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ON SOME SEQUENCE SPACES AND LACUNARY σ-STATISTICAL CONVERGENCE

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Abstract: In this note , we define and study two concepts which arise from the notion of lacunary strong convergence, and invariant means, namely lacunary strong σ-convergence with respect to an Orlicz function and lacunary σ-statistical convergence and established the relationship between these two concepts.
Keywords: Orlicz functions, Invariant Means, lacunary strong convergence, statistical convergence.

1. INTRODUCTION

Let ℓ_∞ and c respectively be the Banach space of bounded and convergent sequences $x = (x_n)$ with norm $\|x\| = \sup_{k\geq 0}|x_k|$, respectively.

A sequence $x=(x_k)\in \ell_\infty$ is said to be almost convergent if all of its Banach limits [1] coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [9] proved that

$$\hat{c} = \{x_k\in \ell_\infty : \lim_m d_{mn}(x) \text{ exists, uniformly in } n\}$$

where $d_{mn}(x) = \dfrac{x_n + x_{n+1} + \ldots + x_{n+m}}{m+1}$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [10] as follows

$$[\hat{c}] = \{x_k\in \ell_\infty : \lim_m d_{mn}(|x-le|) = 0 \text{, uniformly in } n \text{ for some } l\},$$

where $e = (1,1,\ldots)$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x)\geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for each n;

(ii) $\phi(x)=1$, where e = (1,1,1,...) and

(iii) $\phi((x_{\sigma(n)}))=\phi(x)$, for all $x\in \ell_\infty$.

The mapping is one-to-one such that $\sigma^m(n)\neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of mapping σ at n.

Thus, ϕ extends the limit functional on c in the sense that $\phi(x)=\lim x$ for $x\in c$. In case σ is the tranlation mapping $\sigma(n)=n+1$, the σ-mean is often called a Banach limit on ℓ_∞ and V_σ, the set of all bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$,[8].

If $x=(x_n)$, write $Tx=(Tx_n)=(x_{\sigma(n)})$, it can be shown that (see, [19])

$$V_\sigma=\{x\in \ell_\infty : \lim_{m\to\infty} t_{m,n}(x)=L \ \ uniformly \ \ in \ \ n, L=\sigma-limx\},$$

where $t_{m,n}(x)=(m+1)^{-1}(x_n+Tx_n+...+T^m x_n)$.

Just as the concept of almost convergence led naturally to the concept of strong almost convergence, σ- convergence leads naturally to the concept of strong σ-convergence.

A sequence $x=(x_k)$ is said to be strongly σ- convergent(see, [12]) if there exists a number ℓ such that

$$\frac{1}{m}\sum_{k=1}^{m}\left|x_{\sigma^k(n)}-\ell\right|\to 0. \tag{1}$$

as $m\to\infty$, uniformly in n. We write $[V_\sigma]$ as the set of all strongly σ- convergent sequences. When (1) holds we write $[V_\sigma]-limx=\ell$. Taking $\sigma(n)=n+1$, we have $[V_\sigma]=[\hat{c}]$. So that strong σ- convergence generalized the concept of strong almost convergence. Note that $c\subset[V_\sigma]\subset V_\sigma\subset\ell_\infty$.

By a lacunary sequence $\theta=(k_r)$, $r=0,1,2,...$ where $k_0=0$, we shall mean an increasing sequence of non-negative integers with $k_r-k_{r-1}\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_r-k_{r-1}]$, and we let $h_r=k_r-k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [5] as follows

$$N_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} | x_k - \ell | = 0 \quad for\ some\ \ell \right\}.$$

The space N_θ is a BK-space with the norm

$$\| x \| = sup \left(\frac{1}{h_r} \sum_{k \in I_r} | x_k | \right).$$

N_θ^0 denotes the subset of those sequences in N_θ for which $\ell = 0$. $\left(N_\theta^0, \| . \|_\theta \right)$ is also a BK-space. There is a strong connection between N_θ and the space w of strongly Cesaro summable sequences, which is defined by

$$w = \left\{ x = (x_k) : lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell| = 0, \ for\ some\ \ell \right\}.$$

In the special case where $\theta = (2^r)$, we have $N_\theta = w$.

Recently, the concept of lacunary strong σ- convergence was introduced by Savaş[18] which is a generalization of the idea of lacunary strong almost convergence due to Das and Mishra [3]. If $[V_\sigma^\theta]$ denotes the set of all lacunary strongly σ - convergent sequences, then Savaş [15] defined

$$[V_\sigma^\theta] = \left\{ x = (x_n) : lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| x_{\sigma^k(n)} - \ell \right| = 0, for\ some\ \ell, uniformly\ in\ n \right\}.$$

Note that for $\sigma(n) = n+1$, the space $[V_\sigma^\theta]$ in the same as $[AC_\theta]$ which is defined as following by Das and Patel[3].

$$[AC_\theta] = \left\{ x = (x_k) : lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| x_{k+n} - \ell \right| = 0, for\ some\ \ell, uniformly\ in\ n \right\}.$$

We write $[V_\sigma^\theta] = [V_\sigma^\theta]_0$ in case $\ell = 0$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, the function is called a Modulus function defined and discussed by Ruckle [15] and Maddox [11].

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x : \sum_{k=1}^{\infty} [M(|z|/\rho)] < \infty, \ \ for \ \ some \ \ \rho > 0 \right\}$$

The space ℓ_M with the norm

$$||x|| = inf \left\{ \rho > 0 : x : \sum_{k=1}^{\infty} [M(|x_k|/\rho)] \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. An Orlicz function M can always be represented (see, Krounoselskii and Rutitsky [7]) in the general form $M(x) = \int_0^x q(t)dt$, where q, known as the kernel of M, is right differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non-decreasing, $q(t) \to \infty$ as $t \to \infty$. The space ℓ_M is closely related to the sphere ℓ_p which is an Orlicz sequence space with $M(x) = x^p$; $1 \leq p < \infty$.

Quite recently Bhardwaj and Singh [2] introduced the following sequence spaces defined by Orlicz function M and for the sequence $p = (p_k)$ of positive real numbers.

$$[V_\sigma^\theta, M, p] = \left\{ x : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)} - \ell|}{\rho} \right) \right]^{p_k} = 0, \ uniformly \ in \ n \ for \ \ some \ \ \ell > 0, \rho > 0 \right\}$$

$$[V_\sigma^\theta, M, p]_0 = \left\{ x : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} = 0, \ uniformly \ in \ n, for \ \ some \ \ \rho > 0 \right\}$$

$$[V_\sigma^\theta, M, p]_\infty = \left\{ x : \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} < \infty, \ for \ \ some \ \ \rho > 0 \right\}.$$

If $p_k = 1$ for all k, then the above spaces are reduced to $[V_\sigma^\theta, M]$, $[V_\sigma^\theta, M]_0$ and $[V_\sigma^\theta, M]_\infty$. If $x \in [V_\sigma^\theta, M]$ we say that x is lacunary strongly σ-convergent with respect to the Orlicz function M.

Some well – known spaces are obtained by specializing M, θ and p.

(i) If $M(x)=x$, $p_k=1$ for all k and $\theta=(2^r)$ then $[V_\sigma^\theta, M, p]=[V_\sigma]$, (Mursaleen, [12]).

(ii) If $\theta=(2^r)$, then $[V_\sigma^\theta, M, p]=[V_\sigma, M, p]$, $[V_\sigma^\theta, M, p]_0=[V_\sigma, M, p]_0$ and $[V_\sigma^\theta, M, p]_\infty$ $=[V_\sigma, M, p]_\infty$,*(see,* Nuray and Gülcü [13])

(iii) If $M(x)=x$, $p_k=1$ for all k, then $[V_\sigma^\theta, M, p]=[V_\sigma^\theta]$*(see,* Savaş [17]).

In this paper we study and examine the above sequence spaces defined by Orlicz function M and established some new result.

2. MAIN RESULTS

We now have

Theorem 1. $[V_\sigma^\theta, M, p]_0 \subset [V_\sigma^\theta, M, p] \subset [V_\sigma^\theta, M, p]_\infty$.

Proof. Obviously, $[V_\sigma^\theta, M, p]_0 \subset [V_\sigma^\theta, M, p]$. We have

$$|x_k + y_k|^{p_k} \le C(|x_k|^{p_k} + |y_k|^{p_k}) \tag{1}$$

where $C = max(1, 2^{H-1})$, $H = sup\, p_k$. Let $x \in [V_\sigma^\theta, M, p]$. From (1)

$$\frac{1}{h_r}\sum_{k\in I_r} M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right)^{p_k} \le \frac{C}{h_r}\sum_{k\in I_r} M\left(\frac{|x_{\sigma^k(n)} - \ell|}{\rho}\right)^{p_k} + \frac{C}{h_r}\sum_{k\in I_r} M\left(\frac{|\ell|}{\rho}\right)^{p_k}.$$

There exists an integer K_L such that $|\ell| \le K_L$. Hence we have

$$\frac{1}{h_r}\sum_{k\in I_r} M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right)^{p_k} \le \frac{C}{h_r}\sum_{k\in I_r} M\left(\frac{|x_{\sigma^k(n)} - \ell|}{\rho}\right)^{p_k} + C\left[K_L M(\frac{1}{\rho})\right]^H.$$

Thus we get $x \in [V_\sigma^\theta, M, p]_\infty$.

This completes the proof.

Theorem 2. Let M be an Orlicz function and $p=(p_k)$ be any bounded sequence of stricly positive real numbers, then $[V_\sigma, M, p] \subset [V_\sigma^\theta, M, p]$ for every lacunary sequence θ.

Proof. Let $x \in [V_\sigma, M, p]$ and ε>0. There exists a positive integer m_0, number ℓ and $\rho>0$ such that

$$\frac{1}{m}\sum_{k=1}^{m}\left[M(\frac{|x_{\sigma^k(n)} - \ell|}{\rho})\right]^{p_k} < \varepsilon$$

for $m>m_0$, $m = 0, 1, 2, \ldots$. Since θ is lacunary, we can choose $R>0$ such that $r\geq R$ implies $h_r>m_0$ and consequently, $T_r = \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|x_{\sigma^k(n)}-\ell|}{\rho}\right)\right]^{p_k} < \varepsilon$. Thus $x\in[V_\sigma^\theta, M, p]$.
This completes the proof.

A complex number sequence $x=(x_k)$ is said to be statistically convergent to the number ℓ if for every ε>0, (see, [4]),

$$lim_n \frac{1}{n}\left|\{k\leq n: |x_k-\ell|\geq \varepsilon\}\right| = 0.$$

The set of statistically convergent sequences is denoted by s.

Recently, Savaş and Nuray [16] introduced the concept of lacunary σ-statistical convergence as follows:

Definition 1. Let θ be a lacunary sequence. Then a sequence $x=(x_k)$ is said to be lacunary σ- statistically convergent to a number ℓ if for every $\varepsilon>0$,

$$lim_r \frac{1}{h_r} max_{n\geq 0}\left|\{k\in I_r : |x_{\sigma^k(n)}-\ell| \geq \varepsilon\}\right| = 0$$

In this case we write S_σ^θ- $limx = \ell$ or $x_k \to (S_\sigma^\theta)$ and we define

$$S_\sigma^\theta = \{\, x = x_k: \textit{for some } \ell,\ S_\sigma^\theta\text{-}\ limx = \ell \,\}.$$

The set of all lacunary σ-statistically convergent sequences is denoted by S_σ^θ.

Theorem 3. Let M be an Orlicz function. Then $[V_\sigma^\theta, M]_o \subset (S_\sigma^\theta)_o$.

Proof. Suppose $x\in[V_\sigma^\theta, M]_o$ *and* $\varepsilon>0$. Then we have for every n,

$$\sum_{k\in I_r} M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right) \geq \sum_{\substack{k\in I_r\\ |x_{\sigma^k(n)}|\geq\varepsilon}} M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right)$$

$$> M(\varepsilon/\rho).max_{n\geq 0}\left|\{k\in I_r : |x_{\sigma^k(n)}|\geq\varepsilon\}\right|$$

from which it follows that $x\in(S_\sigma^\theta)_o$.

Theorem 4. $(S_\sigma^\theta)_o = [V_\sigma^\theta, M]_o$ if and only if M is bounded.

Proof. Suppose that M is bounded and that $x\in(S_\sigma^\theta)_o$. Since M is bounded there exists an integer K such that $M(x)<K$, for all $x\geq 0$. Then for each n,

$$\frac{1}{h_r}\sum_{k\in I_r} M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right) = \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |x_{\sigma^k(n)}|\geq\varepsilon}} M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right) + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |x_{\sigma^k(n)}|<\varepsilon}} M\left(\frac{|x_{\sigma^k(n)}-\ell|}{\rho}\right)$$

$$\leq \frac{1}{h_r} K.max_{n\geq 0}\left|\left\{k\in I_r : \left|x_{\sigma^k(n)}\right|\geq\varepsilon\right\}\right| + M\left(\varepsilon/\rho\right)$$

and so taking the limit as $r\to\infty$, the result follows.

Conversely, suppose that M is unbounded so that there is a positive sequence $0 < s_1 < s_2 < ... < s_i < ....$ such that $M(s_i)\geq h_i$. Define the sequence $x=(x_i)$ by putting $x_{k_i} = s_i$ for $i = 1, 2, ...$, $x_i = 0$, otherwise. Then, we have $x\in(S_\sigma^\theta)_o$, but $x\notin[V_\sigma^\theta, M]_o$, contradicting, $(S_\sigma^\theta)_o = [V_\sigma^\theta, M]_o$.

This completes the proof.

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