist on the interval (a, b) for* $i = 0, 1, \ldots, r$. *Then the neutrix products* $f^{(k)} \Diamond g$ *exist on the interval (a, b) and*

$$f^{(k)} \Diamond g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left[ f \Diamond g^{(i)} \right]^{(k-i)}$$

$k = 1, 2, \ldots, r.$

PROOF. Let $\phi$ be an arbitrary function in $D$ with support contained in the interval (a, b) and suppose that the neutrix products $f \Diamond g^{(i)}$ exist on the interval (a, b) for $i = 0, 1, \ldots, r$. Put

$$f_n(x) = (f * \delta_n)(x), \quad g_n(x) = (g * \delta_n)(x)$$

Then

$$\langle f \Diamond g, \phi \rangle = \frac{1}{2} N - \lim_{n \to \infty} \langle f_n g + g_n f, \phi \rangle,$$

$$\langle f \Diamond g', \phi \rangle = \frac{1}{2} N - \lim_{n \to \infty} \langle f_n g' + g'_n f, \phi \rangle.$$

Further

$$\begin{aligned} \langle (f \Diamond g)', \phi \rangle &= -\langle f \Diamond g, \phi' \rangle = -\frac{1}{2} N - \lim_{n \to \infty} \langle f_n g + g_n f, \phi' \rangle, \\ &= \frac{1}{2} N - \lim_{n \to \infty} \langle (f_n g)' + (g_n f)', \phi \rangle. \\ &= \frac{1}{2} N - \lim_{n \to \infty} \langle f_n g' + g'_n f, \phi \rangle. + N - \lim_{n \to \infty} \langle g'_n f + g_n f', \phi \rangle \end{aligned}$$

and so

$$N - \lim_{n \to \infty} \langle f'_n g + g_n f', \phi \rangle = \langle (f \Diamond g)', \phi \rangle - \langle f_n g', \phi \rangle.$$

This proves that the neutrix product $f' \Diamond g$ exists and satisfies equation (1) for the case $k = 1$. Thus

$$(f \Diamond g)' = f' \Diamond g + f \Diamond g'. \qquad (2)$$

Now suppose that equation (1) holds for some $k < r$. Then by our assumption the neutrix product $f^{(k)} \Diamond g$ exists and on using equation (2) we have

$$\begin{aligned} [f^{(k)} \Diamond g]' &= f^{(k+1)} \Diamond g + f^{(k)} \Diamond g' \\ &= f^{(k+1)} \Diamond g + \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \Diamond g^{(i+1)}]^{(k+i)} \end{aligned}$$

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^{i} [f \Diamond g]^{(k-i+1)}$$

Thus

$$f^{(k+1)} \Diamond g = \sum_{i=0}^{k} \binom{k}{i} (-1)^{i} [f \Diamond g^{(i)}]^{(k-i-1)} + \sum_{i=1}^{k+1} \binom{k}{i-1} (-1)^{i} [f \Diamond g^{(i)}]^{(k-i+1)}$$
$$= \sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^{i} [f \Diamond g^{(i)}]^{(k-i+1)}$$

Equation (1) now follows by induction.

THEOREM 2. The neutrix products $(x^s \ln |x|) \Diamond \delta^{(r)}(x)$ exists and

$$(x^s \ln |x|) \Diamond \delta^{(r)}(x) = \frac{r!}{(r-s)!} [2c_1(p) + \psi(s)] \delta^{(r-s)}(x) + \\ - (s)! \frac{1}{2} \sum_{s+1}^{r} \binom{r}{i} \frac{(-1)^{i}}{i-s} \delta^{(r-s)}(x), \tag{3}$$

*for* $s = 0, 1, 2, \dots$ and $r = s, s+1, \dots$, where $c_1(p) = \int_0^1 \ln t\, p(t)\, dt$ *and*

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^{r} i^{-1}, & r \geq 1. \end{cases}$$

PROOF. The following Equations were proved by Kilicman and Fisher in [11];

$$(x_+^s \ln x_+) \circ \delta^{(r)}(x) = \frac{(-1)^s r!}{(r-s)!} \left[ c_1(p) + \frac{1}{2}\psi(s) \right] \delta^{(r-s)}(x) + \\ - s! \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{i}}{2(i-s)} \delta^{(r-s)}(x), \tag{4}$$

$$\delta^{(r)}(x) \circ (x_+^s \ln x_+) = \frac{(-1)^s r!}{(r-s)!} \left[ c_1(p) + \frac{1}{2}\psi(s) \right] \delta^{(r-s)}(x), \tag{5}$$

for $s = 1, 2, \dots$, and $r = s, s+1, \dots$. Equation (3) follows from equations (4) and (5) on noting that

$$x^s \ln|x| = x_+^s \ln_+ + (-1)^s x_-^s \ln_-.$$

The proof of the next theorem is straightforward and similar to the proof of theorem 2.

THEOREM 3. *The neutrix product of* $\ln^2|x|$ *and* $\delta^{(r)}(x)$ *exists by definition 3 and*

$$\ln^2|x| \Diamond \delta^{(r)}(x) = [2c_2(p) + \psi(r)]\, \delta^{(r)}(x) \tag{6}$$

for $r = 0, 1, 2, \ldots$, where $c_2(p) = \int_0^1 \ln^2 t\, p(t)\, dt$.

Then it follows from theorem 1 that the neutrix product $x^{-1}\ln|x|\Diamond\delta^{(r)}(x)$ exists on differentiating equation (6) and

$$x^{-1}\ln|x|\Diamond\delta^{(r)}(x) = 0 \tag{7}$$

It now follows by induction that

$$x^{-s}\ln|x|\Diamond\delta^{(r)}(x) = 0. \tag{8}$$

The functions $(x+i0)^r$ and $(x+i0)^r \ln(x+i0)$ are defined as follows, see Gel'fand and Shilov [10].

$(x+i0)^r = x^r,$

$(x+i0)^r \ln(x+i0) = x^r \ln(x+i0) = x^r \ln|x| + (-1)^r i\pi\, x_-^r$

for $r = 0, 1, 2, \ldots$ and the distributions $(x+i0)^{-s}$ and $(x+i0)^{-s}\ln(x+i0)$ are defined by

$$(x+i0)^{-s} = x^{-s} + \frac{(-1)^s i\pi}{(s-1)!}\delta^{(s-1)}(x),$$

$$(x+i0)^{-s}\ln(x+i0) = x^{-s}\ln|x| + (-1)^s i\pi\, x_-^{-s} - \frac{(-1)^s \pi^2}{2(s-1)!}\delta^{(s-1)}(x)$$

for $s = 1, 2, \ldots$ . it follows easily that

$$\frac{d}{dx}(x+i0)^r = r(x+i0)^{r-1}$$

for $r = 0, \pm1, \pm2, \ldots$ and

$$\frac{d}{dx}(x+i0)^r \ln(x+i0) = r(x+i0)^{r-1}\ln(x+i0) + (x+i0)^{r-1}$$

for $r = 0, \pm1, \pm2, \ldots$ . Note here that with Gel'fand and Shilov's definition of $x_-^{-r}$,

$$\frac{dx_-^{-r}}{dx} = r\,x_-^{-r} + \frac{\delta^{(r)}(x)}{r!}$$

for $r = 1, 2, \ldots$ .

The proof of the next theorem is straightforward and we omit the details.

THEOREM 4. *The neutrix product exists $\delta^{(s)}(x)\Diamond\delta^{(r)}(x)$ and*

$$\delta^{(s)}(x)\Diamond\delta^{(r)}(x) = 0, \tag{9}$$

*for* $s, r = 0, 1, 2, \ldots$ .

Also it can be seen easily from equation (3) that

$$x_-^{-s}\lozenge\delta^{(r-1)}(x)=\frac{(r-1)!}{4(r+s-1)!}\delta^{(r+s-1)} \tag{10}$$

In particular

$$\frac{x_-^{-s}\lozenge\delta^{(r-)}(x)}{(r-1)!}+\frac{x_-^{-r}\lozenge\delta^{(s-1)}(x)}{(s-1)!}=\frac{\delta^{(s+r-1)}(x)}{2(r+s-1)!} \tag{11}$$

We now prove the following two theorems by using the definition 3.
THEOREM 5. *The neutrix product* $(x + i0)^{-r} \lozenge (x + i0)^{-s}$ *exists and*

$$(x + i0)^{-r} \lozenge (x + i0)^{-s} = (x + i0)^{-r-s} \tag{12}$$

*for* $r, s = 1, 2, \ldots.$

PROOF. The following equations were proved by Fisher et al. in [8] that

$$x^{-r} \circ x^{-s} = x^{-r-s} \tag{13}$$

$$\ln|x| \circ x^{-s} = x^{-s}\ln|x| = x^{-s}\circ\ln|x| \tag{14}$$

for $r, s = 1, 2, \ldots.$
Since the neutrix product is clearly is clear distributive with respect to addition, it follows that

$$\begin{aligned}(x + i0)^{-r} \lozenge (x + i0)^{-s} &= \left[x^{-r}+\frac{i\pi}{(r-1)!}\delta^{(r-1)}(x)\right]\lozenge\left[x^{-s}+\frac{i\pi}{(s-1)!}\delta^{(s-1)}(x)\right]\\ &= x^{-r}\lozenge x^{-s}\\ &+\frac{i\pi}{(s-1)!}x^{-r}\lozenge\delta^{(s-1)}(x)+\frac{i\pi}{(r-1)!}\delta^{(r-1)}(x)\lozenge x^{-s}\\ &-\frac{\pi^2}{(r-1)!(s-1)!}\delta^{(s-1)}(x)\lozenge\delta^{(r-1)}(x)\end{aligned}$$

and equation (12) follows on using equation (9), (10), (11) and (13).
THEOREM 6. *The neutrix products* $(x + i0)^{-s} \lozenge \lfloor(x+i0)^{-r}\ln(x+i0)\rfloor$ *exist and*

$$(x+i0)^{-s} \lozenge \lfloor(x+i0)^{-r}\ln(x+i0)\rfloor = (x + i0)^{-s-r}\ln(x + i0) \tag{15}$$

*for* $r = 0, 1, \ldots, s-1$ *and* $s = 1, 2, \ldots.$
PROOF. The following equations were proved in [7]

$$\begin{aligned}x^{-r} \circ x_-^{-s} &= \lfloor x_+^{-r}+(-1)^r x_-^{-r}\rfloor \circ x_-^{-s}\\ &= (-1)^r x_-^{-r-s}+\left[\frac{(-1)^r c_1(p)}{(r+s-1)!}-(-1)^s M_{rs}\right]\delta^{(r+s-1)}(x),\end{aligned} \tag{16}$$

$$\begin{aligned}x_-^{-r} \circ x^{-s} &= x_-^{-r}\circ\lfloor x_-^{-r}+(-1)^s x^{-s}\rfloor\\ &= (-1)^s x_-^{-r-s}-\left[\frac{(-1)^s c_1(p)}{(r+s-1)!}+(-1)^r M_{rs}\right]\delta^{(r+s-1)}(x),\end{aligned} \tag{17}$$

where

$$M_{rs}=\sum_{i=0}^{r-1}\binom{r-1}{i}\left[\frac{2c_1(p)}{s+i+1}+\frac{c_1(p)}{s+i}+\frac{\psi(s+i)}{s+i+1}-\frac{1}{(s+i)^2}\right]\frac{(-1)^{r+s+i}}{(r-1)!(s-1)!}$$

It now follows on using equations (16) and (17) that

$$x^{-r} \lozenge x^{-s} = x^{-r-s} \tag{18}$$

for $r, s = 1, 2, \ldots$.

Although the existence of $\ln|x| \lozenge x^{-r}$ can be deduced from equation (14) for $r = 0$ and $s = 1,2, \ldots$. In the following, we are going prove that

$$x^{-r}\ln|x| \lozenge x^{-s} = x^{-r-s}\ln|x| \tag{19}$$

Putting r =0 in equation (19) gives $\ln|x| \lozenge x^{-s}$ and so equation (14) holds in the particular case r = 0 and s = 1, 2,3,. .... Now assume that equation (19) holds for some r and s = 1, 2, 3, .... Then it follows on differentiating equation (19) and using induction method that equation (19) holds for r = 0, 1, 2,....

Since product is distributive then we have

$$(x+i0)^{-s} \lozenge [(x+i0)^{-r}\ln(x+i0)] = \left[x^{-s} + \frac{i\pi}{(s-1)!}\delta^{(s-1)}(x)\right] \lozenge \left[x^{-s}\ln|x| + i\pi\, x_-^{-r} - \frac{\pi^2\delta^{(r-1)}}{2(r-1)!}\right]$$

$$= x^{-s}\lozenge x^{-r}\ln|x| + i\pi x^{-r}\lozenge x_-^{-s} -$$

$$-\frac{\pi^2\delta^{(r-1)}(x)\lozenge x^{-s}}{2(r-1)!} + \frac{i\pi}{2(s-1)!}x^{-s}\ln|x|\lozenge\delta^{(s-1)}(x) -$$

$$-\frac{i\pi^3\delta^{(r-1)}(x)\lozenge\delta^{(s-1)}(x)}{2(r-1)!(s-1)!}$$

Equation (15) follows easily from equations (8), (9), (10), (18) and (19).

## 3. A COMPARISON ON TWO COMMUTATIVE PRODUCTS

The following definition for the product of two distributions was given in [12].

DEFINITON 4. *Let f and g be distributions in D' and let $f_n(x) = (f * \delta_n)(x)$ and $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product f  g of f and g exists and is equal to the distribution h on the interval (a, b) if*

$$N-\lim_{n\to\infty}\langle f_n(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

*for all function $\phi$ in D' with support contained in the interval (a, b), If*

$$\lim_{n\to\infty}\langle f_n(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle,$$

*we simply say that the product $f \cdot g$ exists and equals h, see* [2].

The definition 4 of the neutrix product is clearly commutative. We finally give an example where two commutative neutrix products differ. It was proved by Kilicman in [12] that neutrix products $\ln^p x_+ \;\; \delta^{(s)}(x)$, $\ln^p x_- \;\; \delta^{(s)}(x)$ and $\ln^p |x| \cdot \delta^{(s)}(x)$ exist by definition 4 and

$$\ln^p x_+ \;\; \delta^{(s)}(x) = b_s^p\,\delta^{(s)}(x) \tag{20}$$

$$= (-1)^s \ln^p x_- \;\; \delta^{(s)}(x) \tag{21}$$

$$= \frac{1}{2} \ln^p |x| \quad \delta^{(s)}(x), \tag{22}$$

where

$$b_s^p = \frac{1}{s!} \int_{-1}^{1} u^s p^{(s)}(u) \int_{-1}^{u} \ln^p(v-u)p(u)dudv$$

for s = 0, 1, 2, ..... and p ≥1. Substituting s by r and putting s = 0 in equation (7) gives that

$$\ln x_+ \lozenge \delta^{(s)}(x) = [2c(p) + \psi(s)]\delta^{(s)}(x) \tag{23}$$

Putting particular value p=1 in equation (21) we have

$$\ln x_+ \quad \delta^{(s)}(x) = b_s \delta^{(s)}(x) \tag{24}$$

and comparing the equations (23) and (24) clearly indicates that two products are in general not equal.

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