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APPROXIMATE FIRST INTEGRALS OF A HAMILTONIAN SYSTEM WITH TWO-DEGREES OF FREEDOM

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Abstract – Approximate Noether symmetries of a Hamiltonian system with two-degrees of freedom have been determined by incorporating the resonances. Approximate first integrals corresponding to these symmetries have been obtained by utilizing the approximate version of the Noether's theorem. Analytically obtained result has been compared with the numerically obtained Poincaré's surface of sections.

Keywords –Approximate symmetries, resonances, approximate first integrals, KAM curves

1. INTRODUCTION

Here, we consider a two-degree of freedom system with Hamiltonian

$$H = \frac{1}{2}\left(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + x_3^2 + x_4^2\right) + \varepsilon(x_1^3 x_2 - x_1 x_2^3) \tag{1}$$

where, x_1 and x_2 are canonical positions, x_3 and x_4 are canonical momentas and, ε is a small parameter. Corresponding dynamical system reads as

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & -\omega_2^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ 0 \\ -3x_2 x_1^2 + x_2^3 \\ 3x_1 x_2^2 - x_1^3 \end{bmatrix} \tag{2}$$

Approximate first integrals serve as useful tool to study the structure of the phase space before the onset of chaos. Approximate version of the Noether's theorem can be used for obtaining them. This requires the approximate Noether symmetries. Approximate symmetry analysis has been developed in [1]. In [2], a new method has been given which incorporates the resonances. Here, we will make use of this method.

2. APPROXIMATE NOETHER SYMMETRIES

Dynamical system given in (2) has two parts: unpertubed (linear) and perturbation parts. We first determine the exact Noether symmetries of the unperturbed part. To achieve this goal we now introduce $x=Sz$ into (2) where

$$S = \begin{pmatrix} i/w1 & -i/w1 & 0 & 0 \\ 0 & 0 & i/w2 & -i/w2 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

to obtain

$$\begin{bmatrix} \dfrac{dz_1}{dt} \\ \dfrac{dz_2}{dt} \\ \dfrac{dz_3}{dt} \\ \dfrac{dz_4}{dt} \end{bmatrix} = \begin{bmatrix} i\omega_1 & 0 & 0 & 0 \\ 0 & -i\omega_1 & 0 & 0 \\ 0 & 0 & i\omega_2 & 0 \\ 0 & 0 & 0 & -i\omega_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} - \varepsilon/2 \begin{bmatrix} 3i\dfrac{(z_1-z)^2(z_3-z_4)}{\omega_1^2\omega_2} - i\dfrac{(z_3-z_4)^3}{\omega_2^3} \\ 3i\dfrac{(z_1-z)^2(z_3-z_4)}{\omega_1^2\omega_2} - i\dfrac{(z_3-z_4)^3}{\omega_2^3} \\ -3i\dfrac{(z_1-z)\ (z_3-z_4)^2}{\omega_1\omega_2^2} + i\dfrac{(z_1-z_2)^3}{\omega_1^3} \\ -3i\dfrac{(z_1-z)\ (z_3-z_4)^2}{\omega_1\omega_2^2} + i\dfrac{(z_1-z_2)^3}{\omega_1^3} \end{bmatrix}$$

We now look for Noether symmetries involving homogenous polynomials of the form

$$z_1^{s_1} z_2^{s_2} z_3^{s_3} z_4^{s_4}$$

which satisfy the resonance condition when $w_1 \doteq w_2$

$$s_1 - s_2 + s_3 - s_4 - 1 = 0 \qquad (3)$$

Since the dyamical system in (2) involves cubic monomials we will consider cubic jets satisfying

$$s_1 + s_2 + s_3 + s_4 = 3 \qquad (4)$$

There are twelve Noether symmetries which enjoy these properties:

$$\overline{\mathbf{X}}_0^1 = iz_1 z_3 z_4 \frac{\partial}{\partial z_1} - iz_2 z_3 z_4 \frac{\partial}{\partial z_2}$$
$$\overline{\mathbf{X}}_0^2 = iz_1 z_2 z_3 \frac{\partial}{\partial z_3} - iz_1 z_2 z_4 \frac{\partial}{\partial z_4}$$
$$\overline{\mathbf{X}}_0^3 = iz_1^2 z_2 \frac{\partial}{\partial z_1} - iz_1 z_2^2 \frac{\partial}{\partial z_2}$$
$$\overline{\mathbf{X}}_0^4 = iz_3^2 z_4 \frac{\partial}{\partial z_3} - iz_3 z_4^2 \frac{\partial}{\partial z_4}$$

$$\overline{\mathbf{X}}_0^6 = z_2 z_3^2 \frac{\partial}{\partial z_1} + z_1 z_4^2 \frac{\partial}{\partial z_2} - z_1^2 z_4 \frac{\partial}{\partial z_3} - z_2^2 z_3 \frac{\partial}{\partial z_4}$$

$$\overline{\mathbf{X}}_0^7 = [z_1 z_2 z_3 - z_1^2 z_4]\frac{\partial}{\partial z_1} + [z_1 z_2 z_4 - z_2^2 z_3]\frac{\partial}{\partial z_2} + [-z_1 z_3 z_4 + z_2 z_3^2]\frac{\partial}{\partial z_3} + [-z_2 z_3 z_4 + z_1 z_4^2]\frac{\partial}{\partial z_4}$$
$$\overline{\mathbf{X}}_0^8 = i[z_1 z_2 z_3 + z_1^2 z_4]\frac{\partial}{\partial z_1} - i[z_1 z_2 z_4 + z_2^2 z_3]\frac{\partial}{\partial z_2} + i[z_1 z_3 z_4 + z_2 z_3^2]\frac{\partial}{\partial z_3} - i[z_2 z_3 z_4 + z_1 z_4^2]\frac{\partial}{\partial z_4}$$
$$\overline{\mathbf{X}}_0^9 = -z_1^2 z_4 \frac{\partial}{\partial z_1} - z_2^2 z_3 \frac{\partial}{\partial z_2} + [-2z_1 z_3 z_4 + z_1^2 z_2 + 2z_2 z_3^2]\frac{\partial}{\partial z_3} + [-2z_2 z_3 z_4 + z_1 z_2^2 + 2z_1 z_4^2]\frac{\partial}{\partial z_4}$$
$$\overline{\mathbf{X}}_0^5 = iz_2 z_3^2 \frac{\partial}{\partial z_1} - iz_1 z_4^2 \frac{\partial}{\partial z_2} + iz_1^2 z_4 \frac{\partial}{\partial z_3} - iz_2^2 z_3 \frac{\partial}{\partial z_4}$$
$$\overline{\mathbf{X}}_0^{10} = iz_1^2 z_4 \frac{\partial}{\partial z_1} - iz_2^2 z_3 \frac{\partial}{\partial z_2} + i[2z_1 z_3 z_4 - z_1^2 z_2 + 2z_2 z_3^2]\frac{\partial}{\partial z_3} + i[-2z_2 z_3 z_4 + z_1 z_2^2 - 2z_1 z_4^2]\frac{\partial}{\partial z_4}$$

$$\overline{\mathbf{X}}_0^{11} = z_3^2 z_4 \frac{\partial}{\partial z_1} + z_3 z_4^2 \frac{\partial}{\partial z_2} + [-2z_1 z_3 z_4 + z_2 z_3^2]\frac{\partial}{\partial z_3} + [-2z_2 z_3 z_4 + z_1 z_4^2]\frac{\partial}{\partial z_4} \quad (5)$$
$$\overline{\mathbf{X}}_0^{12} = iz_3^2 z_4 \frac{\partial}{\partial z_1} - iz_3 z_4^2 \frac{\partial}{\partial z_2} + i[2z_1 z_3 z_4 + z_2 z_3^2]\frac{\partial}{\partial z_3} - i[2z_2 z_3 z_4 + z_1 z_4^2]\frac{\partial}{\partial z_4}$$

We now consider their linear combinations of the form

$$\overline{\mathbf{X}_0} = q_1 \overline{\mathbf{X}}_0^1 + q_2 \overline{\mathbf{X}}_0^2 + q_3 \overline{\mathbf{X}}_0^3 + q_4 \overline{\mathbf{X}}_0^4 + q_5 \overline{\mathbf{X}}_0^5 + q_6 \overline{\mathbf{X}}_0^6 + q_7 \overline{\mathbf{X}}_0^7 + q_8 \overline{\mathbf{X}}_0^8 + q_9 \overline{\mathbf{X}}_0^9 + q_{10} \overline{\mathbf{X}}_0^{10} + q_{11} \overline{\mathbf{X}}_0^{11} + q_{12} \overline{\mathbf{X}}_0^{12}$$

We next determine those symmetries which will survive under small perturbations of the nonlinear terms. This can be achieved by calculating (see [2,3,4] for details)

$$\overset{i}{U_1} = g_1^j \overline{\eta}_{0,j}^{i} - \overline{\eta}_0^{j} g_{1,j}^{i} \ .$$

We notice that following resonate monomials appear at the first-order of approximation.

$$
\begin{aligned}
r_{31}(1) = {} & -6iq_7 z_1^3 z_2^2 + 6iq_9 z_1^3 z_2^2 - 3q_1 z_1^2 z_2^2 z_3 - 6q_2 z_1^2 z_2^2 z_3 + 9q_3 z_1^2 z_2^2 z_3 + 9q_5 z_1^2 z_2^2 z_3 - 9iq_6 z_1^2 z_2^2 z_3 + \\
& 6iq_7 z_1 z_2^2 z_3^2 - 6q_8 z_1 z_2^2 z_3^2 + 3iq_9 z_1 z_2^2 z_3^2 - 3q_{10} z_1 z_2^2 z_3^2 + 9iq_{11} z_1 z_2^2 z_3^2 - 9q_{12} z_1 z_2^2 z_3^2 - \\
& 18q_5 z_2^2 z_3^3 + 18iq_6 z_2^2 z_3^3 + 3q_1 z_1^3 z_2 z_4 + 3q_2 z_1^3 z_2 z_4 - 6q_3 z_1^3 z_2 z_4 - 6q_5 z_1^3 z_2 z_4 - 6iq_6 z_1^3 z_2 z_4 + \\
& 24iq_7 z_1^2 z_2 z_3 z_4 - 12iq_9 z_1^2 z_2 z_3 z_4 - 24iq_{11} z_1^2 z_2 z_3 z_4 + 9q_1 z_1 z_2 z_3^2 z_4 + 3q_2 z_1 z_2 z_3^2 z_4 - \\
& 6q_3 z_1 z_2 z_3^2 z_4 - 6q_4 z_1 z_2 z_3^2 z_4 - 24q_5 z_1 z_2 z_3^2 z_4 + 24iq_6 z_1 z_2 z_3^2 z_4 - 6iq_9 z_2 z_3^3 z_4 + \\
& 6q_{10} z_2 z_3^3 z_4 + 6iq_{11} z_2 z_3^3 z_4 - 6q_{12} z_2 z_3^3 z_4 + 6iq_7 z_1^3 z_4^2 + 6q_8 z_1^3 z_4^2 + 9iq_9 z_1^3 z_4^2 + 9q_{10} z_1^3 z_4^2 + \\
& 3iq_{11} z_1^3 z_4^2 + 3q_{12} z_1^3 z_4^2 - 6q_1 z_1^2 z_3 z_4^2 + 3q_3 z_1^2 z_3 z_4^2 + 3q_4 z_1^2 z_3 z_4^2 + 12q_5 z_1^2 z_3 z_4^2 + \\
& 12iq_6 z_1^2 z_3 z_4^2 - 6iq_7 z_1 z_3^2 z_4^2 + 30iq_{11} z_1 z_3^2 z_4^2 - 3q_1 z_3^3 z_4^2 + 3q_4 z_3^3 z_4^2 + 3q_5 z_3^3 z_4^2 - 3iq_6 z_3^3 z_4^2,
\end{aligned}
$$

$$
\begin{aligned}
r_{31}(2) = {} & 6iq_7 z_1^2 z_2^3 - 6iq_9 z_1^2 z_2^3 + 3q_1 z_1 z_2^3 z_3 + 3q_2 z_1 z_2^3 z_3 - 6q_3 z_1 z_2^3 z_3 - 6q_5 z_1 z_2^3 z_3 + 6iq_6 z_1 z_2^3 z_3 - \\
& 6iq_7 z_2^3 z_3^2 + 6q_8 z_2^3 z_3^2 - 9iq_9 z_2^3 z_3^2 + 9q_{10} z_2^3 z_3^2 - 3iq_{11} z_2^3 z_3^2 + 3q_{12} z_2^3 z_3^2 - 3q_1 z_1^2 z_2^2 z_4 - \\
& 6q_2 z_1^2 z_2^2 z_4 + 9q_3 z_1^2 z_2^2 z_4 + 9q_5 z_1^2 z_2^2 z_4 + 9iq_6 z_1^2 z_2^2 z_4 - 24iq_7 z_1 z_2^2 z_3 z_4 + 12iq_9 z_1 z_2^2 z_3 z_4 + \\
& 24iq_{11} z_1 z_2^2 z_3 z_4 - 6q_1 z_2^2 z_3^2 z_4 + 3q_3 z_2^2 z_3^2 z_4 + 3q_4 z_2^2 z_3^2 z_4 + 12q_5 z_2^2 z_3^2 z_4 - 12iq_6 z_2^2 z_3^2 z_4 - \\
& 6iq_7 z_1^2 z_2 z_4^2 - 6q_8 z_1^2 z_2 z_4^2 - 3iq_9 z_1^2 z_2 z_4^2 - 3q_{10} z_1^2 z_2 z_4^2 - 9iq_{11} z_1^2 z_2 z_4^2 - 9q_{12} z_1^2 z_2 z_4^2 + \\
& 9q_1 z_1 z_2 z_3 z_4^2 + 3q_2 z_1 z_2 z_3 z_4^2 - 6q_3 z_1 z_2 z_3 z_4^2 - 6q_4 z_1 z_2 z_3 z_4^2 - 24q_5 z_1 z_2 z_3 z_4^2 - \\
& 24iq_6 z_1 z_2 z_3 z_4^2 + 6iq_7 z_2 z_3^2 z_4^2 - 30iq_{11} z_2 z_3^2 z_4^2 - 18q_5 z_1^2 z_4^3 - 18iq_6 z_1^2 z_4^3 + 6iq_9 z_1 z_3 z_4^3 + \\
& 6q_{10} z_1 z_3 z_4^3 - 6iq_{11} z_1 z_3 z_4^3 - 6q_{12} z_1 z_3 z_4^3 - 3q_1 z_3^2 z_4^3 + 3q_4 z_3^2 z_4^3 + 3q_5 z_3^2 z_4^3 + 3iq_6 z_3^2 z_4^3,
\end{aligned}
$$

$$
\begin{aligned}
r_{31}(3) = {} & 3q_2 z_1^3 z_2^2 - 3q_3 z_1^3 z_2^2 - 3q_5 z_1^3 z_2^2 - 3iq_6 z_1^3 z_2^2 - 6iq_7 z_1^2 z_2^2 z_3 - 42iq_9 z_1^2 z_2^2 z_3 - 12iq_{11} z_1^2 z_2^2 z_3 + \\
& 6q_2 z_1 z_2^2 z_3^2 - 3q_3 z_1 z_2^2 z_3^2 - 3q_4 z_1 z_2^2 z_3^2 - 12q_5 z_1 z_2^2 z_3^2 + 12iq_6 z_1 z_2^2 z_3^2 + 6iq_7 z_2^2 z_3^3 - 6q_8 z_2^2 z_3^3 + \\
& 9iq_9 z_2^2 z_3^3 - 9q_{10} z_2^2 z_3^3 + 3iq_{11} z_2^2 z_3^3 - 3q_{12} z_2^2 z_3^3 - 6iq_9 z_1^3 z_2 z_4 - 6q_{10} z_1^3 z_2 z_4 + 6iq_{11} z_1^3 z_2 z_4 + \\
& 6q_{12} z_1^3 z_2 z_4 - 3q_1 z_1^2 z_2 z_3 z_4 - 9q_2 z_1^2 z_2 z_3 z_4 + 6q_3 z_1^2 z_2 z_3 z_4 + 6q_4 z_1^2 z_2 z_3 z_4 + 24q_5 z_1^2 z_2 z_3 z_4 + \\
& 24iq_6 z_1^2 z_2 z_3 z_4 + 24iq_7 z_1 z_2 z_3^2 z_4 + 72iq_9 z_1 z_2 z_3^2 z_4 + 60iq_{11} z_1 z_2 z_3^2 z_4 - 3q_1 z_2 z_3^3 z_4 - \\
& 3q_2 z_2 z_3^3 z_4 + 6q_4 z_2 z_3^3 z_4 + 6q_5 z_2 z_3^3 z_4 - 6iq_6 z_2 z_3^3 z_4 + 18q_5 z_1^3 z_4^2 + 18iq_6 z_1^3 z_4^2 + 6iq_7 z_1^2 z_3 z_4^2 + \\
& 6q_8 z_1^2 z_3 z_4^2 + 3iq_9 z_1^2 z_3 z_4^2 + 3q_{10} z_1^2 z_3 z_4^2 + 9iq_{11} z_1^2 z_3 z_4^2 + 9q_{12} z_1^2 z_3 z_4^2 + 6q_1 z_1 z_3^2 z_4^2 + \\
& 3q_2 z_1 z_3^2 z_4^2 - 9q_4 z_1 z_3^2 z_4^2 - 9q_5 z_1 z_3^2 z_4^2 - 9iq_6 z_1 z_3^2 z_4^2 - 6iq_7 z_3^3 z_4^2 - 12iq_9 z_3^3 z_4^2 - 18iq_{11} z_3^3 z_4^2,
\end{aligned}
$$

$$
\begin{aligned}
r_{31}(4) = {} & 3q_2 z_1^2 z_2^3 - 3q_3 z_1^2 z_2^3 - 3q_5 z_1^2 z_2^3 + 3iq_6 z_1^2 z_2^3 + 6iq_9 z_1 z_2^3 z_3 - 6q_{10} z_1 z_2^3 z_3 - 6iq_{11} z_1 z_2^3 z_3 + \\
& 6q_{12} z_1 z_2^3 z_3 + 18q_5 z_2^3 z_3^2 - 18iq_6 z_2^3 z_3^2 + 6iq_7 z_1^2 z_2^2 z_4 + 42iq_9 z_1^2 z_2^2 z_4 + 12iq_{11} z_1^2 z_2^2 z_4 - \\
& 3q_1 z_1 z_2^2 z_3 z_4 - 9q_2 z_1 z_2^2 z_3 z_4 + 6q_3 z_1 z_2^2 z_3 z_4 + 6q_4 z_1 z_2^2 z_3 z_4 + 24q_5 z_1 z_2^2 z_3 z_4 - 24iq_6 z_1 z_2^2 z_3 z_4 - \\
& 6iq_7 z_2^2 z_3^2 z_4 + 6q_8 z_2^2 z_3^2 z_4 - 3iq_9 z_2^2 z_3^2 z_4 + 3q_{10} z_2^2 z_3^2 z_4 - 9iq_{11} z_2^2 z_3^2 z_4 + 9q_{12} z_2^2 z_3^2 z_4 + \\
& 6q_2 z_1^2 z_2 z_4^2 - 3q_3 z_1^2 z_2 z_4^2 - 3q_4 z_1^2 z_2 z_4^2 - 12q_5 z_1^2 z_2 z_4^2 - 12iq_6 z_1^2 z_2 z_4^2 - 24iq_7 z_1 z_2 z_3 z_4^2 - \\
& 72iq_9 z_1 z_2 z_3 z_4^2 - 60iq_{11} z_1 z_2 z_3 z_4^2 + 6q_1 z_2 z_3^2 z_4^2 + 3q_2 z_2 z_3^2 z_4^2 - 9q_4 z_2 z_3^2 z_4^2 - 9q_5 z_2 z_3^2 z_4^2 + \\
& 9iq_6 z_2 z_3^2 z_4^2 - 6iq_7 z_1^2 z_4^3 - 6q_8 z_1^2 z_4^3 - 9iq_9 z_1^2 z_4^3 - 9q_{10} z_1^2 z_4^3 - 3iq_{11} z_1^2 z_4^3 - 3q_{12} z_1^2 z_4^3 - 3q_1 z_1 z_3 z_4^3 - \\
& 3q_2 z_1 z_3 z_4^3 + 6q_4 z_1 z_3 z_4^3 + 6q_5 z_1 z_3 z_4^3 + 6iq_6 z_1 z_3 z_4^3 + 6iq_7 z_3^2 z_4^3 + 12iq_9 z_3^2 z_4^3 + 18iq_{11} z_3^2 z_4^3,
\end{aligned}
$$

These resonant monomials can be killed by choosing

$$
\begin{aligned}
& q_2 = q_1 \quad , q_3 = q_1 \quad , q_4 = q_1 \, , q_{10} = -q_8/2 \, , q_{12} = -q_8/2 \\
& q_5 , q_6 , q_7 , q_9 , q_{11} = 0 \quad .
\end{aligned}
\tag{6}
$$

This tantamounts to saying that symmetries corresponding to those parameter which vanish in (6) are broken due to perturbations. When we consider second-order symmetries we encounter the following resonant monomials:

$$\begin{aligned} r_{32}(1) = {} & 168\, q_8 z_1^3 z_2^3 z_3 - 168\, q_8 z_1 z_2^3 z_3^3 - 126\, q_8 z_1^4 z_2^2 z_4 - 252\, q_8 z_1^2 z_2^2 z_3^2 z_4 - 126\, q_8 z_2^2 z_3^4 z_4 + \\ & 168\, q_8 z_1^3 z_2 z_3 z_4^2 - 168\, q_8 z_1 z_2 z_3^3 z_4^2 + 42\, q_8 z_1^4 z_4^3 + 84\, q_8 z_1^2 z_3^2 z_4^3 + 42\, q_8 z_3^4 z_4^3, \end{aligned}$$

$$\begin{aligned} r_{32}(2) = {} & -126\, q_8 z_1^2 z_2^4 z_3 + 42\, q_8 z_2^4 z_3^3 + 168\, q_8 z_1^3 z_2^3 z_4 + 168\, q_8 z_1 z_2^3 z_3^2 z_4 - 252\, q_8 z_1^2 z_2^2 z_3 z_4^2 + \\ & 84\, q_8 z_2^2 z_3^3 z_4^2 - 168\, q_8 z_1^3 z_2 z_4^3 - 168\, q_8 z_1 z_2 z_3^2 z_4^3 - 126\, q_8 z_1^2 z_3 z_4^4 + 42\, q_8 z_3^3 z_4^4, \end{aligned}$$

$$\begin{aligned} r_{32}(3) = {} & -42\, q_8 z_1^4 z_2^3 - 84\, q_8 z_1^2 z_2^3 z_3^2 - 42\, q_8 z_2^3 z_3^4 + 168\, q_8 z_1^3 z_2^2 z_3 z_4 - 168\, q_8 z_1 z_2^2 z_3^3 z_4 + \\ & 126\, q_8 z_1^4 z_2 z_4^2 + 252\, q_8 z_1^2 z_2 z_3^2 z_4^2 + 126\, q_8 z_2 z_3^4 z_4^2 + 168\, q_8 z_1^3 z_3 z_4^3 - 168\, q_8 z_1 z_3^3 z_4^3, \end{aligned}$$

$$\begin{aligned} r_{32}(4) = {} & -42\, q_8 z_1^3 z_2^4 + 126\, q_8 z_1 z_2^4 z_3^2 + 168\, q_8 z_1^2 z_2^3 z_3 z_4 + 168\, q_8 z_2^3 z_3^3 z_4 - 84\, q_8 z_1^3 z_2^2 z_4^2 + \\ & 252\, q_8 z_1 z_2^2 z_3^2 z_4^2 - 168\, q_8 z_1^2 z_2 z_3 z_4^3 - 168\, q_8 z_2 z_3^3 z_4^3 - 42\, q_8 z_1^3 z_4^4 + 126\, q_8 z_1 z_3^2 z_4^4, \end{aligned}$$

It can be easily seen that these monomials disappear when we set $q_8 = 0$. This allows to obtain following second-order approximate symmetry of the form

$$\mathbf{X} = \eta_i \frac{\partial}{\partial x_i}, \qquad \eta_i = \eta_i^0 + \varepsilon \eta_i^1 + \varepsilon^2 \eta_i^2 \tag{7}$$

$$\begin{aligned} \eta_1 = {} & \frac{x_3 (x_3^2 + x_4^2 + x_1^2 \omega_1^2 + x_2^2 \omega_1^2)}{4 \omega_1} + \frac{1}{32 \omega_1^5} \\ & (\epsilon(-15 x_3^4 x_4 + 3 x_4^5 - 12 x_1 x_2 x_3^3 \omega_1^2 - 18 x_1^2 x_3^2 x_4 \omega_1^2 + 6 x_2^2 x_4^3 \omega_1^2 + 4 x_1^3 x_2 x_3 \omega_1^4 - \\ & \quad 16 x_1 x_2^3 x_3 \omega_1^4 - 3 x_1^4 x_4 \omega_1^4 + 3 x_2^4 x_4 \omega_1^4)) + \frac{1}{256 \omega_1^9} \\ & (\epsilon^2 (18 x_3^7 + 57 x_3^5 x_4^2 + 50 x_3^3 x_4^4 + 19 x_3 x_4^6 + 54 x_1^2 x_3^5 \omega_1^2 - 123 x_2^2 x_3^5 \omega_1^2 + 300 x_1 x_2 x_3^4 x_4 \omega_1^2 - \\ & \quad 6 x_1^2 x_3^3 x_4^2 \omega_1^2 - 132 x_2^2 x_3^3 x_4^2 \omega_1^2 + 348 x_1 x_2 x_3^2 x_4^3 \omega_1^2 - 66 x_1^2 x_3 x_4^4 \omega_1^2 - 3 x_2^2 x_3 x_4^4 \omega_1^2 + 60 x_1 x_2 \\ & \quad x_4^5 \omega_1^2 + 30 x_1^4 x_3^3 \omega_1^4 + 18 x_1^2 x_2^2 x_3^3 \omega_1^4 + 18 x_2^4 x_3^3 \omega_1^4 - 72 x_1^3 x_2 x_3^2 x_4 \omega_1^4 + 108 x_1 x_2^3 x_3^2 x_4 \omega_1^4 + \\ & \quad 9 x_1^4 x_3 x_4^2 \omega_1^4 + 108 x_1^2 x_2^2 x_3 x_4^2 \omega_1^4 + 9 x_2^4 x_3 x_4^2 \omega_1^4 + 36 x_1^3 x_2 x_4^3 \omega_1^4 - 24 x_1 x_2^3 x_4^3 \omega_1^4 - 6 x_1^6 x_3 \omega_1^6 - \\ & \quad 75 x_1^4 x_2^2 x_3 \omega_1^6 + 150 x_1^2 x_2^4 x_3 \omega_1^6 + 31 x_2^6 x_3 \omega_1^6 - 84 x_1^5 x_2 x_4 \omega_1^6 + 20 x_1^3 x_2^3 x_4 \omega_1^6 - 84 x_1 x_2^5 x_4 \omega_1^6)), \end{aligned}$$

$$\begin{aligned} \eta_2 = {} & \frac{x_4 (x_3^2 + x_4^2 + x_1^2 \omega_1^2 + x_2^2 \omega_1^2)}{4 \omega_1} + 32 \epsilon \omega_1^5 (-3 x_3^5 + 15 x_3 x_4^4 - 6 x_1^2 x_3^3 \omega_1^2 + \\ & \quad 18 x_2^2 x_3 x_4^2 \omega_1^2 + 12 x_1 x_2 x_4^3 \omega_1^2 - 3 x_1^4 x_3 \omega_1^4 + 3 x_2^4 x_3 \omega_1^4 + 16 x_1^3 x_2 x_4 \omega_1^4 - 4 x_1 x_2^3 x_4 \omega_1^4) + \\ & 256 \epsilon^2 \omega_1^9 (19 x_3^6 x_4 + 50 x_3^4 x_4^3 + 57 x_3^2 x_4^5 + 18 x_4^7 + 60 x_1 x_2 x_3^5 \omega_1^2 - 3 x_1^2 x_3^4 x_4 \omega_1^2 - 66 x_2^2 x_3^4 x_4 \omega_1^2 + \\ & \quad 348 x_1 x_2 x_3^3 x_4^2 \omega_1^2 - 132 x_1^2 x_3^2 x_4^3 \omega_1^2 - 6 x_2^2 x_3^2 x_4^3 \omega_1^2 + 300 x_1 x_2 x_3 x_4^4 \omega_1^2 - 123 x_1^2 x_4^5 \omega_1^2 + \\ & \quad 54 x_2^2 x_4^5 \omega_1^2 - 24 x_1^3 x_2 x_3^3 \omega_1^4 + 36 x_1 x_2^3 x_3^3 \omega_1^4 + 9 x_1^4 x_3^2 x_4 \omega_1^4 + 108 x_1^2 x_2^2 x_3^2 x_4 \omega_1^4 + 9 x_2^4 x_3^2 x_4 \omega_1^4 + \\ & \quad 108 x_1^3 x_2 x_3 x_4^2 \omega_1^4 - 72 x_1 x_2^3 x_3 x_4^2 \omega_1^4 + 18 x_1^4 x_4^3 \omega_1^4 + 18 x_1^2 x_2^2 x_4^3 \omega_1^4 + 30 x_2^4 x_4^3 \omega_1^4 - 84 x_1^5 x_2 x_3 \omega_1^6 + \\ & \quad 20 x_1^3 x_2^3 x_3 \omega_1^6 - 84 x_1 x_2^5 x_3 \omega_1^6 + 31 x_1^6 x_4 \omega_1^6 + 150 x_1^4 x_2^2 x_4 \omega_1^6 - 75 x_1^2 x_2^4 x_4 \omega_1^6 - 6 x_2^6 x_4 \omega_1^6), \end{aligned}$$

$$\eta_3 =$$
$$-\frac{1}{4} x_1 \omega_1 (x_3^2 + x_4^2 + x_1^2 \omega_1^2 + x_2^2 \omega_1^2) + \frac{1}{32\omega_1^3}(\epsilon(3 x_2 x_3^4 + 12 x_1 x_3^3 x_4 - 3 x_2 x_4^4 - 6 x_1^2 x_2 x_3^2 \omega_1^2 + 8 x_2^3 x_3^2 \omega_1^2 +$$
$$12 x_1^3 x_3 x_4 \omega_1^2 - 24 x_1^2 x_2 x_4^2 \omega_1^2 + 2 x_2^3 x_4^2 \omega_1^2 - 25 x_1^4 x_2 \omega_1^4 + 5 x_2^5 \omega_1^4)) + \frac{1}{256\omega_1^7}$$
$$(\epsilon^2(-18 x_1 x_3^6 - 60 x_2 x_3^5 x_4 + 3 x_1 x_3^4 x_4^2 - 116 x_2 x_3^3 x_4^3 + 66 x_1 x_3^2 x_4^4 - 60 x_2 x_3 x_4^5 + 41 x_1 x_4^6 - 30 x_1^3 x_3^4 \omega_1^2 -$$
$$9 x_1 x_2^2 x_3^4 \omega_1^2 + 72 x_1^2 x_2 x_3^3 x_4 \omega_1^2 - 36 x_2^3 x_3^3 x_4 \omega_1^2 - 18 x_1^3 x_3^2 x_4^2 \omega_1^2 - 108 x_1 x_2^2 x_3^2 x_4^2 \omega_1^2 -$$
$$108 x_1^2 x_2 x_3 x_4^3 \omega_1^2 + 24 x_2^3 x_3 x_4^3 \omega_1^2 - 18 x_1^3 x_4^4 \omega_1^2 - 9 x_1 x_2^2 x_4^4 \omega_1^2 + 18 x_1^5 x_3^2 \omega_1^4 + 150 x_1^3 x_2^2 x_3^2 \omega_1^4 -$$
$$150 x_1 x_2^4 x_3^2 \omega_1^4 + 420 x_1^4 x_2 x_3 x_4 \omega_1^4 - 60 x_1^2 x_2^3 x_3 x_4 \omega_1^4 + 84 x_2^5 x_3 x_4 \omega_1^4 - 93 x_1^5 x_4^2 \omega_1^4 -$$
$$300 x_1^3 x_2^2 x_4^2 \omega_1^4 + 75 x_1 x_2^4 x_4^2 \omega_1^4 + 30 x_1^7 \omega_1^6 - 9 x_1^5 x_2^2 \omega_1^6 + 222 x_1^3 x_2^4 \omega_1^6 - 3 x_1 x_2^6 \omega_1^6)),$$
$$\eta_4 = -\frac{1}{4} x_2 \omega_1 (x_3^2 + x_4^2 + x_1^2 \omega_1^2 + x_2^2 \omega_1^2) +$$
$$\frac{1}{32\omega_1^3}(\epsilon(3 x_1 x_3^4 - 12 x_2 x_3 x_4^3 - 3 x_1 x_4^4 - 2 x_1^3 x_3^2 \omega_1^2 + 24 x_1 x_2^2 x_3^2 \omega_1^2 -$$
$$12 x_2^3 x_3 x_4 \omega_1^2 - 8 x_1^3 x_4^2 \omega_1^2 + 6 x_1 x_2^2 x_4^2 \omega_1^2 - 5 x_1^5 \omega_1^4 + 25 x_1 x_2^4 \omega_1^4)) + \frac{1}{256\omega_1^7}$$
$$(\epsilon^2(41 x_2 x_3^6 - 60 x_1 x_3^5 x_4 + 66 x_2 x_3^4 x_4^2 - 116 x_1 x_3^3 x_4^3 + 3 x_2 x_3^2 x_4^4 - 60 x_1 x_3 x_4^5 - 18 x_2 x_4^6 - 9 x_1^2 x_2 x_3^4 \omega_1^2 -$$
$$18 x_2^3 x_3^4 \omega_1^2 + 24 x_1^3 x_3^3 x_4 \omega_1^2 - 108 x_1 x_2^2 x_3^3 x_4 \omega_1^2 - 108 x_1^2 x_2 x_3^2 x_4^2 \omega_1^2 - 18 x_2^3 x_3^2 x_4^2 \omega_1^2 -$$
$$36 x_1^3 x_3 x_4^3 \omega_1^2 + 72 x_1 x_2^2 x_3 x_4^3 \omega_1^2 - 9 x_1^2 x_2 x_4^4 \omega_1^2 - 30 x_2^3 x_4^4 \omega_1^2 + 75 x_1^4 x_2 x_3^2 \omega_1^4 - 300 x_1^2 x_2^3 x_3^2 \omega_1^4 -$$
$$93 x_2^5 x_3^2 \omega_1^4 + 84 x_1^5 x_3 x_4 \omega_1^4 - 60 x_1^3 x_2^2 x_3 x_4 \omega_1^4 + 420 x_1 x_2^4 x_3 x_4 \omega_1^4 - 150 x_1^4 x_2 x_4^2 \omega_1^4 +$$
$$150 x_1^2 x_2^3 x_4^2 \omega_1^4 + 18 x_2^5 x_4^2 \omega_1^4 - 3 x_1^6 x_2 \omega_1^6 + 222 x_1^4 x_2^3 \omega_1^6 - 9 x_1^2 x_2^5 \omega_1^6 + 30 x_2^7 \omega_1^6)).$$

3. APPROXIMATE FIRST INTEGRALS

Following [2,3] we can proceed to obtain the approximate first integral corresponding to approximate Noether symmetry given in [7]. We now consider

$$X \rfloor \Omega = dI + O(\varepsilon^3) \tag{8}$$

where,

$$\Omega = dx_1 , dx_3 + dx_2 , dx_4 .$$

Here, ?? is the exterior and I is the first integral. This leads to following equations:

$$\frac{\partial I_0}{\partial x_1} = -\eta_3^0 \ , \ \frac{\partial I_0}{\partial x_2} = -\eta_4^0 \ , \frac{\partial I_0}{\partial x_3} = \eta_1^0 \ , \ \frac{\partial I_0}{\partial x_4} = \eta_2^0 \ . \tag{9}$$

By integrating (9) we find the following second-order approximate first integral.

$$
\begin{aligned}
I = {} & \frac{x_3^4+2x_3^2x_4^2+x_4^4+2x_1^2x_3^2\omega_1^2+2x_2^2x_3^2\omega_1^2+2x_1^2x_4^2\omega_1^2+2x_2^2x_4^2\omega_1^2+x_1^4\omega_1^4+2x_1^2x_2^2\omega_1^4-x_2^4\omega_1^4}{16\omega_1} + \\
& \frac{1}{256\,\omega_1^5}(\epsilon(24x_3x_4^5-24x_1x_2x_3^4\omega_1^2-48x_1^2x_3^3x_4\omega_1^2+48x_2^2x_3x_4^3\omega_1^2+ \\
& \quad 24x_1x_2x_4^4\omega_1^2+16x_1^3x_2x_3^2\omega_1^4-64x_1x_2^3x_3^2\omega_1^4-24x_1^4x_3x_4\omega_1^4+24x_2^4x_3x_4\omega_1^4+ \\
& \quad 64x_1^3x_2x_4^2\omega_1^4-16x_1x_2^3x_4^2\omega_1^4+40x_1^5x_2\omega_1^6-40x_1x_2^5\omega_1^6))+\frac{1}{1024\,\omega_1^9} \\
& (\epsilon^2(9x_3^8+38x_3^6x_4^2+50x_3^4x_4^4+38x_3^2x_4^6+9x_4^8+36x_1^2x_3^6\omega_1^2-82x_2^2x_3^6\omega_1^2+240x_1x_2x_3^5x_4\omega_1^2- \\
& \quad 6x_1^2x_3^4x_4^2\omega_1^2-132x_2^2x_3^4x_4^2\omega_1^2+464x_1x_2x_3^3x_4^3\omega_1^2-132x_1^2x_3^2x_4^4\omega_1^2-6x_2^2x_3^2x_4^4\omega_1^2+ \\
& \quad 240x_1x_2x_3x_4^5\omega_1^2-82x_1^2x_4^6\omega_1^2+36x_2^2x_4^6\omega_1^2+30x_1^4x_3^4\omega_1^4+18x_1^2x_2^2x_3^4\omega_1^4+18x_2^4x_3^4\omega_1^4- \\
& \quad 96x_1^3x_2x_3^3x_4\omega_1^4+144x_1x_2^3x_3^3x_4\omega_1^4+18x_1^4x_3^2x_4^2\omega_1^4+216x_1^2x_2^2x_3^2x_4^2\omega_1^4+18x_2^4x_3^2x_4^2\omega_1^4+ \\
& \quad 144x_1^3x_2x_3x_4^3\omega_1^4-96x_1x_2^3x_3x_4^3\omega_1^4+18x_1^4x_4^4\omega_1^4+18x_1^2x_2^2x_4^4\omega_1^4+30x_2^4x_4^4\omega_1^4- \\
& \quad 12x_1^6x_3^2\omega_1^6-150x_1^4x_2^2x_3^2\omega_1^6+300x_1^2x_2^4x_3^2\omega_1^6+62x_2^6x_3^2\omega_1^6-336x_1^5x_2x_3x_4\omega_1^6+ \\
& \quad 80x_1^3x_2^3x_3x_4\omega_1^6-336x_1x_2^5x_3x_4\omega_1^6+62x_1^6x_4^2\omega_1^6+300x_1^4x_2^2x_4^2\omega_1^6-150x_1^2x_2^4x_4^2\omega_1^6- \\
& \quad 12x_2^6x_4^2\omega_1^6-15x_1^8\omega_1^8+6x_1^6x_2^2\omega_1^8-222x_1^4x_2^4\omega_1^8+6x_1^2x_2^6\omega_1^8-15x_2^8\omega_1^8)),
\end{aligned}
$$

KAM curves can be obtained by setting $x_2=0$ and

$$x_4=\sqrt{2h-w_1^2x_1^2-x_3^2}$$

in approximate first integral I and drawing contour lines (see Figure 1a). As it can be easily seen from Figure 1b, numerically obtained KAM curves have the same structure. Both figures display four elliptic and two hyperbolic fixed points. Countur lines around the hyperbolic fixed points are, indeed, closed orbits, but this does not appear in Figure 1a due to low quality of the program package.

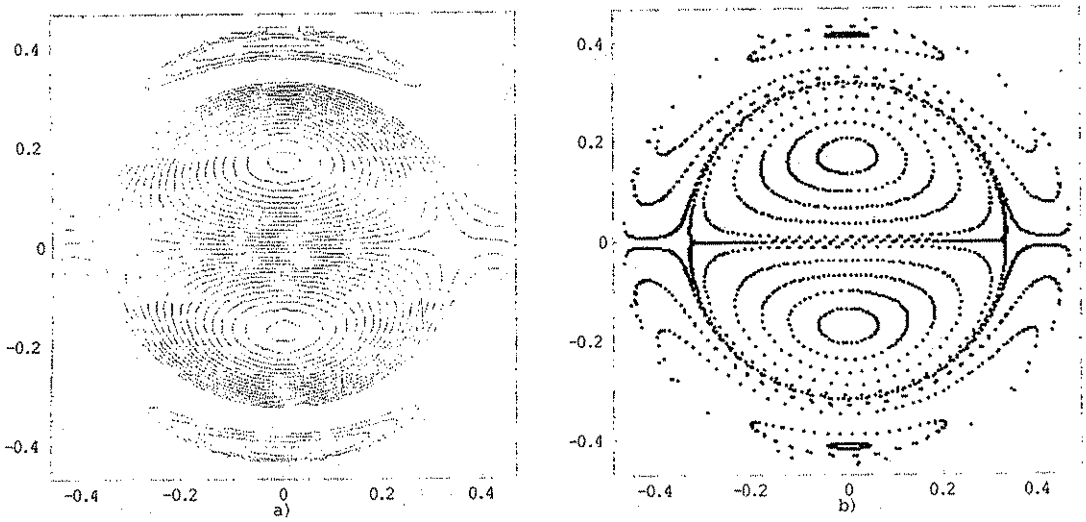

Figure 1. Analytical versus numerical results obtained for h=0.1, $\omega_1^2 = 0.9, \varepsilon = 0.1$.

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