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RECOVERING IMAGES FROM TRAVELTIME DATA

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Abstract- We study here a mathematical problem of retrieving slowness (reciprocal of velocity) distribution of a medium from a set of the measured first arrivals traveltimes data between sources and receivers within the medium.
Keywords- Tomography, Nonlinear Integral Equations, and Inverse Problems.

1. INTRODUCTION

Classical methods of tomography provide information for reconstruction of function from a set of line integrals. In medical X-ray tomography, such integrals are provided by a measurement of the amplitude attenuation for X-rays passing through the body. When backprojected along the known ray path, the attenuation data provide a picture of an inhomogeneous density distribution that can be interpreted for purpose of diagnosis.
Outside of the field of medicine, tomography has many uses [2]. In seismic tomography, rays can be used to probe the earth, analogously to X-rays in medical tomography [11]. However, there are some major differences. In medical tomography both the source and receiver positions are known to high accuracy, and the X- rays traverse straight paths from source to receiver. In contrast, only the receiver location is known to high accuracy for seismic ray. This is because the epicentre (the projection of the earthquake source on the surface of the Earth) is not often known to sufficient accuracy. Moreover, the rays traverse curved paths whose shape depends on the wavespeed structure of the Earth. Therefore, medical tomography techniques are not directly applicable to the study of the Earth, due to often-cited problems of limited coverage, nonlinearity and low signal to noise ratio. The effect of noisy traveltime data on tomographic images, and useful methods for dealing with this, are subjects of this study.
When a sound wave or seismic wave is sent into a medium, it takes time for the influence of the wave to progress from a point close to the source to a more distant point. The time taken by the wave to travel from one point of interest to the next is called the traveltime. For a medium in which there are no physical or chemical changes during the passage of the wave, it has a definite speed when it travels between any two points in the medium. This speed is called the average wave speed or wave velocity.
Let P denote arbitrary paths connecting a given source and receiver in a slowness model s. We shall refer to p as a trial ray path. We define a functional τ^p that yields the traveltime along path p. Letting s be the continuous slowness distribution $s(x)$, we have

$$\tau^p(s) = \int_p s(x)\,dl^p \tag{1}$$

where dl^p denotes the infinitesimal distance along p. Fermat's principle (Fermat, 1891) states that the correct ray path between two points is the one of least overall traveltime, i.e. it minimizes $\tau^{p}(s)$ with respect to path p. Let us define τ^{*} to be the functional that yields the traveltime along the Fermat ray path:

$$\tau^{*} = \min_{p \in P} \tau^{P}(s) \tag{2}$$

where p^{*} denotes the set of all continuous paths connecting the given source and receiver. If more than one path produces the same minimum traveltime value, then p^{*} denotes any particular member in this set of minimizing paths. Substituting Equation (1) into Equation (2), we have Fermat's principle of least time

$$\tau^{*} = \int_{P^{*}} s(x)dl^{P^{*}} = \min_{P} \int_{P} s(x)dl^{P} \tag{3}$$

The traveltime functional τ^{*} is stationary with respect to small variations in the path p^{*}. The task of tomography is to find a function $s(x)$ given the integrals τ^{*} over a family of manifold p.

Tomography, or more generally, the inversion for varying velocity structures using data collected on bounding surface, had been firstly developed in medical field [**11**]. Uniqueness of recovery of s from τ was established by Radon in 1917 [**3**]. But in 1960s, Cormack and Hounsfield, who developed an effective numerical and medical technique for exploring the interior of the human body for diagnostic purposes, have made the applied importance of this problem clear. Aki [**4**] was first to use seismic data in their 3-D study of the earth's crust. After this study, the inversion for varying velocity structures using seismic traveltime data has become an important geophysical tool [8, 9, 10].

2. MATHEMATICAL PROBLEM

Let $\mathbf{t}$ be the measured travel time m-vector such that $\mathbf{t}^{T} = \{t_1, t_2, \ldots, t_m\}$, where t_i is the traveltime along the i^{th} ray path (a superscript T implies transpose of a vector). We form our model of two-dimensions by dividing the rectangular region enclosed by our sources and receivers into rectangular cells of constant slowness.

Then, s is the model slowness n-vector such that $\mathbf{s}^{T} = \{s_1, s_2, s_3, \ldots, s_n\}$. The relationship between $\mathbf{s}$ and $\mathbf{t}$ is generally given by Equation (3). Let l_{ij} be the length of the i^{th} ray path passing through the j^{th} cell and be defined in the form:

$$l_{ij} = \int_{P \cap cell\ j} dl^{P_i} \tag{4}$$

Then, Equation (3) becomes

$$t_i = \sum_{j=1}^{n} l_{ij} s_j, \ (i = 1, 2, \ldots, m). \tag{5}$$

In the vector- matrix notation, this equation can be written in the following form:

$$\mathbf{t} = \mathbf{M}\mathbf{s} \tag{6}$$

where the matrix $\mathbf{M}$ is a $(m \times n)$ matrix whose the i^{th} row and the j^{th} column is given by l_{ij}. Then this is the basic equation of forward modelling for ray equation analysis. In other words, it can be considered as a numerical approximation to Equation (3). The mathematical problem is therefore to find **s** from **t** and it is called mathematical inversion.

3. RAY-TRACING

The most expensive step in any traveltime inversion or tomography algorithm is the forward modelling step associated with ray tracing through the current best estimate of the wave speed model. It is therefore essential to make a good choice of ray tracing algorithm for the particular application under consideration. However, the choice of the ray-tracing algorithm depends on a model of slowness. The model representation of the slowness chosen in this paper is cells of constant slowness. Therefore, we shall consider a bending method as a ray-tracing method [6].

Let (x_S, y_S) and (x_R, y_R) be coordinates of a known source and receiver pair, respectively. We are interested in the *ith* ray path p^i that connects the source - receiver pair while satisfying Fermat's principle (the first arrival necessarily follows the path of minimum traveltime). We therefore seek the ray path p^i that minimizes

$$t_i = \int_{p^i} s dl^{p^i} \tag{7}$$

Let L be the horizontal distance between two vertical boreholes. Call the abscissa x and the ordinate y. We assumed that the ray path connecting the source -receiver pairs can be expressed by a single valued function $y(x)$ (or $x(y)$). In ray bending, we begin with an initial ray $y_0(x)$ and seek a perturbation $\delta y(x)$ to the initial ray such that the traveltime along the perturbed ray is reduced. Typically, the initial ray is taken to be a straight line:

$$y_0(x) = \left(\frac{y_R - y_S}{x_R - x_S}\right)(x - x_S) + y_S . \tag{8}$$

The perturbation is to be a harmonic series of the form:

$$\delta y(x; a_1, a_2, ..., a_K) = \sum_{k=1}^{K} a_k \sin\left(\frac{k\pi x}{L}\right). \tag{9}$$

Only sine and not cosine terms are used because the end points of the ray remain unperturbed. The i^{th} travel time along the i^{th} perturbed ray path, defined by

$$y^i(x; \alpha_1, \alpha_2, ..., \alpha_K) = y_0{}^i(x) + \delta y^i(x; \alpha_1, \alpha_2, ..., \alpha_K),$$

is given by the traveltime functional:

$$t_i(a_1, a_2, ..., a_K) = \int_{x_S}^{x_R} s(x, y^i(x)) \sqrt{1 + \left(\frac{dy^i(x)}{dx}\right)^2} dx . \tag{10}$$

The problem is now reduced to the determination of the coefficients a_k's that minimize t_i. The value for K in Equation (9) for calculating the traveltime of the ray depends only on the resolution of our tomographic model. For a relatively low resolution, it is necessary to seek the general bend in rays, so we only need to determine a few a_k coefficients. However, the determination of these coefficients is difficult since they depend nonlinearly on the traveltimes t_i. To simplify this problem, we chose to ignore the Snell's law at cells boundaries and assumed that $K = 2$. Then, the problem is simply to minimize the traveltime functional given in Equation (10) with respect to the coefficients a_1 and a_2. To implement this, we used the Simplex method [5]. Starting with three points which have traveltimes $\mathbf{t}^1, \mathbf{t}^2$ and $\mathbf{t}^3$, the algorithm seeks to replace the point with the largest traveltime by a smaller one and then other moves are made such as checking values between the original vertex and the reflected vertex or expansion (contraction) of the triangle. When an improved vertex is found, the vertices are relabelled and the process starts over for the new triangle. If no improvement (or improvement less than a preset threshold) is attained or a fixed number of iterations is executed, the process terminates for this ray path.

4. THE METHOD

In agreement with Berryman [1,2], the forward problem in (6) can be replaced with the following feasibility constraints:

$$(\mathbf{Ms})_i \geq t_i, \quad (i = 1,2,3,\ldots,m) \tag{12}$$

This arises from Fermat's principle and it implies that first arrival rays follow the path with minimum traveltimes for a model $\mathbf{s}$. Thus, if $\mathbf{s}$ is a true model then any ray path matrix $\mathbf{M}$ must satisfy these constraints. Therefore, set of models that violate (12) along any path matrix $\mathbf{M}$ is called a nonfeasible set. The true solution to the inverse problem in (6) is then obtained if and only if all inequalities in (12) become identities for some choice of the model $\mathbf{s}$. Moreover, for the m-feasible constraints the limiting equality is an equation for the hyperplane in the n-dimensional model space. The feasible region is bounded by these hyperplanes and by the planes determined by the positivity constraint,

$$s_j > 0, \quad (j = 1,2,3,\ldots,n). \tag{13}$$

It can easily be shown that the constraints in (12) and (13) imply that the feasible region of the model space is convex [2]. Hence, for a fixed ray path matrix $\mathbf{M}$ the set of all feasible models includes all models either in the feasible region or on the feasibility boundary determined by $\mathbf{M}$ and $\mathbf{t}$.

For any combination of the ray-path matrix $\mathbf{M}$, slowness vector $\mathbf{s}$ and the measured traveltimes $\mathbf{t}$, the number of rays violating the constraints (12) is called the feasibility violation number, determined by

$$VN_M(\mathbf{s}) = \sum_{i=1}^{m} \delta[t_i - (\mathbf{Ms})_i] \tag{14}$$

where δ is a step function. Following Lanzcos [7], a generalised eigenvalue problem is given in the form:

$$\begin{bmatrix} 0 & \mathbf{M} \\ \mathbf{M}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \tag{15}$$

where $\mathbf{u}$ and $\mathbf{v}$ are m and n vectors of ones, respectively. The matrix on the right is defined in terms of diagonal matrices $\mathbf{L}$ and $\mathbf{C}$ whose diagonal elements are the row sum L_i and the column sum C_j of the matrix $\mathbf{M}$, respectively. The quantity L_i is the total length of the path i. The quantity C_j is the total ray path segments passing through the cell j. It is called the coverage of cell j. Any cell with $C_j = 0$ is uncovered and therefore lies outside the span of the data for the current choice of ray paths. We retain only covered cells in the reduced slowness vector $\tilde{\mathbf{s}}$ with $\tilde{n} \le n$. By deleting the corresponding columns in the matrix $\mathbf{M}$, the size of the ray path matrix $\mathbf{M}$ is reduced. For the simplicity, it is assumed that $\tilde{n} = n$ in the following discussions. An analogous eigenvalue problem providing for high contrast reconstruction is given in the form:

$$\begin{bmatrix} 0 & \mathbf{M} \\ \mathbf{M}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_\lambda \\ \mathbf{x}_\lambda \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{w}_\lambda \\ \mathbf{x}_\lambda \end{bmatrix} \tag{16}$$

where, for $\lambda = 1$, $\mathbf{w}_1 = \mathbf{u}$ and $\mathbf{x}_1 = \mathbf{s}_b$,

$$T_{ii} = \sum_{j=1}^{n} l_{ij} s_j \text{ and } D_{jj} = \sum_{i=1}^{m} \frac{l_{ij}}{s_j} . \tag{17}$$

By writing (16) in the canonical form, we have

$$\begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{\frac{-1}{2}} & 0 \\ 0 & \mathbf{D}^{\frac{-1}{2}} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{M} \\ \mathbf{M}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\frac{-1}{2}} & 0 \\ 0 & \mathbf{D}^{\frac{-1}{2}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{D}^{\frac{-1}{2}}\mathbf{M}\mathbf{T}^{\frac{-1}{2}} \\ \mathbf{T}^{\frac{-1}{2}}\mathbf{M}^{\mathrm{T}}\mathbf{D}^{\frac{-1}{2}} & 0 \end{bmatrix} \tag{18}$$

and

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{\frac{1}{2}}\mathbf{u} \\ \mathbf{D}^{\frac{1}{2}}\mathbf{s}_b \end{bmatrix} \tag{19}$$

Thus, the eigenvalue problem given in (16) is transformed into

$$\begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}. \tag{20}$$

As seen above, with normalisation the current slowness model $\mathbf{s}_b$ gives rise to the unique eigenvector with the highest eigenvalue and that eigenvalue is unity. Given a set of transmitter-receiver pairs and any model slowness $\mathbf{s}_b$, Fermat's principle can be used to find the ray-path matrix $\mathbf{M}$ associated with $\mathbf{s}_b$ and with any slowness $\gamma\mathbf{s}_b$ $(\gamma > 0)$ in the same direction as $\mathbf{s}_b$. If the normalised data is given by

$$\mathbf{y} = \mathbf{T}^{\frac{-1}{2}}\mathbf{t} \tag{21}$$

then the problem is to find γ such that

$$\varphi(\gamma) = [\mathbf{y} - \mathbf{A}\gamma\mathbf{z}_b]^T[\mathbf{y} - \mathbf{A}\gamma\mathbf{z}_b] \tag{22}$$

achieves its minimum. This value is found to be

$$\gamma = \frac{\mathbf{z}_b{}^{\mathrm{T}}\mathbf{A}^T\mathbf{y}}{\mathbf{z}_b{}^T\mathbf{A}^T\mathbf{A}\mathbf{z}_b} = \frac{\mathbf{z}_b{}^T\mathbf{A}^T\mathbf{y}}{\mathbf{z}_b{}^T\mathbf{z}_b}, \tag{23}$$

since $\mathbf{A}^T\mathbf{A}\mathbf{z}_b = \mathbf{z}_b$. Having found optimal slowness $\mathbf{s}_b = \gamma\mathbf{s}_b$ in the given direction, an attempt is made to improve the model by finding another direction in the slowness vector space that gives better fit to the traveltime data. To do this, we consider a second weighted least-squares problem:

$$\phi(\mathbf{z}) = [\mathbf{y} - \mathbf{A}\mathbf{z}]^T [\mathbf{y} - \mathbf{A}\mathbf{z}] + \mu(\mathbf{z} - \mathbf{z}_b)^T (\mathbf{z} - \mathbf{z}_b), \tag{24}$$

where μ is a damping parameter. The minimum of (24) occurs at $\mathbf{z} = \mathbf{z}_\mu$ where $\mathbf{z}_\mu$ satisfies

$$(\mathbf{A}^T\mathbf{A} + \mu\mathbf{I})(\mathbf{z}_\mu - \mathbf{z}_b) = \mathbf{A}^T\mathbf{y} - \mathbf{z}_b \tag{25}$$

The matrix $(\mathbf{A}^T\mathbf{A} + \mu\mathbf{I})$ is nonsingular for $\mu > 0$, so we use conjugate-gradient method [12] to solve (26) for $\mathbf{z}_\mu$. Thus, the point $\mathbf{s}_\mu = \mathbf{D}^{-\frac{1}{2}}\mathbf{z}_\mu$ in the slowness vector space is obtained. It can be shown that both of points, $\mathbf{s}_b$ and $\mathbf{s}_\mu$ lie in the nonfeasible part of the vector space. If the solution of (6) exists, it must lie on the feasibility boundary. So $\mathbf{s}_b$ and $\mathbf{s}_\mu$ are used to find a point on this boundary that is optimum in the sense that it is as consistent as possible with the ray path matrix, with the travel time measurements and with the feasibility constraints. As mentioned previously, the feasible region is convex. Therefore, there exists a point $\mathbf{s}_l$ between points $\mathbf{s}_b$ and $\mathbf{s}_\mu$ that is closer to the feasible region than the either of two end points. This can easily be found by computing the feasibility violation number and by choosing the model that gives a minimum violation number when we move in the direction $(\mathbf{s}_\mu - \mathbf{s}_b)$ from $\mathbf{s}_b$. Then, we get

$$[\mathbf{s}_l]' = \mathbf{s}_b + \alpha(\mathbf{s}_\mu - \mathbf{s}_b) \tag{26}$$

As α gets smaller, it is expected that the inversion method is not providing any further improvement so that a threshold for α of 0.25 is used to stop searching. Once we find $\mathbf{s}_l$ and then scale it up to the point, denoted as $\mathbf{s}_f$, in the same direction lying in the feasibility boundary. It is not hard to see that these three points, $\mathbf{s}_b, \mathbf{s}_\mu$ and $\mathbf{s}_f$ are distinct unless we found the exact solution of the inverse problem. We conclude that $\mathbf{s}_b = \mathbf{s}_\mu$ if and only if $\mathbf{s}_b$ is an exact solution of (6). So unless we have already solved the problem, these three points form a triangle and the size of the triangle gives us an estimate how far we are from the solution.

An iterative method that uses above ideas is implemented and it is used for retrieving slowness distribution from artificially generated traveltime data in the following section.

5. COMPUTER SIMULATIONS

We assumed that a model slowness structure consists of 8×16 cells that have normalised slowness one and allowed that the model slowness has a low speed anomaly on bottom and a high-speed anomaly on the top. We then parameterised the model by γ, where the slow region had a slowness of $1+\gamma$ and the fast region had a slowness of $\frac{1}{1+\gamma}$. Thus, changes in the value of γ will provide changes in the contrast of the model slowness. Hence, 256 rays travelling from left to right and 64 rays travelling from bottom to top were used for computing the traveltime data shown in Figures 1-2 (b) by using ray-tracing method introduced in the paper.

Then, our computer program, written in C++ programming language, ran on a personal computer and used the data to retrieve the slowness structures in Figures 1 -2(a). After about 20 iterations, the reconstructed slowness distributions were plotted and they are

shown in Figures 1-2(c). Visually, reconstructed slowness has similar structure with model slowness. High and low speed anomalies were detected and their boundaries were clearly recovered. These results agree with that of Berryman.
As expected, the greater contrast in the model increases bending of the ray-paths so that it leads to deteriorate the reconstructions.

6. CONCLUSIONS

In agreement with Berryman, the results presented in this paper indicate that the method can be a useful tool for retrieving slowness distributions from the traveltimes for the first arrivals. However, an increase in the contrast will deteriorate the results. In addition, ray tracing process consumes a large amount of CPU time. Therefore, new developments in the ray-tracing algorithm and incorporating the present algorithm with a principle of maximum entropy will be subject to our future work.

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REFERENCES

[1] Thomas J.Yorkey and James G. Berryman, An Iterative Algorithm for Traveltime Tomography Between Boreholes. *Lawrence Livermore National Laboratory* Doc Number: **210339**

[2] James Berryman, Stable Iterative Reconstruction Algorithm for Nonlinear Traveltime Tomography, *Inverse Problems*: **6**, 21-42, 1990

[3] J. Radon, Über die Bestimmung von Funktionen durch ihre Integralweite langs gewisser Mannigfaltigkeiten, *Ber.Verh.Saechs.Akad.wiss, Leipzig. Math.Phys.Kl*: **69**, 262-277, 1917.

[4] Aki, K., Lee, W.H.K., Determination of three dimensional velocity anomalies under a seismic array using first P-arrival time from local earthquakes, 1.A homogeneous initial model, *J.Geophys.Res*: **81**, 4381-4399, 1976.

[5] Nelder, J.A., Mead, R., A simplex method for function minimization, *Computer J*: **7**, 308-313, 1965.

[6] W.A. Prothero , W.J. Taylor and J.A. Eickemeyer, A fast two-point, three-dimensional ray tracing algorithm using a simple search method, *Bull. Seismol. Soc. Am.*: **78,** 1190-1198, 1988.

[7] C. Lanczos, An iteration method for the solution of the eigenvalue problem of linear differential equations and integral operators, *J. Res. Nat. Bur. Stand.*; **45**, 255-282, 1950.

[8] Jonathan M. Lees and Robert S. Crosson, Tomographic inversion for three dimensional velocity structure at Mount St. Helens using earthquake data, *Journal of Geophysical Research*: **94**, No B5, 5716-5728, 1989

[9] J.A.Hole, Nonlinear high-resolution three dimensional seismic traveltime tomography, *Journal of Geophysical Research*: **97**, No B5, 6553-6562, 1991.

[10] R. Jeffrey Lytle and Kris A. Dines, Iterative ray tracing between boreholes for underground image reconstruction, *IEEE Transactions on Geoscience and Remote Sensing*: **18**, 234-240, 1980.

[11] W.H.K. Lee, V. Pereyra, *Seismic Tomography/Theory and Practice,* Edited by H.M Iyer&K.Hirahara, Chapman&Hall, London, 1993.
[12] K.E. Atkinson, *Introduction to Numerical Analysis,* Wiley, New York, USA, 1997.

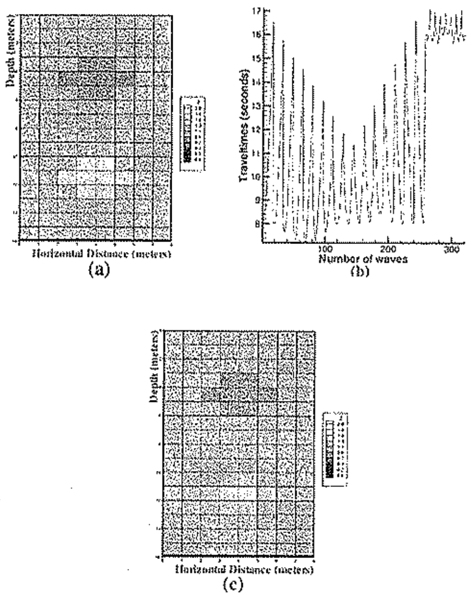

Figure 2. (a) Model slowness distribution with 40% contrast. (b) Traveltime Data. (c) Reconstructed slowness distribution

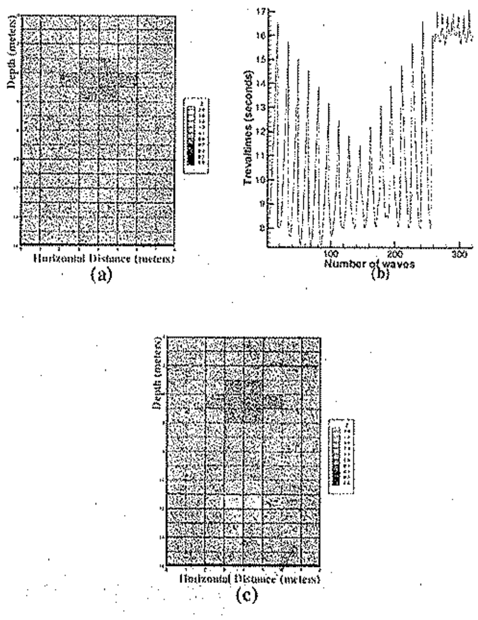

Figure 1. (a) Model slowness distribution with 20% contrast. (b) Traveltime Data. (c) Reconstructed slowness distribution