...atical & Computational Applications, Vol. 4, No. 2, pp. 189-194, 1999
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ON SOME EXTENSIONS OF DYNAMICAL HOMEOMORPHISM AND DYNAMICAL ISOMORPHISM IN DYNAMICAL SYSTEM

Necdet BİLDİK
Celal Bayar University, Faculty of Arts and Sciences, Department of Mathematics, Manisa -TURKEY

Mustafa İNÇ
Fırat University, Faculty of Arts and Sciences, Department of Mathematics, Elazığ-TURKEY

Abstract- In this paper, the definitions and the theorems concerning with the invariant sets, w- and α- limit sets are given and proved in the new Dynamical Systems. By using of the definition of P^+ and P^- stability , some remarkable unknown results are obtained.

Finally, the definition of Dynamical Isomorphism and Dynamical Homeomorphism are given and related original theorems are presented and proved.

INTRODUCTION

A phase transformation on a metric space X is defined to be mapping $\pi : X \times I_* \to X$ where I_* is the usual topological group of integers subject to the conditions **[3]**:

i) [Identity property]

$$\pi(x, 0) = x \quad \text{for all } x \in X.$$

ii) [Group property]

$$\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2) \text{ for all } x \in X \text{ and all } t_1, t_2 \in I_*.$$

iii) π is continuous.

If X is a metric space, I_* is a topological group of integers and π is a phase transformation, then the system (X, I_*, π) is called a dynamical system.

THEOREM 1.

For each t in I_* , the mapping π^t is a homeomorphism of X onto X , that is a bijection

PROOF:

The group property (ii) implies that $(\pi^t)^{-1} = \pi^{-t}$, and therefore both π^t and $(\pi^t)^{-1}$ are continuous by the continuity property (iii). Furthermore, π^t is an injection. Indeed let π be a continuous flow on X and let define $\pi^t(x) = \pi(x, t)$. Let also $\pi(x_1, t) = \pi(x_2, t)$ for any $x_1, x_2 \in X$. Then by the group property one has

$$\pi(\pi(x_1, t), -t) = \pi(x_1, t - t) = \pi(x_1, 0) = x_1$$

and

$$\pi(\pi(x_2, t), -t) = \pi(x_2, t - t) = \pi(x_2, 0) = x_2.$$

Thus this implies that $x_1 = x_2$ because of the definition

$$\pi(\pi(x_1, t), -t) = \pi(\pi(x_2, t), -t).$$

Finally, π^t is surjective for if $y \in X$, then $y = \pi^t(x)$ where x is given by $x = \pi(y, -t)$. Namely,

$$\pi^t(x) = \pi(x, t) = \pi(\pi(y, -t), t) = \pi(y, t - t) = \pi(y, 0) = y.$$

THEOREM 2.

For each x in X , the mapping π_x is a homeomorphism of X onto X , that is a bijection.

PROOF : The proof of the theorem is similar to the Theorem 1.

DEFINITION 2.

Let (X_1, I_*, π_1) and (X_2, I_*, π_2) be two dynamical systems. $f : X_1 \rightarrow X_2$ is called a dynamical homeomorphism providing

$$f(\pi_1(x, t)) = \pi_2(f(x), t) \text{ for all } x \in X \text{ and } t \in I_*$$ **[2]**.

If f is a homeomorphism providing $f(xt) = f(x)t$ then f is called a dynamical isomorphism, the systems (X_1, I_*, π_1) and (X_2, I_*, π_2) are then said to be Isomorphic Dynamical Systems **[2]**.

DEFINITION 3.

A point x in X is said to be critical point for the Dynamical System if $\pi(x, t) = x$ for all t or a point x in X is said to be critical point of the motion if

$$\lim_{n \to \infty} \pi(x, t_n) = x$$

DEFINITION 4.

Let a motion $\pi(x, t)$ be given in the metric space X. We consider a certain positive half-trajectory $\pi(x; 0, +\infty)$. We take any increasing sequence of values of t :

$$0 \leq t_1 < t_2 < \ldots < t_n < \ldots \quad \lim_{n \to \infty} t_n = +\infty$$

If the sequence of points

$$\pi(x, t_1), \pi(x, t_2), \pi(x, t_3), \ldots, \pi(x, t_n), \ldots$$

has a limit point y then we shall call this point a w-limit point of the motion $\pi(x, t)$. The sets of the w- limit point is defined a w-limit set and denoted as Ω_x. Namely w- limit sets is described as

$$\Omega_x = \{y \in X : y = \lim \pi(x, t_n) \text{ for some sequence } (t_n) \text{ with } t_n \rightarrow +\infty\}.$$

Analogously , any limit point y of a negative half - trajectory $\pi(x; -\infty, 0)$ is called an α- limit point of the motion $\pi(x, t)$ **[4]**.

The sets of the α- limit point is defined an α- limit set and denoted as A_x. Namely α- limit sets is described as

$$A_x = \{y \in X : y = \lim \pi(x, t_n) \text{ for some sequence } (t_n) \text{ with } t_n \rightarrow -\infty\}$$ **[5]**.

THEOREM 3.

Let f be a homeomorphism , $t_0 \in R$ be constant. If $y = \pi(x, t_0)$, then $\Omega_y = \Omega_x$.

PROOF:

Let $z \in \Omega_x$. We need to show $z \in \Omega_y$. Since $z \in \Omega_x$, then there exist increasing sequence (t_n) with $t_n \to +\infty$ such that

$$z = \lim_{n\to\infty} \pi(x, t_n).$$

If we apply the function f to the both sides of the equation $z = \lim_{n\to\infty} \pi(x, t_n)$, then

$$f(z) = \lim_{n\to\infty} \pi(f(x), t_n)$$

is found. Since f is a homeomorphism, then

$$f(z) = f\left(\lim_{n\to\infty} \pi(x, t_n)\right).$$

Substituting $x = \pi(y, -t_0)$, then

$$f(z) = f\left(\lim_{n\to\infty} \pi(\pi(y, -t_0), t_n)\right)$$

is obtained. Using the group property one has

$$f(z) = f\left(\lim_{n\to\infty} \pi(y, t_n - t_0)\right)$$

Now let assume that $t_n - t_0 = t'_n$. Since $t'_n \to +\infty$ for $t_n \to +\infty$ and f is a homeomorphism, then

$$f(z) = \lim_{n\to\infty} \pi(f(y), t_n)$$

is found. Thus $z \in \Omega_y$.

On the other hand let us $z \in \Omega_y$. We need to show $z \in \Omega_x$. Since $z \in \Omega_x$, then there exist increasing sequence (t_n) with $t_n \to +\infty$ such that

$$z = \lim_{n\to\infty} \pi(y, t_n).$$

Applying the function f to the both sides of the equation $z = \lim_{n\to\infty} \pi(y, t_n)$, then

$$f(z) = \lim_{n\to\infty} \pi(f(y), t_n)$$

is obtained. Since f is a homeomorphism , then

$$f(z) = f\left(\lim_{n\to\infty} \pi(y, t_n)\right)$$

Substituting $y = \pi(x, t_0)$, then

$$f(z) = f\left(\lim_{n\to\infty} \pi(\pi(x, t_0), t_n)\right)$$
$$= f\left(\lim_{n\to\infty} \pi(x, t_n + t_0)\right).$$

Now let assume that $t_n + t_0 = t''_n$. Since $t''_n \to +\infty$ for $t_n \to +\infty$, then

$$f(z) = f\left(\lim_{n\to\infty} \pi(x, t''_n)\right)$$

is found. Since f is a homeomorphism

$$f(z) = \lim_{n\to\infty} \pi(f(x), t''_n)$$

is obtained. Thus $z \in \Omega_x$. This completes the proof of the theorem.

THEOREM 4.

If y is a critical point , then $\Omega_x = A_x = \{y\}$ such that f is a homeomorphism.

PROOF:

We know that y is a critical point providing $y = \lim_{n\to\infty} \pi(x, t_n)$. By Definition 3, we may write that $\pi(y, t_0) = y$. Since f is homeomorphism, then

$$f(x) = \pi(f(x), t_0).$$

On the other hand it is also known that

$$x = \pi(y, -t_0), \quad y = \pi(x, t_0)$$

by [1]. Then $y = \lim_{n\to\infty} \pi(x, t_n)$. Applying the function f to the both sides of the equation,

$$f(y) = f\left\{\lim_{n\to\infty} \pi(x, t)\right.$$

is found. Since f is a homeomorphism , then

$$f(y) = \lim_{n\to\infty} \pi(f(x), t_0) = f(x)$$

is obtained. Since f is homeomorphism, this implies that x = y. The second part of the proof of the theorem can be done similar.

Hence $\Omega_x = A_x = \{y\}$.

DEFINITION 5.

A point x is called positively stable according to POISSON (written stable P^+) if , for any neighborhood U of the point x and for any T > 0 , there can be found a value $t \geq T$ such that $\pi(x, t) \in U$. Analogously, if there can be found a $t \leq -T$ such that $\pi(x, t) \in U$ then the point x is negatively stable according to Poisson (P^-).

A point stable to Poisson both as $t \to +\infty$ and as $t \to -\infty$ is called (simply) stable according to Poisson (Stable P) [6].

THEOREM 5.

A point x is P^+- stable if and only if there exist an increasing sequence (t_n) with $\lim_{n\to\infty} t_n = +\infty$ such that $\lim_{n\to\infty} \pi(x, t_n) = x$.

PROOF:

Let a point x is P^+- stable. Then there can be found $t_n > 0$ with $\lim_{n\to\infty} t_n = +\infty$ for any sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots\ldots. > \varepsilon_n > \ldots\ldots with \lim_{n\to\infty} \varepsilon_n = 0$ such that

$$d(x, \pi(x, t_n)) < \varepsilon_n. \qquad (1)$$

Since $\lim_{n\to\infty} t_n = +\infty$ and by Equation (1) then it is clear that

$$\lim_{n\to\infty} \pi(x, t_n) = x.$$

Namely there exist a sequence (t_n) with $\lim\limits_{n\to\infty} t_n = +\infty$ such that $f(x) = \lim\limits_{n\to\infty} \pi(f(x), t_n)$

Conversely, if there exist an increasing sequence (t_n) with $\lim\limits_{n\to\infty} t_n = +\infty$, such that $\lim\limits_{n\to\infty} \pi(x, t_n) = x$.

Then it can be obtained directly that a point x is P^{+} - stable.

THEOREM 6.

If a point x is P^{+} - stable then every point of the trajectory $\pi(x; I_*)$ is also P^{+} - stable.

PROOF:

Consider an arbitrary point $\pi(x, t)$. of the trajectory. By properties (ii) and (iii) of a dynamical system we have

$$\lim_{n\to\infty} \pi(x, t + t_n) = \pi(x, t)$$

[1] i.e., the point $\pi(x, t)$ is P^{+} - stable. Since for every point of $\pi(x, t)$ is P^{+} - stable, then P^{+} can obviously be written thus:

$$\pi(x; I_*) \subset \overline{\pi(x; 0, +\infty)} ;$$

the condition for stability

$$P^{-} : \pi(x; I_*) \subset \overline{\pi(x; -\infty, 0)} .$$

Alternatively we can say: $x \in \Omega_x$ or $x \in A_x$ (where $\overline{\pi(x; -\infty, 0)}$ is the closure of the negative semi-trajectory).

THEOREM 7.

If the motion $\pi(x, t)$ is P^{+} - stable, then $\Omega_x = \overline{\pi(x, I_*)}$.

PROOF:

Let the motion $\pi(x, t)$ is P^{+} - stable. Then by Theorem 6, all points of its trajectory are w-limits for it i.e., $\pi(x; I_*) \subset \Omega_x$

Since Ω_x is a closed set, from the last inclusion there follows

$$\overline{\pi(x; I_*)} \subset \Omega_x$$

On the other hand the relation holds

$$\Omega_x \subset \overline{\pi(x; 0, +\infty)} , \quad A_x \subset \overline{\pi(x; -\infty, 0)} \tag{2}$$

since the closure of a semi - trajectory contains all its limit points.

Comparing this with the inverse inclusion (2) , which always holds , we have for a motion stable P^{+} : $\Omega_x = \overline{\pi(x; I_*)}$. Similarly if the motion $\pi(x, t)$ is P^{-} - stable then it is easy to show that

$$A_x = \overline{\pi(x; I_*)} .$$

THEOREM 8.

If the motion $\pi(x, t)$ is P^-- stable then

$$\Omega_x = A_x = \overline{\pi(x; I_*)}.$$

PROOF:

If the motion π (x , t) is P^- - stable, then by Theorem 7

$$\Omega_x \subset \overline{\pi(x; I_*)}, \quad \text{and} \quad \overline{\pi(x; I_*)} \subset A_x$$

is obtained. Since $A_x = \overline{\pi(x; I_*)}$, then

$$\Omega_x \subset A_x = \overline{\pi(x; I_*)} \tag{3}$$

On the other hand , If the motion $\pi(x, t)$ is P^-- stable, then by Theorem 7,

$$\overline{\pi(x; I_*)} \subset \Omega_x, \quad \text{and} \quad A_x \subset \overline{\pi(x; I_*)}$$

is obtained. Since $\Omega_x = \overline{\pi(x; I_*)}$, then

$$A_x \subset \Omega_x = \overline{\pi(x; I_*)}. \tag{4}$$

Therefore $\Omega_x = A_x = \overline{\pi(x; I_*)}$ are found by Equation 3 and Equation 4. This completes the proof of the theorem.

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