Mathematical & Computational Applications, Vol. 8, No. 1 , pp. 63-70, 2003

INVARIANT SOLUTIONS OF THE BLACK-SCHOLES EQUATION

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Abstract- As the Black-Scholes equation can be transformed into the one-dimensional linear heat equation via two sets of transformations, an optimal system of one-dimensional subalgebras for the one-dimensional heat equation is exploited to obtain two classes of optimal systems of one-dimensional subalgebras for the well-known Black-Scholes equation of the mathematics of finance. Two methods for the derivation of the two classes of optimal systems of group-invariant solutions for this model are available. We present the simpler approach.

Keywords- Black-Scholes equation, optimal system, invariant solution.

1. INTRODUCTION

The classification of group-invariant solutions of differential equations by means of the so-called optimal systems is one of the main applications of Lie group analysis to differential equations. The method was first conceived by Ovsiannikov [1]. The general idea behind the method is discussed in his papers [2,3] and also by Chupakin [4] and Ibragimov *et al* [5]. Coggeshall and Meyer-ter-Vehn [6] applied this method to a problem in hydrodynamics. The results by Olver [7] who obtained an optimal system of one-dimensional subalgebras for the one-dimensional heat equation lays the foundation for part of this work. We also utilize the transformations that reduce the Black-Scholes equation [8] to the one-dimensional heat equation given in Gazizov and Ibragimov [9].

We can, in principle, always construct a family of group-invariant solutions corresponding to a subgroup of a symmetry group admitted by a given differential equation. Since there are almost always an infinite number of such subgroups it is not feasible to list all the group-invariant solutions. An effective and systematic way of classifying these solutions is to obtain optimal systems of subalgebras of the symmetry Lie algebra. This leads to, under symmetry transformations, non-similar invariant solutions. As the Black-Scholes equation is transformable to the heat equation via point transformations, one would expect the same classes of invariant solutions for this equation as for the heat equation. However, there are two sets of reduction transformations from the Black-Scholes to the heat equation that need to be taken into account. So the problem is not as simple as that. The purpose of this paper is to present the two classes of optimal systems of group-invariant solutions of the Black-Scholes model. We can use two approaches: a direct approach which requires solution of differential equations and a method whereby an appeal is made to the heat invariant solutions. We apply the latter method.

The outline of the work is as follows. Section 2 deals with the transformation of the Black-Scholes equation to the heat equation. In Section 3 we briefly present the results of Olver [7] on optimal system of one-dimensional subalgebras admitted by the heat equation including the classes of invariant solutions, as these are intrinsic to our work. This section is also devoted to the computation of the two classes of optimal systems of group invariant solutions of the Black-Scholes equation.

2. TRANSFORMATION OF THE BLACK-SCHOLES TO THE HEAT EQUATION

In their paper, Gazizov and Ibragimov [9] derived two sets of transformation formulas which transform the Black-Scholes equation (A, B and C are constants)

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0 \tag{2.1}$$

i, ...e classical heat equation

$$v_\tau = v_{yy}. \tag{2.2}$$

These ...sformations, for arbitrary constants a_i, b_i, c_i and d_i, are written as

$$\begin{aligned} \bar{t} &= a_1 - ta_4^2, \quad a_4 \neq 0, \\ \bar{x} &= \frac{\pm\sqrt{2}a_4 \ln x}{A} + a_4 d_2 t + b_2, \\ \bar{u} &= ub_3 x^{\frac{B}{A^2} - \frac{1}{2} \pm \frac{d_2}{\sqrt{2}A}} \exp\left[c_3 t + \frac{td_2^2}{4}\right], \quad b_3 \neq 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \bar{t} &= \frac{4a_4^2}{t + 2a_1} + a_3, \quad a_4 \neq 0, \\ \bar{x} &= \frac{\pm 2\sqrt{2}a_4 \ln x}{A(t + 2a_1)} - \frac{4a_4 b_1}{t + 2a_1} + b_2, \\ \bar{u} &= ub_3\sqrt{t + 2a_1}\, x^{\frac{B}{A^2} - \frac{1}{2} \pm \frac{\sqrt{2}b_1}{A(t+2a_1)}} \\ &\times \exp\left(-c_3 t - \frac{\ln^2 x + 2A^2 b_1^2}{2A^2(t + 2a_1)}\right), \quad b_3 \neq 0 \end{aligned} \tag{2.4}$$

in which A, B and C satisfy

$$2B = 2D + A^2, \quad c_3 = \frac{D^2}{2A^2} + C. \tag{2.5}$$

Also, the symmetries admitted by the Black-Scholes equation (see [9]) are given by

$$
\begin{aligned}
X_1 &= \partial_t, \\
X_2 &= x\partial_x, \\
X_3 &= u\partial_u, \\
X_4 &= 2t\partial_t + (\ln x + Dt)x\partial_x + 2Ctu\partial_u, \\
X_5 &= A^2tx\partial_x + (\ln x - Dt)u\partial_u, \\
X_6 &= 2A^2t^2\partial_t + 2A^2tx\ln x\partial_x + \left((\ln x - Dt)^2 + 2A^2t^2C - A^2t\right)u\partial_u, \\
X_\phi &= \phi(x,t)\partial_u
\end{aligned} \tag{2.6}
$$

and the heat equation symmetries generate an infinite dimensional Lie algebra ([7], [13])

$$
\begin{aligned}
Y_1 &= \partial_\tau, \\
Y_2 &= \partial_y, \\
Y_3 &= v\partial_v, \\
Y_4 &= 2\tau\partial_\tau + y\partial_y, \\
Y_5 &= 2\tau\partial_y - yv\partial_v, \\
Y_6 &= 4\tau^2\partial_\tau + 4y\tau\partial_y - v(2\tau + y^2)\partial_v, \\
Y_\varphi &= \varphi\partial_v,
\end{aligned} \tag{2.7}
$$

where

$$\varphi(\tau, y) = \phi(t,x)e^{-Ct},$$

with φ an arbitrary solution of the heat equation and ϕ solves the Black-Scholes equation. Only the finite dimensional subalgebra L_6 of the infinite algebra of operators (2.7) is considered in the subsequent calculations in Section 3 as the infinite part does not provide invariant solutions. The easiest and best way in which the two classes of group-invariant solutions can be realized is to use the fact that the operators (2.6) can be transformed to the heat vectors (2.7) under the two transformations (2.3) and (2.4). Also, this idea will enable us to obtain the two optimal systems of one-dimensional subalgebras admitted by the Black-Scholes equation.

In general, the transformed operators are obtained from

$$X_i = X_i(\bar{x}^j)\frac{\partial}{\partial \bar{x}^j}, \tag{2.8}$$

where $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3\right) = \left(\bar{t}, \bar{x}, \bar{u}\right)$ are the transformed variables.

The use of (2.3) and (2.8) results in the operators (2.6) being expressed in terms of the six basis vectors admitted by the heat equation as (the calculations are straightforward albeit very tedious):

$$\begin{aligned}
X_1 &= a_4^2 Y_1 + a_4 d_2 Y_2 + \left(\frac{d_2^2 - 4c_3}{4}\right) Y_3, \\
X_2 &= \pm \frac{\sqrt{2}a_4}{A} Y_2 + \left(\frac{2D \pm \sqrt{2}Ad_2}{2A^2}\right) Y_3, \\
X_3 &= Y_3, \\
X_4 &= Y_4 - 2a_1 Y_1 - b_2 Y_2 - \left(\frac{Ad_2 \pm \sqrt{2}D}{2Aa_4}\right) Y_5, \\
X_5 &= \mp \frac{A}{\sqrt{2}a_4} Y_5 \pm \frac{A\sqrt{2}a_1}{a_4} Y_2 \mp \frac{Ab_2}{\sqrt{2}a_4} Y_3, \\
X_6 &= \frac{A^2(b_2^2 - 2a_1)}{2a_4^2} Y_3 + \frac{2A^2 a_1}{a_4^2} Y_4 + \frac{A^2 b_2}{a_4^2} Y_5 - \frac{A^2}{2a_4^2} Y_6
\end{aligned} \tag{2.9}$$

Similarly if the transformations (2.4) are used we obtain

$$\begin{aligned}
X_1 &= -\frac{a_3^2}{4a_4^2} Y_1 - \frac{a_3 b_2}{4a_4^2} + \left(\frac{b_2^2 - 2a_3 - 16c_3 a_4^2}{16a_4^2}\right) Y_3 + \frac{a_3}{4a_2^2} Y_4 + \frac{b_2}{8a_4^2} Y_5 - \frac{1}{16a_4^2} Y_6, \\
X_2 &= \pm \frac{a_3}{\sqrt{2}Aa_4} Y_2 + \left(\frac{4Da_4 \pm \sqrt{2}b_2}{4A^2 a_4}\right) Y_3 \pm \frac{1}{2\sqrt{2}Aa_4} Y_5, \\
X_3 &= Y_3, \\
X_4 &= \left(2a_3 + \frac{a_1 a_3^2}{a_4^2}\right) Y_1 + \left(b_2 + \frac{a_1 a_3 b_2}{a_4^2} + \frac{\pm\sqrt{2}a_1 a_3 D - Ab_1 a_3}{Aa_4} \pm \frac{\sqrt{2}D}{A} - \frac{b_2 a_3}{a_4}\right) Y_2 \\
&+ \left(\frac{A \pm \sqrt{2}Db_1}{A} \mp \frac{\sqrt{2}Db_2 a_1}{2Aa_4} + \frac{b_1 b_2}{2a_4} + \frac{a_1(2a_3 - b_2^2)}{4a_4^2}\right) Y_3 - \left(1 + \frac{a_1 a_3}{a_4^2}\right) Y_4 \\
&+ \left(\frac{b_1}{2a_4} - \frac{a_1 b_2}{2a_4^2} \mp \frac{\sqrt{2}Da_1}{2Aa_4}\right) Y_5 + \frac{a_1}{4a_4^2} Y_6, \\
X_5 &= \left(2a_4 \pm \frac{2Aa_1 a_3}{a_4}\right) Y_2 \pm \left(\frac{2Ab_1 - Aa_1 b_1}{\sqrt{2}a_4}\right) Y_3 \mp \frac{Aa_1}{\sqrt{2}a_4} Y_5, \\
X_6 &= 2A^2 \left(\frac{a_1^2 a_3^2 + 2a_3 a_4^3 + 2a_1 a_3 a_4 + 4a_4^4}{a_4^2}\right) Y_1 \\
&+ 2A^2 \left(\frac{2a_1 a_3 a_4 b_1 - a_1^2 a_3 b_2 + 4a_4^3 b_1 - 2a_1 a_4^2 b_2}{a_4^2}\right) Y_2 \\
&+ A^2 \left(\frac{4a_4^2 b_1^2 + a_1(b_2^2 - 2a_3) - 4a_1 a_4^2 - 4a_1 a_4 b_1 b_2}{a_4^2}\right) Y_3 + 2A^2 \left(\frac{a_1^2 a_3 + a_1 a_4^2}{a_4^2}\right) Y_4 \\
&+ A^2 \left(\frac{a_1^2 b_2 - 2a_1 a_4 b_1}{a_4^2}\right) Y_5 - A^2 \frac{a_1^2}{2a_4^2} Y_6.
\end{aligned} \tag{2.10}$$

It is possible to express $Y_1, Y_2, \ldots, Y_6$ in terms of $X_1, X_2, \ldots, X_6$. For example from (2.9)

$$Y_2 = \pm\frac{A}{\sqrt{2a_4}}X_2 - \frac{\pm\sqrt{2}D + Ad_2}{2Aa_4}X_3. \tag{2.11}$$

Similarly, from (2.10), we can write the same operator as

$$Y_2 = \frac{\sqrt{2}A\left(\pm 4Aa_1a_4b_2 - (a_1-2)b_1X_3\right) \mp 2a_4\left(2a_1(A^2X_2 - DX_3) + X_5\right)}{(2\sqrt{2}-4)Aa_1a_3 \mp 4a_4^2}. \tag{2.12}$$

In general, the inversion from the X_is to the Y_is is messy. We shall make use of the particular choices of the constants a_i, b_i, c_i and d_i in Section 3 that renders this simple.

3. OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRAS OF THE HEAT AND BLACK-SCHOLES

The optimal system of the one-dimensional subalgebras of symmetries admitted by the heat equation is (see [7])

$$\Theta_1 = \{Y_4 + aY_3, Y_2 + Y_6 + aY_3, Y_2 + Y_5, Y_2 + aY_3, Y_1, Y_3;\ a \in R\} \tag{3.1}$$

The invariant solutions corresponding to each of the elements are respectively [7]

$$\begin{aligned}
u(x,t) &= t^a \exp(-\frac{x^2}{8x})(k_1U[2a+\frac{1}{2},\frac{x}{\sqrt{2t}}] + k_2V[2a+\frac{1}{2},\frac{x}{\sqrt{2t}}]),\\
u(x,t) &= \frac{1}{(4t^2+1)^{\frac{1}{4}}}\exp\left(-\frac{tx}{4t^2+1} - \frac{a}{2}\arctan 2t\right)\left(k_1W\left[-\frac{a}{2},\frac{x}{\sqrt{8t^2+2}}\right]\right.\\
&\quad + \left. k_2W\left[-\frac{a}{2},\frac{-x}{\sqrt{8t^2+2}}\right]\right),\\
u(x,t) &= \exp\left(xt + \frac{3}{2}t^3\right)\left(k_1Ai[x+t^2] + k_2Bi[x+t^2]\right)\\
u(x,t) &= \begin{cases} k\exp(at)\cosh(\sqrt{a}x+\delta)), & a>0\\ k_1x + k_2, & a=0\\ k_3\exp(at)\cos(\sqrt{-a}x+\delta), & a<0\end{cases}
\end{aligned} \tag{3.2}$$

Note that there is no invariant solution corresponding to Y_3, and as for Y_1 it is constant.

The method is illustrated with the case in which the mapping (2.4) is used. For the particular choice of constants $a_1 = a_3 = b_1 = b_2 = c_3 = 0, a_4 = \frac{1}{4}, b_3 = 1,$ these transformations may be written as follows:

$$\begin{aligned}
\bar{t} &= \frac{1}{t}, \\
\bar{x} &= \frac{\sqrt{2}\ln x}{At}, \\
\bar{x} &= u\sqrt{t}\,x^{\frac{B}{A^2}-\frac{1}{2}-\frac{\ln x}{2tA^2}}.
\end{aligned} \tag{3.3}$$

Thus, in terms of the heat basis, the symmetries admitted by the Black-Scholes equation which in this case have A, B, and C being related by (2.5) as $c_3 = 0$, take the form

$$\begin{aligned}
X_1 &= -Y_6, \\
X_2 &= \frac{1}{\sqrt{2}A}Y_5 + (\frac{B}{A^2}-\frac{1}{2})Y_3, \\
X_3 &= Y_3, \\
X_4 &= \frac{\sqrt{2}D}{A}Y_2 + Y_3 - Y_4, \\
X_5 &= \sqrt{2}AY_2, \\
X_6 &= -2A^2Y_1.
\end{aligned} \tag{3.4}$$

It is not difficult to see that, for example after inversion, $Y_4 + aY_3$ is transformed into $-X_4 + \frac{D}{A^2}X_5 + (1+a)X_3$. In a similar manner we obtain the following optimal system of one-dimensional subalgebras admitted by the Black-Scholes equation:

$$\begin{aligned}
\Theta = \{ & -X_4 + \frac{D}{A^2}X_5 + (1+a)X_3, -X_1 + aX_3 - \frac{1}{2A^2}X_6, \\
& \sqrt{2}AX_2 - (\frac{\sqrt{2}B}{A^2} - \frac{1}{\sqrt{2}})X_3 + \frac{1}{2A^2}X_6, \frac{1}{2A^2}X_6 + aX_3, \frac{1}{\sqrt{2}A}X_5, \\
& X_3, \ a \in R\}
\end{aligned} \tag{3.5}$$

The invariant solutions that correspond to $-X_4 + \frac{D}{A^2}X_5 + (1+a)X_3$ are obtained via the transformations (3.3) applied to the invariant solutions that correspond to the operator $Y_4 + aY_3$ admitted by the heat equation. Thus

$$u(x,t) = t^{-a-\frac{1}{2}}\exp\left(-\frac{\ln x^2}{4A^2t^3}\right)x^{\left(\frac{1}{2}-\frac{A}{B}+\frac{\ln x}{2A^2t}\right)}\left(k_1U\left[2a+1/2, \frac{\ln x}{A\sqrt{t}}\right] + k_2V\left[2a+1/2, \frac{\ln x}{A\sqrt{t}}\right]\right),$$

where U and V are parabolic cylindrical functions, is the invariant solution corresponding to $-X_4 + \frac{D}{A^2}X_5 + (1+a)X_3$. The construction of group-invariant solutions for each of the one-dimensional subalgebras in the optimal system (3.5) proceeds in the same fashion. For each of the other elements in the optimal system we have respectively

$$u(x,t) = (4+t^2)^{-\frac{1}{4}} \exp\left[-\frac{1}{2} a \arctan\left(\frac{2}{t}\right) - \frac{2\ln^2 x}{A^2 t(4+t^2)}\right] x^{(\frac{1}{2}-\frac{A}{B}+\frac{\ln x}{2A^2 t})}$$

$$\times (k_1 U[-a/2, \frac{-\ln x}{A\sqrt{4+t^2}}] + k_2 V[-a/2, \frac{\ln x}{A\sqrt{4+t^2}}]),$$

$$u(x,t) = \frac{1}{\sqrt{t}} x^{\left(\frac{2}{1}-\frac{A}{B}+\frac{\ln x}{2A^2 t}\right)} \left(k_1 A_i \left[\frac{A+\sqrt{2}t\ln x}{At^2}\right] + k_2 B_i \left[\frac{A+\sqrt{2}t\ln x}{At^2}\right] \right)$$

where U and V are parabolic cylindrical functions and Ai and Bi are the Airy functions, and

$$u(x,t) = \begin{cases} \frac{1}{\sqrt{t}} x^{\left(\frac{1}{2}-\frac{A}{B}+\frac{\ln x}{2A^2 t}\right)} \exp\left(\frac{a}{t}\right) \cosh\left(\frac{\sqrt{2a}\ln x}{At} + \delta\right), & a > 0 \\ \frac{1}{At^{\frac{3}{2}}} x^{\left(\frac{1}{2}-\frac{A}{B}+\frac{\ln x}{2A^2 t}\right)} \left(k_1 \sqrt{2} \ln x + k_2 At\right), & a = 0 \\ \frac{1}{\sqrt{t}} x^{\left(\frac{1}{2}-\frac{A}{B}+\frac{\ln x}{2A^2 t}\right)} \exp\left(\frac{a}{t}\right) \cos\left(\frac{\sqrt{-2a}\ln x}{At} + \delta\right), & a < 0. \end{cases}$$

One can use this method to obtain the two general classes of optimal systems of invariant solutions of the Black-Scholes model. The tedium in having to do this in general results from the lengthy inversions of the operators *Ys* in terms of the *Xs* notwithstanding the lengthy expressions for the invariant solutions.

4. DISCUSSIONS

We have utilized the optimal system of one-dimensional subalgebras of the one-dimensional heat equation to deduce that there are two classes of optimal systems of invariant solutions of the Black-Scholes equation. This is dictated by the two sets of transformations that reduce the Black-Scholes to the heat equation. As a matter of fact, we mentioned two methods to obtain the optimal systems of invariant solutions. The first method depends upon solutions of differential equations that at times became tedious and hence we resorted to another easier approach. The second method is more efficient and relied on the inversions of the operators from the heat basis to the Black-Scholes as well as reliance on the heat invariant solutions.

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