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AN EFFECTIVE ALGORITHM FOR EULER SYSTEM IN A CLASS OF DISCONTINUOUS FUNCTIONS

Mahir Rasulov and Erhan Coşkun

Department of Mathematics, Karadeniz Technical University, 61080, Trabzon, Turkey

Abstract- In this article, an effective algorithm is proposed for solving a Cauchy problem for Euler system in a class of discontinuous functions. To this end, an auxiliary system which is equivalent to, but has more advantages then, Euler system is introduced. The auxiliary system requires less smoothness assumptions on the solution, hence, is better for the numerical applications.

1. INTRODUCTION

Let $\Re^3$ denote the usual Euclidean space of points (x, y, t), and $\Re^3_+$ the subspace of $\Re^3$ with t>0. We consider the Euler system of equations defined on $\Re^3_+$ which models the flow of incompressible, irrotational and low viscosity fluid subject to a constant pressure[1],[3],[4]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=0\,, \tag{1.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=0, \tag{1.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{1.3}$$

where u, v are the components of the velocity vector in the x- and y- directions, respectively. We consider the Cauchy problem for (1.1)-(1.3)

$$u(x,y,0)=u_0(x,y)\,, \tag{1.4}$$

$$v(x,y,0)=v_0(x,y)\,. \tag{1.5}$$

By use of the method of characteristics, the solution of the problem (1.1)-(1.5) can be given as

$$u(x,y,t)=u_0(\xi,\eta)\,, \tag{1.6}$$

$$v(x,y,t)=v_0(\xi,\eta)\,. \tag{1.7}$$

Here, the relations for the characeristics may be given as

$$\xi = x-ut, \quad \eta = y-vt\,. \tag{1.8}$$

When the Jacobian is different from zero, i.e.,

$$\frac{D(u_0, v_0)}{D(\xi, \eta)} \neq 0,$$

at the roots $t = t_1$, and $t = t_2$ of the polynomial

$$\Delta = 1 + t\left(\frac{\partial u_0}{\partial \xi} + \frac{\partial v_0}{\partial \eta}\right) + t^2 \frac{D(u_0, v_0)}{D(\xi, \eta)}$$

the derivatives of the functions $u(x, y, t)$ and $v(x, y, t)$ with respect to all variables approaches to positive infinity. In another words, the problem (1.1)-(1.5) does not have a classical solution. In addition, from the relations (1.6) and (1.7) , being the closed form expressions for the solution, an expilicit form of the solution can't be obtained in general. For this reason, one has to resort to numerical techniques for solving the problem (1.1)-(1.5).

If the initial functions $u_0(x, y)$ and $v_0(x, y)$ have both positive and negative slopes, at the points where $u > 0$ and $v > 0$ the solution will grow according to the the magnitute of these quantities.

Therefore, at the first point when $t > \min\{t_1, t_2\}$, there will be a breaking. This then will raise the issue of determinining the physically meaningful solution.

2. AN AUXILIARY PROBLEM

In some instances, it is possible to render a single-valued solution having first type of discontinuities from the multi-valued continuous solution. To this end, one has to generalize the concept of classical solution to the so-called weak solution. In order to determine the weak solution of the problem (1.1)-(1.5), first we introduce the operator defined as

$$M(t) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

It is easy to see that the kernel of this operator is given by

$$KerM = \{h = w[\psi(x, y)], \quad w \in C^1(Q)\}$$

where $w-$ is an arbitrary function and $\psi(x, y)$ is a stream function satisfying the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Theorem 1 If the functions u and v satisfy the equation(1.3), then the operator M^{-1} is independent of time.

Indeed, we have

$$0 = M\varphi = u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial y} = 0.$$

Here w - is a first order differentiable function and $\varphi = w[\psi(x,y)]$. If $M\varphi = h$, then from the Lagrange system

$$\frac{dx}{u} = \frac{dy}{v} = \frac{d\varphi}{h}$$

we obtain $\psi(x,y) = c$, $\varphi = c_1$, and the general solution $\varphi = w[\psi(x,y)]$.

To determine the weak solution of the problem, as in [5], we consider the following auxiliary problem.

$$\frac{\partial \pi(x,y,t)}{\partial t} + u = h(x,y), \tag{2.1}$$

$$\frac{\partial \phi(x,y,t)}{\partial t} + v = h(x,y). \tag{2.2}$$

The initial conditions for the equations (2.1) and (2.2) are

$$\pi(x,y,0) = \pi_0(x,y), \tag{2.3}$$

$$\phi(x,y,0) = \phi_0(x,y). \tag{2.4}$$

Here $h(x,y) \in \ker M$ is an arbitrary function, and the arbitrary functions $\pi_0(x,y)$, $\phi_0(x,y)$ are such that

$$u_0 \frac{\partial \pi_0}{\partial x} + v_0 \frac{\partial \pi_0}{\partial y} = u_0(x,y),$$

$$u_0 \frac{\partial \phi_0}{\partial x} + v_0 \frac{\partial \phi_0}{\partial y} = v_0(x,y).$$

Teorem 2 If the functions π and ϕ are the solutions of the problem (2.1)-(2.4), then the functions defined by

$$u = M_0\pi, \quad v = M_0\phi \text{ with } M_0 = M(0)$$

are the solutions of the problem (1.1)-(1.5).

The auxiliary problem has the following advantages.

(i) The equations (2.1) and (2.2) do not involve derivatives of any order for the functions u and v.

(ii) The functions π, ϕ are much smoother than u, v, more precisely, if the functions u,v are differentiable to the order of k, then π,ϕ are differentiable to the order of $k+1$, $(0 \le k < 1)$.

These advantages makes it possible to approximate the problem (2.1)-(2.4) by finite differences.

Discretizing the problem (2.1)-(2.4) by finite differences, we have [2],[5]

$$\Pi_{i,j,k+1} = \Pi_{i,j,k} - U_{i,j,k} + h(x_i, y_j),$$
$$\Phi_{i,j,k+1} = \Phi_{i,j,k} - V_{i,j,k} + h(x_i, y_j), \quad (k = 0,1,2,\dots)$$
$$\Pi_{i,j,0} = \pi_0(x_i, y_j),$$
$$\Phi_{i,j,0} = \phi_0(x_i, y_j).$$

Theorem 3 For any i, j, k, the identities

$$U_{i,j,k+1} = U_{i,j,k}\Pi_x + V_{i,j,k}\Pi_y,$$

$$V_{i,j,k+1} = U_{i,j,k}\Phi_x + V_{i,j,k}\Phi_y$$

hold.

As can be seen, the algorithm given above is rather compact and economical.

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