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OPTIMAL SYSTEMS AND INVARIANT SOLUTIONS FOR A CLASS OF SOIL WATER EQUATIONS

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Abstract- We construct an optimal system of one-dimensional subalgebras for a class of soil water equations and then use it to obtain an optimal system of two-dimensional subalgebras. We also present a few group-invariant solutions of rank one corresponding to an optimal system of two-dimensional subalgebras.

Keywords - Optimal system, group-invariant solution, subalgebra.

1. INTRODUCTION

A mathematical model was developed to simulate soil water infiltration, redistribution, and extraction in a bedded soil profile overlaying a shallow water table and irrigated by a line source drip irrigation system. The governing partial differential equation can be written as

$$C(\psi)\psi_t = \left(K(\psi)\psi_x\right)_x + (K(\psi)(\psi_z - 1))_z - S(\psi), \tag{1.1}$$

where ψ is soil moisture pressure head, $C(\psi)$ is specific water capacity, $K(\psi)$ is unsaturated hydraulic conductivity, $S(\psi)$ is a sink or source term, x is the horizontal and z is the vertical axis which is considered positive downward (see [1] and [2]). This equation has been studied by many researchers (see, e.g., [2] and references therein) and analytic and numerical solutions have been obtained for particular functions C, K and S. Group classification of these equations with respect to admitted point transformation group was done by Baikov *et al* [3]. Conservation laws for some classes of equation (1.1) were obtained and their association with the generators of Lie symmetries were given in Kara and Khalique [4].

Soil water equation (1.1) is linked with the heat conduction type equation

$$u_t = \left(k(u)u_x\right)_x + \left(k(u)u_z\right)_z + l(u)u_z + p(u), \tag{1.2}$$

where $u = \int C(\psi)d\psi$, $k(u) = \dfrac{K(\psi)}{C(\psi)}$, $l(u) = -\dfrac{K'(\psi)}{C(\psi)}$, $p(u) = -S(\psi)$.

For the proof see [3].

In their paper, Baikov *et al* [3] utilized the procedure for the construction of solutions which are invariant with respect to two-dimensional subalgebras that reduces equation (1.1) to an ordinary differential equation. Two particular equations of form (1.1) were considered and their invariant solutions were obtained.

The problem of constructing the optimal system of subalgebras plays a very important role in the group analysis of differential equations. It can be used in the classification of group-invariant solutions of differential equations (see Ovsiannikov [5], [6] and [7]). The method is also described in detail in the papers by Ibragimov *et al* [8] and Chupakhin [9]. For the definitions of optimal system of subalgebras and optimal system of group invariant solutions the reader is referred to Ovsiannikov [6], Ibragimov [10] or Olver [11]. Optimal system of one-dimensional subalgebras for the one-dimensional heat equation can be found in Olver [11]. Recently, Pooe *et al* [12] obtained two classes of optimal systems of group-invariant solutions for the Black-Scholes equation.

In this paper, following Ovsiannikov [6], we will construct optimal system of one-dimensional subalgebras for equation (1.2) for a particular type of coefficients $k(u)$, $l(u)$ and $p(u)$. Using this result we will then compute optimal system of two-dimensional subalgebras. Lastly, we present optimal system of group-invariant solutions of rank one for that particular equation.

2. CONSTRUCTION OF OPTIMAL SYSTEM OF ONE AND TWO-DIMENSIONAL SUBALGEBRAS

In this section we use the method given in Ovsiannikov [6] and first construct an optimal system of one-dimensional subalgebras for a particular case of equation (1.2) when $k(u) = u^{-1}$, $l(u) = 0$ and $p(u) = u$, namely,

$$u_t = \left(\frac{u_x}{u}\right)_x + \left(\frac{u_z}{u}\right)_z + u. \tag{2.1}$$

According to the classification result (see Ibragimov [10]), equations (1.2) admits the principal Lie algebra L_p (i.e., the Lie algebra of the Lie transformation group admitted by equation (1.2) for arbitrary functions $k(u)$, $l(u)$ and $p(u)$) of (translations) point symmetries

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial z} \tag{2.2}$$

and equation (2.1) admits, in addition to (2.2), the rotation symmetry

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

and the two symmetries

$$X_5 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u} \quad \text{and} \quad X_6 = e^t\frac{\partial}{\partial t} + e^t u\frac{\partial}{\partial u}.$$

The commutator table for these six operators is given below.

Table

$[X_\alpha, X_\beta]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	0	X_6
X_2	0	0	0	$-X_3$	X_2	0
X_3	0	0	0	X_2	X_3	0
X_4	0	X_3	$-X_2$	0	0	0
X_5	0	$-X_2$	$-X_3$	0	0	0
X_6	$-X_6$	0	0	0	0	0

Consider now the algebra L_6 with basis $X_1, X_2, \ldots, X_6$. Then the Lie algebra L_6^A is spanned by the following operators:

$$E_\mu = c_{\mu\nu}^{\lambda} e^\nu \frac{\partial}{\partial e^\lambda}, \qquad \mu = 1,2,\ldots,6$$

where $c_{\mu\nu}^{\lambda}$ are the structure constants (see e.g., Ibragimov [10]). Using the commutator table we calculate the structure constants. Hence the algebra L_6^A is spanned by

$$E_1 = e^6\frac{\partial}{\partial e^6},$$

$$E_2 = -e^4\frac{\partial}{2e^3} + e^5\frac{\partial}{2e^2},$$

$$E_3 = e^4\frac{\partial}{\partial e^2} + e^5\frac{\partial}{\partial e^3},$$

$$E_4 = e^2 \frac{\partial}{\partial e^3} - e^3 \frac{\partial}{\partial e^2},$$

$$E_5 = -e^2 \frac{\partial}{\partial e^2} - e^3 \frac{\partial}{\partial e^3},$$

$$E_6 = -e^1 \frac{\partial}{\partial e^6}.$$

By solving the Lie equations for the above operators, we obtain the corresponding transformations for each of these six operators. Taking their composition, we obtain the following six-parameter group G^A of inner automorphisms of L_6 or the adjoint group of G:

$$\begin{aligned}
\overline{e^1} &= e^1, \\
\overline{e^2} &= b_5 e^2 - b_4 e^3 + b_3 e^4 + b_2 e^5, \\
\overline{e^3} &= b_4 e^2 + b_5 e^3 - b_2 e^4 + b_3 e^5, \\
\overline{e^4} &= e^4, \\
\overline{e^5} &= e^5, \\
\overline{e^6} &= -b_6 e^1 + b_1 e^6.
\end{aligned} \tag{2.3}$$

Here $b_1, b_2, \ldots, b_6$ are arbitrary real parameters with $b_1 > 0$, $b_4^2 + b_5^2 \neq 0$ and $b_5 \neq 0$. Because we want to construct one dimensional subalgebras we now have to see how an arbitrary operator

$$X = \sum_{\mu=1}^{6} e^{\mu} X_{\mu}$$

of the algebra L_6 is transformed to its simplest form using the above transformations. After some calculations it turns out that the optimal system of one-dimensional subalgebras is

$\{X_2, X_3, X_5, X_6, X_1 \pm X_2, X_1 \pm X_3, X_2 \pm X_6, X_5 \pm X_6, X_3 \pm X_6, X_4 + \alpha X_5, X_1 + \alpha X_4 + \beta X_5,$ $X_4 + \beta X_5 \pm X_6; \alpha, \beta \in R\}$. Now using the discrete symmetries $x \to -x$ and $z \to -z$ of

equation (2.1) we can delete the operators $X_1 - X_2, X_1 - X_3, X_2 - X_6, X_3 - X_6$ from the above set. We also note that the operators X_2 and X_3 belong to the same class of one-dimensional subalgebras. Same is also true for the operators $X_1 + X_2$ and $X_1 + X_3$. Hence finally, the optimal system of one-dimensional subalgebras is

$$\{X_2, X_5, X_6, X_1 + X_2, X_2 + X_6, X_5 \pm X_6, X_4 + \alpha X_5, X_1 + \alpha X_4 + \beta X_5, X_4 + \beta X_5 \pm X_6; \alpha, \beta \in R\}.$$

We now use the above optimal system of one-dimensional subalgebras to construct optimal system of two-dimensional subalgebras. Consider an operator $Y_1 = X_1 + X_2$, say, form the optimal system of one-dimensional subalgebras and let $Y_2 = e^1 X_1 + ... + e^6 X_6$. Using the commutator table we simplify Y_2 such that Y_1 and Y_2 form a two-dimensional subalgebra. The following two subalgebras arise:

$$< X_1 + X_2; e^2 X_2 + e^3 X_3 > \text{ and } < X_1 + X_2; X_6 >.$$

We now simplify the first subalgebra. Under the general transformations (2.3), this subalgebra is transformed to

$$< X_1 + b_5 X_2 + b_4 X_3 - b_6 X_6; (b_5 e^2 - b_4 e^3) X_2 + (b_4 e^2 + b_5 e^3) X_3 >.$$

One now chooses the arbitrary parameters such that the subalgebra has the simplest form. It can be shown that two subalgebras, namely,

$$< X_1 + X_2; X_3 > \text{ and } < X_1; X_2 >.$$

are obtained. Thus by considering $X_1 + X_2$ we obtain the following three two-dimensional subalgebras:

$$< X_1 + X_2; X_3 >, \; < X_1 + X_2; X_6 >; \; < X_1; X_2 >.$$

Likewise, by taking the remaining operators of the optimal system of one-dimensional subalgebras and proceeding in the above manner, we obtain the following optimal system of two dimensional subalgebras:

$$< X_1; X_2 >, < X_2; X_6 >, < X_2; X_5 >, < X_2; X_3 >, < X_2; X_5 \pm X_6 >,$$
$$< X_2; X_1 + \alpha X_5 >, < X_2 + X_6; X_1 - X_5 >, < X_3; X_1 + X_2 >, < X_3; X_2 + X_6 >,$$
$$< X_4, X_5 >, < X_4 + \alpha X_5; X_1 + \beta X_5 >, < X_4 + \alpha X_5; X_5 \pm X_6 >, < X_4 + \alpha X_5; X_6 >,$$
$$< X_4 + \alpha X_6; X_5 \pm X_6 >, < X_4 \pm X_6; X_5 >, < X_5; X_6 >, < X_5; X_1 + \alpha X_4 >,$$
$$< X_6; X_1 + X_2 >, < X_6; X_1 + \alpha X_4 + \beta X_5 >.$$

(2.4)

We now show how one constructs the group-invariant solutions corresponding to the two-dimensional subalgebras of the optimal system (2.4). Consider, for example, the subalgebra $< X_2 + X_6; X_3 >$. We calculate a basis of invariants $I(t, x, z, u)$ by solving the equations

$$(X_2 + X_6) I = 0, \quad X_3 I = 0. \tag{2.5}$$

The second equation gives us three functionally independent solutions

$$J_1 = t \ , \ J_2 = x, \ J_3 = u .$$

Writing the action of $X_2 + X_6$ on the space of J_1, J_2, J_3 and using the first equation of system (2.5), we obtain the following two functionally independent invariants:

$$I_1 = x + e^{-t}, \ I_2 = ue^{-t} .$$

Hence we obtain the invariant solution

$$u = e^t \phi(x + e^{-t})$$

where ϕ satisfies the second-order ordinary differential equation

$$\phi\phi'' - (\phi')^2 + \phi'\phi^2 = 0$$

whose solution is given by

$$\phi = \frac{c_1 c_2}{c_2 - e^{-c_1(x+e^{-t})}} .$$

Thus the group-invariant solution under the subalgebra $< X_2 + X_6; X_3 >$ is

$$u = \frac{c_1 c_2 e^t}{c_2 - e^{-c_1(x+e^{-t})}} .$$

Similarly, the group-invariant solution under the two-dimensional subalgebra $< X_2; X_5 >$ can be calculated and the solution is given by

$$u = \frac{ce^t - 2}{z^2} .$$

However, in many cases the reduced ordinary differential equation for ϕ is not easily solved. For example, in the case of the two-dimensional subalgebra $< X_2 + X_6; X_1 - X_5 >$ the equation satisfied by ϕ is

$$(1+\xi^2)\left(\frac{\phi'}{\phi}\right)' + 2\xi\frac{\phi'}{\phi} + \phi' + 1 = 0, \quad \xi = \frac{x + e^{-t}}{z} .$$

It would be of interest to look at the other cases that arise and to obtain solutions of the reduced equation. This would then give us all group-invariant solutions of rank one corresponding to optimal system (2.4) of two-dimensional subalgebras.

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