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GOURSAT PROBLEM FOR THE FACTORIZABLE HYPERBOLIC EQUATION IN TWO INDEPENDENT VARIABLES

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Abstract- For the scalar linear hyperbolic partial differential equations (PDEs) in two independent variables to be factorizable, the Laplace invariants h or k must be zero. In this paper, we find the Riemann function for the Goursat problem using the Lie group theoretical method where the hyperbolic equation involved is factorized. What emerges is that the ordinary differential equation (ODE) whose solution gives the Riemann function for the Goursat problem is factorizable. Finally, an example is given as application of the method.

Keywords- Hyperbolic equation, Laplace invariants, Goursat problem.

1. INTRODUCTION

Consider the linear hyperbolic PDE

$$L[u] \equiv u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = f(t,x) \tag{1}$$

and its adjoint equation

$$L^*[v] \equiv v_{tx} - (av)_t - (bv)_x + cv = 0 \tag{2}$$

together with the following boundary conditions on the characteristics:

$$v\,|_{t=t_0} = \exp\left(\int_{x_0}^{x} a(t_0,\eta)d\eta\right), \qquad v\,|_{x=x_0} = \exp\left(\int_{t_0}^{t} b(\xi\,,x_0)d\xi\right), \tag{3}$$

where v is an auxiliary function. The initial value problem (2)-(3) is known as the Goursat problem. Existence of a unique solution is guaranteed throughout the entire first quadrant. The solution v of the Goursat problem (2)-(3) is known as the Riemann function and if the function v is found, the solution of the Cauchy problem under the consideration is obtained by using the formula given in [3, p. 43].

Six different well-known methods to find the Riemann function for special type of hyperbolic equations are given and discussed extensively in Copson's paper [1]. Further, Ibragimov [2, 3] established another method (seventh) to construct the Riemann function for the Goursat problem based on the Lie group theoretical approach and the use of Ovsiannikov's invariants [5]. In this paper, we also utilize a Lie group theoretical method. We propose to construct the Riemann function for the Goursat problem when the hyperbolic equation is factorizable.

We organize this paper as follows. In Section 2, we give several results which are related to our main topic in the paper. We state the main theorem for the Goursat problem and provide an example in Section 3. Finally, comments are made in Section 4.

2. PRELIMINARIES

In this section, we provide the proposition for the hyperbolic equation to be factorizable and the relevant known classification results are stated to suit our needs. In fact, we add the factorized hyperbolic PDEs in Theorem 2.1 below.

P sition 2.1 The linear second-order hyperbolic PDE

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0 \tag{4}$$

admits factorization in terms of first-order operators if and only if the Laplace invariants h and k satisfy $h \equiv a_t + ab - c = 0$ and $k \equiv b_x + ab - c = b_x - a_t$ or $k = 0$ and $h = -(b_x - a_t)$.

The above Proposition 2.1 was stated in [4]. The proof is straightforward.

We state the following theorem for the group classification of equation (4) which is factorizable form.

Theorem 2.1 If

$$p = \frac{h}{k} = 0 \ \ (i.e., h = 0) \ \text{and} \ q = \frac{\partial_t \partial_x \ln k}{k} = \text{const.}, \ k \neq 0, \tag{5}$$

Then the equation (4) for $q \neq 0$, is equivalent to the equation

$$u_{tx} + \frac{1}{2}u_t - \frac{2}{q}\frac{1}{t+x}u_x - \frac{1}{q}\frac{1}{t+x}u = 0, \tag{6}$$

which in factorized form is

$$\left(\frac{\partial}{\partial t}-\frac{2}{q}\frac{1}{t+x}\right)\left(\frac{\partial}{\partial x}+\frac{1}{2}\right)u=0$$

or for $q=0$ to the equation

$$u_{tx}+xu_x=0 \tag{7}$$

which in factorized form is

$$\left(\frac{\partial}{\partial t}+x\right)\left(\frac{\partial}{\partial x}\right)u=0.$$

If

$$p=\frac{k}{h}=0\ (i.e.,\ k=0)\ \text{ and } q=\frac{\partial_t\partial_x \ln h}{h}=\text{const.}, \tag{8}$$

then the equation (4) for $q\neq 0$ is equivalent to the equation

$$u_{tx}-\frac{2}{q}\frac{1}{t+x}u_t+\frac{1}{2}u_x-\frac{1}{q}\frac{1}{t+x}u=0, \tag{9}$$

which in factorized form is

$$\left(\frac{\partial}{\partial x}-\frac{2}{q}\frac{1}{t+x}\right)\left(\frac{\partial}{\partial t}+\frac{1}{2}\right)u=0$$

or for $q=0$ to the equation

$$u_{tx}+tu_t=0 \tag{10}$$

which in factorized form is

$$\left(\frac{\partial}{\partial x}+t\right)\left(\frac{\partial}{\partial t}\right)u=0.$$

Proof: Proof is similar to the analogues result given in [5].

3. GOURSAT PROBLEM FOR FACTORIZABLE HYPERBOLIC EQUATIONS

It is possible to construct the Goursat problem for factorizable hyperbolic equations.

Theorem 3.1 Let the equation (1) be written as

$$L[u] \equiv u_{tx} + au_t + bu_x + cu = \left(\frac{\partial}{\partial t} + b\right)\left(\frac{\partial}{\partial x} + a\right)u - hu = 0 \tag{11}$$

or

$$L[u] \equiv u_{tx} + au_t + bu_x + cu = \left(\frac{\partial}{\partial x} + a\right)\left(\frac{\partial}{\partial t} + b\right)u - ku = 0, \tag{12}$$

where h and k are the Laplace invariants. Then they have joint invariants

$$p = \frac{h}{k} = 0 \quad \text{and} \quad q = \frac{\partial_t \partial_x (\ln k)}{k} = \text{const.},$$

or

$$p = \frac{k}{h} = 0 \quad \text{and} \quad q = \frac{\partial_t \partial_x (\ln h)}{h} = \text{const.}$$

Moreover, the Goursat problem for (11) or (12)

$$L^*[v] \equiv v_{tx} - (av)_t - (bv)_x + cv = 0,$$

$$v\,|_{t=t_0} = \exp\left(\int_{x_0}^{x} a(t_0, \eta)\,d\eta\right), \qquad v\,|_{x=x_0} = \exp\left(\int_{t_0}^{t} b(\xi\ , x_0)\,d\xi\right), \tag{13}$$

admits a one-parameter group and the Riemann function is obtained from the second-order ODE which is also factorizable.

Proof: We give the proof for the first case of the theorem. The second case can be proved in a similar fashion. In the first case $h = 0$. Let $q_* = 0$ (here q_* is the corresponding invariant of the adjoint equation) and the adjoint equation of (11) have the form

$$L^*[v] \equiv v_{tx} + xv_x = 0. \tag{14}$$

Then the boundary conditions (13) are

$$v\,|_{t=t_0} = 1, \quad v\,|_{x=x_0} = \exp(x_0(t_0 - t)) \tag{15}$$

and the Goursat problem (14)-(15) admits the operator

$$X = (t-t_0)\frac{\partial}{\partial t} - (x-x_0)\frac{\partial}{\partial x} - x_0(t-t_0)v\frac{\partial}{\partial v}. \tag{16}$$

This operator has two invariants, viz.

$$V = v\exp(x_0(t-t_0)) \text{ and } \mu = (t-t_0)(x-x_0).$$

We seek the solution of the Goursat problem in the invariant form

$$v = \exp(x_0(t-t_0))V(\mu), \quad \mu = (t-t_0)(x-x_0). \tag{17}$$

Substitution of this equation into (14) yields

$$\mu V'' + (1+\mu)V' = 0 \tag{18}$$

which can be rewritten in factorizable form

$$\left(\frac{d}{d\mu}+1\right)\left(\mu\frac{d}{d\mu}\right)V = 0$$

and the characteristic data (15) becomes $V(0)=1$. Thus, the Riemann function for this Goursat problem is given by the solution of the above ODE (18) with initial condition $V(0)=1$.

Let $q_* \neq 0$ and the adjoint equation have the form (6):

$$v_{tx} + \frac{1}{2}v_t - \frac{2}{q_*}\frac{1}{t+x}v_x - \frac{1}{q_*}\frac{1}{t+x}v = 0, \quad q_* = \text{const}. \tag{19}$$

In this case the conditions (13) assume the form

$$v|_{t=t_0} = \exp\left(\frac{1}{2}(x_0-x)\right), \quad v|_{x=x_0} = \left(\frac{t+x_0}{t_0+x_0}\right)^{\frac{2}{q_*}} \tag{20}$$

and the Goursat problem (19)-(20) admits the operator

$$X = (t-t_0)(x-x_0)\frac{\partial}{\partial t} - (x+t_0)(x-x_0)\frac{\partial}{\partial x} + \left(\frac{1}{2}(x+t_0)(x-x_0) + \frac{2}{q_*}(t-t_0)\right)v\frac{\partial}{\partial v}.$$

The invariants are

$$V = v\left(\frac{t+x_0}{t_0+x_0}\right)^{-\frac{2}{q_*}} \exp\left(\frac{1}{2}(x-x_0)\right) \text{ and } \mu = \frac{(t-t_0)(x-x_0)}{(t+x_0)(x+t_0)}.$$

Hence, the invariant solution is of the form

$$v = \left(\frac{t+x_0}{t_0+x_0}\right)^{\frac{2}{q_*}} \exp\left(\frac{1}{2}(x_0-x)\right)V(\mu), \quad \mu = \frac{(t-t_0)(x-x_0)}{(t+x_0)(x+t_0)}. \tag{21}$$

The Goursat problem (19)-(20) is then reduced to the solution of the ODE

$$\mu(1-\mu)V'' + \left[1-\left(1-\frac{2}{q_*}\right)\mu\right]V' = 0 \tag{22}$$

with the condition $V(0)=1$. The above ODE (22) can be factorized as

$$\left((1-\mu)\frac{d}{d\mu} + \frac{2}{q_*}\right)\left(\mu\frac{d}{d\mu}\right)V = 0.$$

Example 3.1 We consider the equation

$$u_{tx} + xu_t + xu_x + x^2u = 0. \tag{23}$$

We have $h=0$, $k=1$ and $p=0$, $q=0$. So we can invoke Theorem 3.1. The adjoint equation has the form

$$v_{tx} - xv_t - xv_x + (x^2-1)v = 0 \tag{24}$$

together with the boundary conditions (13)

$$v|_{t=t_0} = \exp\left(\frac{1}{2}(x^2 - x_0{}^2)\right), \qquad v|_{x=x_0} \exp[x_0(t-t_0)]. \tag{25}$$

The symmetry generator of (24) is given by

$$X = (C_1 t + C_2)\frac{\partial}{\partial t} - (-C_1 x + C_3)\frac{\partial}{\partial x} + ((t+x)C_3 - C_1 x^2 + C_4)v\frac{\partial}{\partial v} \tag{26}$$

and contains arbitrary constants C_1, C_2, C_3 and C_4. By following the same procedure to calculate these constants as we have done in the Theorem 3.1, the Goursat problem (24)–(25) admits the one-parameter group generated by

$$X = (t - t_0)\frac{\partial}{\partial t} - (x - x_0)\frac{\partial}{\partial x} + [x_0(t-t_0) - x(x-x_0)]v\frac{\partial}{\partial v}. \tag{27}$$

This group has two functionally independent invariants

$$V = v\exp\left[(t_0 - t)x_0 + \frac{1}{2}(x_0^2 - x^2)\right], \quad \mu = (t-t_0)(x-x_0).$$

Hence, the invariant solution can be written as

$$v = V(\mu)\exp\left[(t - t_0)x_0 + \frac{1}{2}(x^2 - x_0^2)\right].$$

Substitution into the equation (24) gives

$$\mu V'' + (1-\mu)V' = 0, \tag{28}$$

with the condition $V(0) = 1$ from the characteristic data (25). The above equation (28) can be written in factorizable form

$$\left(\frac{d}{d\mu} - 1\right)\left(\mu\frac{d}{d\mu}\right)V = 0$$

which is easily solvable.

4. COMMENTS

We have provided and proved the theorem to obtain the Riemann function for the Goursat problem when the hyperbolic equation is factorizable. In this case, it was noted that the ODE whose solution gives the Riemann function for the Goursat problem under consideration, is in factorizable form and easily lends itself to solution.

We have also observed that one cannot construct the Riemann function for the Goursat problem by the Lie theory method when the hyperbolic equation admits less than a four-dimensional Lie algebra of point symmetries.

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