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THE SIMPLE RECURRENT FORMULAS TO FIND A SEQUENCES OF NUMBERS SATISFYING EQUATION $X^2 + (X+1)^2 = Z^2$, AND THE PROPERTIES OF THESE INTEGERS

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Abstract: In this study, simple recurrent formulas have been found to form the sequence of Phythogorean triple. These three integers, any two of which are relatively prime numbers, are set up basic Phythogorean triple. The properties of these triples have been examined and classified.

Keywords: Phythogorean triple, recurrent formulas,

1. INTRODUCTION

In our previous article(1) related to this subject the presented algorithm is a little complicated to form of Phythogorean triples, (x, x+1, z). New recurrent formulas that we found are quite simple and these facilitates are to form sequence of Phythogorean triple. The Phythogorean triple has an important attitude in theory of integers.

If x, y, and z being positive integers are the sides of a right-angled triangle, this (x, y, z) triple is called a Phythogorean triple.
There are shown some Phythogorean triple examples, obtained by the Phythogorean method:

"When $n \neq 1$ and a positive odd integer, and at the same time $x = n, y = \frac{1}{2}(n^2+1)$ and $z = \frac{1}{2}(n^2+1)$ then (x, y, z) is determined as Phythogorean triple."

Table 1

X	3	5	7	9	...	17	19	...
Y	4	12	24	40	...	144	180	...
Z	5	13	25	41	...	145	181	...

The numbers satisfying the equation $x^2 + (x+1)^2 = z^2$ (1) are obtained by the help of Pell numbers' squares $[1-3]$:

$x_n = P_{n+1}^2 - P_n^2;\quad z_n = P_{n+1}^2 + P_n^2$

here $P_0 = 0; P_1 = 1; P_{n+2} = 2P_{n+1} + P_n$.

2 .THE RECURRENT FORMULAS AND PROPERTIES OF THE TRIPLES

The equation given above being written as $x^2+y^2=z^2$ (1') for the sequences of numbers satisfying this equation to be obtained the simple recurrent formulas have been found

$$\left.\begin{array}{l} x_{n+2}=6x_{n+1}-(x_n-2)\,;\ x_1=0\,;\ x_2=3 \\ y_{n+2}=6y_{n+1}-(y_n+2)\,;\ y_1=1\,;\ y_2=4 \\ z_{n+2}=6z_{n+1}-z_n \qquad ;\ z_1=1\,;\ z_2=5 \end{array}\right\} \quad (2)$$

The important pecularities of relations between the numbers of sequences written by the help of the formulas (2) were revealed.

Since $y=x+1$, $x_{n+1}+y_n=x_n+y_{n+1}$ (3) is obtained.

For the two successive terms of the sequences revealed the equation (1') is written as:

$$\left.\begin{array}{l} x_{n+1}^2+y_{n+1}^2=z_{n+1}^2 \\ x_n^2+y_n^2 \qquad =z_n^2 \end{array}\right\} \quad (4)$$

If the second expression subtract from the first and carry out some simplification:

$$x_{n+1}^2-y_n^2+y_{n+1}^2-x_n^2=z_{n+1}^2-z_n^2 \Rightarrow (x_{n+1}-y_n)(x_{n+1}+y_n)+(y_{n+1}-x_n)(y_{n+1}+x_n) \Rightarrow$$
$$=(z_{n+1}-z_n)(z_{n+1}+z_n)$$

can be obtained.

If considering (3) in this formula:

$$(x_{n+1}+y_n)\left[x_{n+1}+y_{n+1}-(x_n+y_n)\right]=(z_{n+1}-z_n)(z_{n+1}+z_n) \quad (5)$$

From the formula (5) two equation can be obtained :

$$\left.\begin{array}{l} x_n+y_{n+1}=x_{n+1}+y_n=z_{n+1}-z_n \\ x_{n+1}+y_{n+1}-(x_n+y_n)=z_{n+1}+z_n \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l} x_n+y_{n+1}=x_{n+1}+y_n=z_{n+1}+z_n \\ x_{n+1}+y_{n+1}-(x_n+y_n)=z_{n+1}-z_n \end{array}\right\} \quad (6')$$

It is seen that the system (6') is not possible, if considering equation (4)

So we can estimate the system (6).

And now writing Phytagorean theorem for n+2 term, considering that y=x+1, and using the recurrent formulas (2):

$$x_{n+2}^2+y_{n+2}^2=z_{n+2}^2 \Rightarrow [6x_{n+1}-(x_n-2)]^2+[6y_{n+1}-(y_n+2)]^2=(6z_{n+1}-z_n)^2 \Rightarrow$$
$$36x_{n+1}^2-12x_{n+1}(x_n-2)+(x_n-2)^2+36y_{n+1}^2-12y_{n+1}(y_n+2)+(y_n+2)^2=36z_{n+1}^2-12z_{n+1}z_n+z_n^2 \Rightarrow$$
$$-12x_{n+1}x_n+24x_{n+1}+x_n^2-4x_n+4-12y_{n+1}y_n-24y_{n+1}+y_n^2+4y_n+4=-12z_{n+1}z_n+z_n^2 \Rightarrow$$
$$-12(x_{n+1}x_n+y_{n+1}y_n)-24(y_{n+1}-x_{n+1})+4(y_n-x_n)+8=-12z_{n+1}z_n \Rightarrow$$
$$12(x_{n+1}x_n+y_{n+1}y_n)+24-4-8=12z_{n+1}z_n \Rightarrow x_{n+1}x_n+y_{n+1}y_n=z_{n+1}z_n-1 \qquad (7)$$

Equation (7) is obtained.

Taking into account the first equation of system (6), raising it to square, and carrying out some simplifications, we can obtain:

$$(z_{n+1} - z_n)^2 = (x_{n+1} + y_n)^2 \Rightarrow z_{n+1}^2 - 2z_{n+1}z_n + z_n^2 = x_{n+1}^2 + 2x_{n+1}y_n + y_n^2 \Rightarrow$$

$$2z_{n+1}z_n = z_{n+1}^2 - x_{n+1}^2 + z_n^2 - y_n^2 - 2x_{n+1}y_n = y_{n+1}^2 + x_n^2 - 2x_{n+1}y_n =$$

$$(x_{n+1}+1)^2 + (y_n - 1)^2 - 2x_{n+1}y_n = x_{n+1}^2 + 2x_{n+1} + 1 + y_n^2 - 2y_n + 1 - 2x_{n+1}y_n \Rightarrow$$

$$2(z_{n+1}z_n - 1) = x_{n+1}^2 + 2x_{n+1}y_n + y_n^2 + 2x_{n+1} - 2y_n - 2x_{n+1}y_n - 2x_{n+1}y_n =$$

$$(x_{n+1} + y_n)^2 - 2x_{n+1}(y_n - 1) - 2y_n(x_{n+1} + 1) = (x_{n+1} + y_n)^2 - 2x_{n+1}x_n - 2y_n y_{n+1} =$$

$$(x_{n+1} + y_n)^2 - 2(x_{n+1}x_n + y_{n+1}y_n)$$

Considering equation (7) in the last expression

$$2(z_{n+1}z_n - 1) = (x_{n+1} + y_n)^2 - 2(z_{n+1}z_n - 1) \Rightarrow$$

$$4(z_{n+1}z_n - 1) = (x_{n+1} + y_n)^2 \quad \Rightarrow z_{n+1}z_n - 1 = \left(\frac{x_{n+1} + y_n}{2}\right)^2 = \left(\frac{x_n + y_{n+1}}{2}\right)^2 \tag{8}$$

is obtained.

If again raising the first equation of system (6) to second power , and taking into account equation (1)

$$(z_{n+1} + z_n)^2 = (x_{n+1} + y_n)^2 \Rightarrow z_{n+1}^2 - 2z_{n+1}z_n + z_n^2 = x_{n+1}^2 + 2x_{n+1}y_n + y_n^2 \Rightarrow$$

$$x_n^2 + y_{n+1}^2 = 2(x_{n+1}y_n + z_{n+1}z_n) \tag{9}$$

By the same way

$$x_{n+1}^2 + y_n^2 = 2(x_n y_{n+1} + z_{n+1}z_n) \tag{9'}$$

is obtained.

And now using formulas (8) and (9):

$$\left(\frac{x_{n+1} + y_n}{2}\right)^2 = z_{n+1}z_n - 1 \Rightarrow x_n^2 + y_{n+1}^2 + 2x_n y_{n+1} = 4(z_{n+1}z_n - 1) \Rightarrow$$

$$x_{n+1}y_n + z_{n+1}z_n + x_n y_{n+1} = 2z_{n+1}z_n - 2 \Rightarrow x_{n+1}y_n + x_n y_{n+1} = z_{n+1}z_n - 2 \tag{10}$$

But taking into account formulas (7) and (10)

$$x_{n+1}y_n + x_n y_{n+1} = x_{n+1}x_n + y_{n+1}y_n - 1 \tag{11}$$

is obtained.

And now raising the second equation of the system (6) to the square and carrying out some algebraic conversions:

$$[x_{n+1}+y_{n+1}-(x_n+y_n)]^2=(z_{n+1}+z_n)^2 \Rightarrow$$
$$(x_{n+1}+y_{n+1})^2+(x_n+y_n)^2-2(x_{n+1}+y_{n+1})(x_n+y_n)=z_{n+1}^2+2z_{n+1}z_n+z_n^2 \Rightarrow$$
$$x_{n+1}^2+y_{n+1}^2+2x_{n+1}y_{n+1}+x_n^2+y_n^2+2x_n y_n-2(x_{n+1}+y_{n+1})(x_n+y_n)=z_{n+1}^2+z_n^2+2z_{n+1}z_n \Rightarrow$$
$$x_{n+1}y_{n+1}+x_n y_n-x_{n+1}x_n-x_{n+1}y_n-y_{n+1}x_n-y_{n+1}y_n=z_{n+1}z_n \Rightarrow$$
$$x_{n+1}(y_{n+1}-y_n)-x_n(y_{n+1}-y_n)=z_{n+1}z_n+x_{n+1}x_n+y_{n+1}y_n$$

Taking into account (7) in the last equation

$$(x_{n+1}-x_n)(y_{n+1}-y_n)=2z_{n+1}z_n-1 \qquad (12)$$

and the formula $y_{n+1}=x_{n+1}+1$ being used

$$(x_{n+1}-x_n)^2=(y_{n+1}-y_n)^2=2z_{n+1}z_n-1 \qquad (12')$$

is obtained.

At last, let the following limit be calculated:

$$k=\lim_{n\to\infty}\frac{z_n}{z_{n+1}}=\lim_{n\to\infty}\frac{z_n}{6z_n-z_{n-1}}=\frac{1}{6-\lim\limits_{n\to\infty}\frac{z_{n-1}}{z_n}}=\frac{1}{6-k}\Rightarrow$$

$$k^2-6k+1=0$$

$$k=3\pm\sqrt{8}=\left(\sqrt{2}\right)^2\pm 2\sqrt{2}+1=\left(\sqrt{2}\pm 1\right)^2$$

Here the sign "+" is available for $\lim\limits_{n\to\infty}\frac{z_{n+1}}{z_n}$

Summary: And now let the pecularities, found and proved, be listed.

1. $x_{n+1}+y_n=y_{n+1}+x_n=z_{n+1}-z_n$
2. $x_{n+1}+y_{n+1}-(x_n+y_n)=z_{n+1}+z_n$
3. $x_{n+1}x_n+y_{n+1}y_n=z_{n+1}z_n-1$
4. $\left(\frac{x_n+y_{n+1}}{2}\right)^2=\left(\frac{x_{n+1}+y_n}{2}\right)^2=z_{n+1}z_n-1$
5. $\begin{cases} x_{n+1}^2+y_n^2-2x_n y_{n+1}=2z_{n+1}z_n \Rightarrow x_{n+1}^2+y_n^2=2(x_n y_{n+1}+z_n z_{n+1}) \\ x_n^2+y_{n+1}^2-2x_{n+1}y_n=2z_{n+1}z_n \Rightarrow x_n^2+y_{n+1}^2=2(x_{n+1}y_n+z_n z_{n+1}) \end{cases}$
6. $x_n y_{n+1}+x_{n+1}y_n=z_{n+1}z_n-2$
7. $x_n y_{n+1}+x_{n+1}y_n=x_{n+1}x_n+y_{n+1}y_n-1$
8. $(x_{n+1}-x_n)(y_{n+1}-y_n)=2z_{n+1}z_n-1=(x_{n+1}-x_n)^2=(y_{n+1}-y_n)^2$
9. $\lim\limits_{n\to\infty}\frac{z_n}{z_{n+1}}=\left(\sqrt{2}-1\right)^2;\lim\limits_{n\to\infty}\frac{z_{n+1}}{z_n}=\left(\sqrt{2}+1\right)^2$

If consider that $y_n=x_n+1$, these pecularities may be listed as:

1. $x_{n+1}+x_n+1=z_{n+1}-z_n$

2. $x_{n+1} - x_n = \dfrac{z_{n+1} + z_n}{2}$
3. $2x_{n+1}x_n + x_{n+1} + x_n = z_{n+1}z_n - 2$
4. $\left(\dfrac{x_{n+1} + x_n + 1}{2}\right)^2 = z_{n+1}z_n - 1$
5. $(x_{n+1} - x_n)^2 = z_{n+1}z_n - 1$
6. $2x_{n+1}x_n + x_{n+1} + x_n = \left(\dfrac{x_{n+1} + x_n + 1}{2}\right)^2 - 1$
7. $\sqrt{2z_{n+1}z_n - 1} = \dfrac{z_{n+1} + z_n}{2}$

Comparing (1) with $x^2 + y^2 = (y+d)^2$, $d \in N$, (13), [5,6], at the certain significances of d, one can see that the numbers satisfying these equations are the same. In order to find the required meanings of d, writing the following system of equations and solving it, considering formulas (2) for z_{n+1} and x_{n+1} we can obtain:

$$\begin{cases} z_{n+1} - x_{n+1} = 1 + d_{n+1} \Rightarrow 6z_n - z_{n-1} - 6x_n + x_{n-1} + 2 = 1 + d_{n+1} \Rightarrow z_n - x_n = 1 + d_n \\ z_{n-1} - x_{n-1} = 1 + d_{n-1}\,;\ 6(z_n - x_n) - (z_{n-1} - x_{n-1}) - 3 = d_{n+1} \Rightarrow \\ d_{n+1} = 6d_n + 6 - 1 - d_{n-1} - 3 = 6d_n - d_{n-1} + 2 \end{cases}$$

$$d_{n+1} = 6d_n - (d_{n-1} - 2) \qquad (14)$$

Here $d_1 = 0$ $d_2 = 1$ are taking into account.
0,1,8,49,288,1681,.....
In order to verify the pecularities shown for the numbers given above , and compare the meaning of d, we form following table:

Table 2

n	x_n	$y_n = x_n + 1$	$z_n = x_n + 1 + d_n$	d_n
1	0	1	1	0
2	3	4	5	1
3	20	21	29	8
4	119	120	169	49
5	696	697	985	288
6	4059	4060	5741	1681

Discussion

Definition: 1) The triple, which is coprime in pairs, is called the basic Pythagorean triple.

2) The triples which satisfy the equations, $x^2 + y^2 = (y + d)^2$ and $x^2 + (x + 1)^2 = z^2$ are called hypotenuse – right side (HR) and right side-right side (RR) integers, respectively.

Let's see the relation between these triples. HR numbers are valid in all integer values of **d:** the triples that is only suitable for the value **d = 1** form the basic Pythagorean triple; others (for $d \neq 1$) contain the basic and non- basic triples.
In RR triples, basic Pythagorean triple exists only for the value **k = 1**; for the other values of **k**, none of triples are basic Pythagorean triples. This sequence of triples is proportional with the elements of basic Pythagorean triples,
Examples:

$$(3,4,5) \longrightarrow \begin{cases} k=2 & (6,8,10) \\ k=3 & (9,12,15) \\ k=4 & (12,16,20) \\ \dots\dots & \end{cases}$$

$$(20,21,29) \longrightarrow \begin{cases} k=2 & (40,82,58) \\ k=3 & (60,63,87) \\ k=4 & (80,84,116) \\ \dots\dots & \end{cases}$$

....

We can see the relation between HR and RR triples in the *fig 1*. (Note: diameters of circles approach to infinity in the *fig 1*)

Which rows of triples do RR triple contain in the HR triple sequences that follow the **d values**? To answer this question, let's separate the value of **d** into 2 groups and show the integer of row.

Sequence	Row no:			Row no:	
$d_3 = 8 = 3^2 - 1$	3		$d_2 = 1 = 1^2$	1	z_1
$d_5 = 288 = 17^2 - 1$	17	and	$d_4 = 49 = 7^2$	5	z_2
$d_7 = 9800 = 99^2 - 1$	99		$d_6 = 1681 = 41^2$	29	z_3
$d_9 = 332928 = 577^2 - 1$	577		$d_8 = 57121 = 239^2$	169	z_4
....			$d_{10} = 1940449 = 1393^2$	985	z_5

....

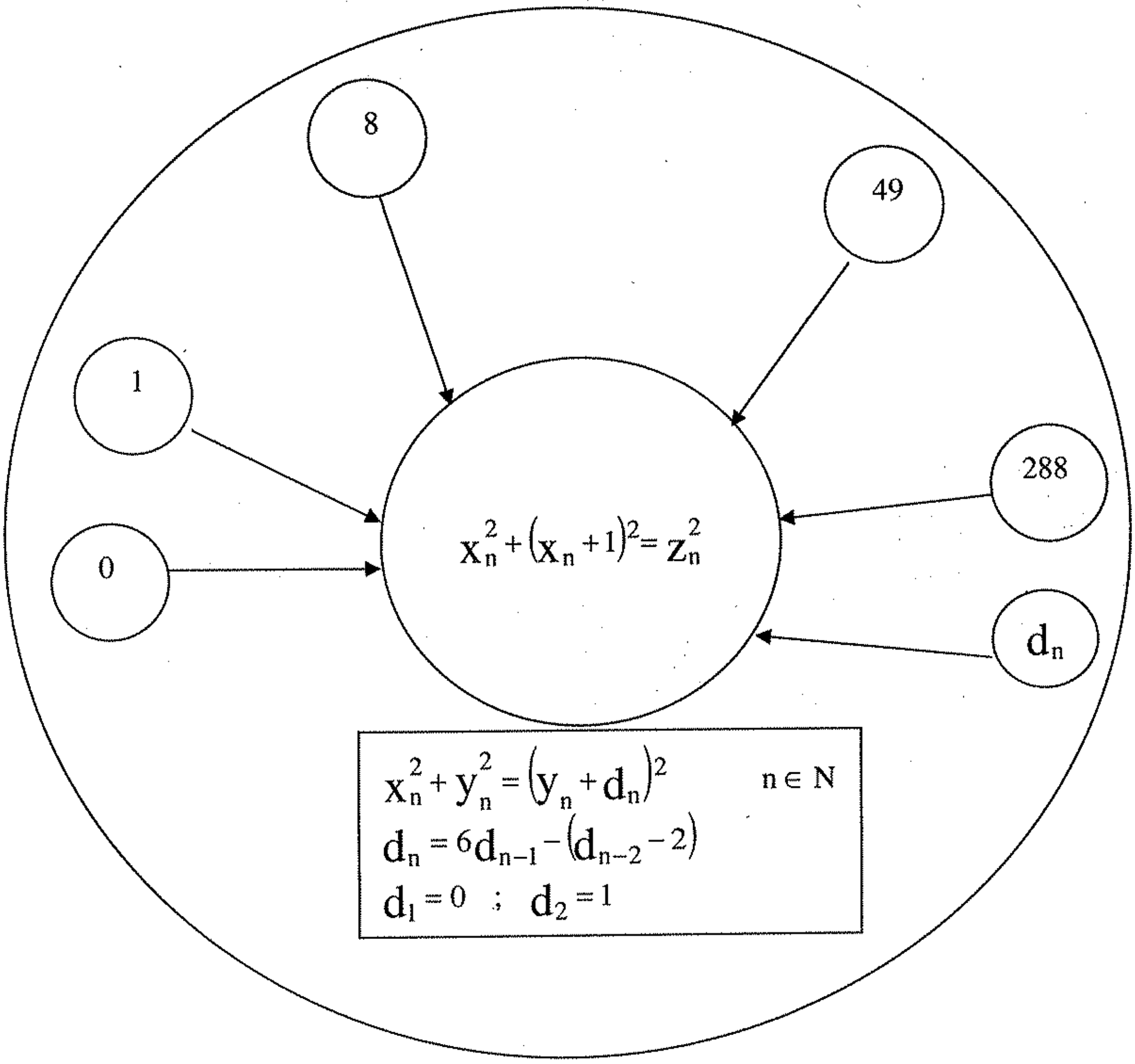

Figure 1.

CONCLUSION

These interesting properties give the special Pythagorean triples that satisfy the equation

$x^2+(x+1)^2=z^2$.

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