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SOME RESULTS ON A STOCHASTIC PROCESS WITH A DISCRETE CHANCE INTERFERENCE

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Abstract- In this study, some problems connected with stochastic process with a discrete chance interference X(t) are investigated. In particular, one-dimensional distribution functions of the process are obtained and under some weak assumptions, the ergodic theorem for this process is given. As a result, the explicit form of ergodic distribution function is derived. Moreover, the double transform of distribution function of additive functional of the process X(t) is derived. Furthermore, asymptotic behaviour of the additive functional is investigated as $t \to \infty$. Based on these results characteristic function of ergodic distribution of the process $X(t)$ is obtained by using a joint distribution of random variables N and S_N. In addition, the first and second moments of ergodic distribution of $X(t)$ are expressed in terms of the moments of random variable N.

1. INTRODUCTION

It is known that a number of very interesting problems in the fields of queues, reliability, stock control theories, mathematical biology etc. are often given by means of the stochastic processes with a discrete chance interference. Many good monographs and articles in this field exist in literature (see, for instance, [1]-[17]).

But most of these studies are dedicated to some boundary functional of these stochastic processes. It is well known that boundary functional plays an essential role in the study of probabilistic problems connected with the random walks. Additive functional is as important as the boundary functional but unfortunately, they are not studied enough in the literature. Because of that, in this study, a stochastic process with a discrete chance interference is constructed and additive functional of this process is considered.

2. CONSTRUCTION OF THE PROCESS AND ITS ADDITIVE FUNCTIONAL

Let $\{(\xi_i, \eta_i, \zeta_i)\}$, $(i = 1,2,\cdots)$ be a sequence of independent and identically distributed triples of random variables defined on any probability space $(\Omega, \Im, P)$ such that ξ_i, η_i, ζ_i are independent random variables, where ξ's and ζ_i's take on positive values.

Suppose that the distribution functions of ξ_1, η_1 and ς_1 are known, i. e.,

$$\Phi(t) = P\{\xi_1 \le t\},\ \ F(x) = P\{\eta_1 \le x\},\ \ \pi(z) = P\{\varsigma_1 \le z\};\ t, z \ge 0,\ \ x \in (-\infty, \infty).$$

Define renewal process $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i\ ,\ S_n = \sum_{i=1}^{n} \eta_i\ ,\ n = 1,2,\cdots,\ T_0 = S_0 = 0,$$

and a sequence of integer valued random variables N_n as

$$N_{n+1} = \min\{k \geq N_n + 1 : \zeta_n + S_k - S_{N_n} \leq 0\ ,\ n = 1,2,\cdots,\ N_0 = 0\ ,\ \zeta_0 = z > 0\ .$$

Let $\tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i$, $n = 1,2,\cdots$, $\tau_0 = 0$ and $\nu(t) = \max\{n \geq 0 : T_n \leq t\}$.

We can now construct desired stochastic process $X(t)$ as follows

$$X(t) = \zeta_n + S_{\nu(t)} - S_{N_n},\ \text{if}\ \ \tau_n \leq t < \tau_{n+1},\ n = 0,1,2,\cdots,\ \zeta_0 = z > 0\ .$$

We call this process the stochastic process with a discrete chance interference.
We define

$$J_f(t) = \int_0^t f(X(u))du\ ,$$

where $f(x)$ is a measurable bounded function defined on the interval $[0,\infty)$. Here $J_f(t)$ is called an additive functional of the process $X(t)$.

In this study, some problems related to X(t) and $J_f(t)$ are investigated.

NOTATIONS. The following notations will be used throughout this study.

$\Phi_n(t) = \Phi^{*n}(t), n \geq 1, \Phi_0(t) = \varepsilon(t) = 1$ if $t \geq 0$, $\varepsilon(t) = 0$ if $t < 0$.

$\Delta\Phi_n(t) = \Phi_{n-1}(t) - \Phi_n(t),\ n \geq 1$.

$a_n(x,z) = P\{z + S_i \geq 0,\ i = \overline{1,n},\ z + S_n \leq x\},\ n \geq 1,\ a_0(x,z) = \varepsilon(x - z)$.

$b_n(z) = P\{z + S_i \geq 0, i = \overline{1, n-1}, z + S_n < 0\}, n \geq 1, b_0(z) = 0$.

$A(x,z) = \sum_{n=0}^{\infty} a_n(x,z)$.

For arbitrary measurable bounded functions $M_i(t,x,z),\ (i = 1,2)$ let

$$M_i(t,x,*) = \int_0^{\infty} M_i(t,x,z)d\pi(z),\ M_1(t,x,z) * M_2(t,x,z) = \int_0^t M_2(t-s,x,z)d_sM_1(s,x,z).$$

For the function $M_i(t,x,z),\ (i = 1,2)$, $\tilde{\tilde{M}}_i(\lambda,\mu,z)$ $(M_i^{**}(\lambda,\mu,z))$ is the Laplace-Fourier double transform (Laplace-Stieltjes and Fourier- Stieltjes) with respect to t and x, respectively.

3. THE DISTRIBUTION FUNCTIONS OF THE PROCESS $X(t)$ AND ADDITIVE FUNCTIONAL $J_f(t)$

Let us define

$Q(t,x,z) = P_z\{X(t) \leq x\} \equiv P\{X(t) \leq x / X(0) = z\}$

and

$Q_f(t,x,z) = P_z\{J_f(t) \leq x\}$, where $0 < t, x, z < \infty$.

We can formulate the main results of this section as follows.

THEOREM 3.1. One dimensional distribution function $Q(t,x,z)$ of the process $X(t)$ is expressed by the probability characteristics of the renewal process $\{T_n\}$ and random walk $\{S_n\}$ as

$$Q(t,x,z) = G(t,x,z) + R(t,z) * U(t) * G(t,x,*),$$

where

$$G(t,x,z)=\sum_{n=0}^{\infty}a_n(x,z).\Delta\Phi_n(t);\quad R(t,z)=\sum_{n=1}^{\infty}b_n(z)\Phi_n(t);$$

$$R(t)=R(t,*)=P\{\tau_1\le t\};\quad U(t)=\sum_{n=0}^{\infty}R^{*n}(t).$$

PROOF. It is easily seen that the random variables $\tau_1,\tau_2,\ldots$ defined above are formed a sequence of stopping moments for a process X(t). Besides under the assumptions of Theorem 3.1 $\{\varsigma_n\}$, (n = 1,2,...) is a sequence of independent and identically distributed random variables. Considering these properties, the following integral equation of renewal type for one dimensional distribution function of X(t) can be written as

$$Q(t,x,z)=G(t,x,z)+R(t,z)*Q(t,x,*).$$

Now solving this integral equation of renewal type (see, for instance, [4], p.351), we obtain the final expression for $Q(t,x,z)$. This completes the proof. ♦

THEOREM 3.2. The Laplace-Fourier double transform of one-dimensional distribution functions of additive functional $J_f(t)$ of process $X(t)$ has the following form

$$\tilde{\tilde{Q}}_f(\lambda,\mu,z)=\tilde{\tilde{G}}_f(\lambda,\mu,z)+\frac{R_f^{**}(\lambda,\mu,z)\tilde{\tilde{G}}_f(\lambda,\mu,*)}{1-R_f^{**}(\lambda,\mu,*)},$$

where $G_f(t,x,z)$ and $R_f(t,x,z)$ are expressed in terms of the probability characteristics of $\{T_n\}$ and $\{S_n\}$ as follows

$$G_f(t,x,z)=\sum_{n=0}^{\infty}P\left\{z+S_k\ge 0,k=\overline{1,n},T_n\le t<T_{n+1},\sum_{k=1}^{n}f(z+S_{k-1})\xi_k+(t-T_n)f(z+S_n)<x\right\},$$

$$R_f(t,x,z)=\sum_{n=1}^{\infty}P\left\{z+S_k\ge 0,k=\overline{1,n-1};z+S_n<0,T_n\le t,\sum_{k=1}^{n}f(z+S_{k-1})\xi_k<x\right\}.$$

PROOF. It is not difficult to see that some integral equation of renewal type for distribution functions $Q_f(t,x,z)$ of additive functional $J_f(t)$ of process X(t) can be formulated. In such formulated integral equation, applying the Laplace-Fourier double transform with respect to t and x, respectively, we can obtain the final expression for $\tilde{\tilde{Q}}_f(\lambda,\mu,z)$. Here $G_f(t,x,z)$ and $R_f(t,x,z)$ may be expressed by the probability characteristics of renewal process $\{T_n\}$ and random walk $\{S_n\}$. This completes the proof. ♦

In Theorem 3.2, we obtained an exact expression for a double transform of one-dimensional distribution functions of $J_f(t)$. But it is very difficult to use this formula in practical needs. For this reason, the features of $J_f(t)$ need to be explored in detail. Further study of $J_f(t)$ is closely related to the study of ergodic property of process X(t). That is, if the process $X(t)$ is ergodic then the ratio $J_f(t)/t$ exists as $t\to\infty$ and the limit of this ratio is not random with probability 1. Moreover, under some weak conditions the process $X(t)$ is ergodic. Let's state this result.

THEOREM 3.3. Let the initial sequence of random pairs (ξ_n,η_n) satisfy the following supplementary conditions

1) $E\xi_1<\infty$; 2) $E\eta_1<0$; 3) η_1 is not an arithmetic random variable.

Then the process $X(t)$ is ergodic and for any measurable bounded function $f(x)$ defined on the interval $[0,\infty)$ the following relation holds

$$\lim_{t\to\infty}\frac{J_f(t)}{t}\equiv S_f=\frac{1}{A(\infty;*)}\int_0^\infty f(x)d_xA(x,*).$$

PROOF. The ergodic theorem of Smith's "key renewal theorem" type exists in the literature for a general class of processes with a discrete chance interference (see, for instance, [5], p.243). But to verify the assumptions of such theorems are satisfied is rather difficult in the concrete cases. Therefore, it is desired one consider a restricted class of these processes in order to offer weaker and obvious conditions for the ergodicity of such processes.

For this reason, in this study, we consider a special class of processes with a discrete chance interference and offer the ergodicity conditions for this class which are weak enough and obvious. It is required to show that under the conditions (1)-(3) of Theorem 3.3 all assumptions of the ergodic theorem (see, [5], p.243) are satisfied mentioned earlier.

For this purpose, it is necessary to verify the following two assumptions.

Assumption 1. A chose of a sequence of ascending random epochs is required such that the values of the process at these moments form an imbedded Markov chain which is ergodic and has a stationary distribution.

For this aim it suffices to consider the sequence of random moments $\tau_1,\tau_2,...$ defined above, which are the stopping times. On the other hand, the values of process $X(t)$ at these points $\aleph_n=X(\tau_n+0)$, $n\geq 1$, form an imbedded Markov chain.

Since $X(\tau_n+0)=\varsigma_n$, $n\geq 1$, is a sequence of independent and identically distributed random variables then the imbedded Markov chain $\{\aleph_n\}$ is ergodic with a stationary distribution function $\pi(z)=P\{\varsigma_1<z\}$. Therefore, the first assumption of general ergodic theorem is satisfied.

Assumption 2. The finiteness of expectation of τ_1 (i.e., $E\tau_1<\infty$) is required.

If a drift of process is negative (i.e., $E\eta_1<0$) then a finiteness of expectation of random variable τ_1 is derived from the studies of A. A. Borovkov [2], T. I. Nasırova [13] and etc.

Thus, under the conditions of Theorem 3.3, all assumptions of general ergodic theorem for processes with a discrete chance interference are satisfied. Therefore, the process $X(t)$ is ergodic. This completes the proof. ♦

Substituting the indicator-function for the set $(-\infty,x]$ instead of $f(x)$, from Theorem 3.3 the following Corollary is derived immediately.

COROLLARY. The explicit form of ergodic distribution function of $X(t)$ is given as

$$Q(x)=\lim_{t\to\infty}P\{X(t)\leq x\}=\frac{A(x,*)}{A(\infty,*)},\ x\in[0,\infty).$$

4. CONNECTION BETWEEN PROBABILITY CHARACTERISTICS OF PROCESS $X(t)$ AND RANDOM PAIR (N, S_N)

It is not easy to calculate $A(x,z)$, but a great number of papers has investigated certain boundary functional of random walk $\{S_n\}, n\geq 1$. For this reason, we will try to express the probability characteristics of $X(t)$ by means of a joint distribution function of a certain random pair (N, S_N), defined below. According to W. Feller (see, for instance, [4], p.598-600), we consider the random walk $\{S_n\}, n\geq 1$, with initial state $S_0=0$ in the interval $B=(-z,\infty)$, where z is any fixed positive real number.

Let $N(z)$ be a first moment of the exit of the random walk $\{S_n\}, n \geq 1$ from the interval B, i.e. $N(z) = \min\{n \geq 1 : \; S_n \leq -z\}$, $z > 0$ In the below, sometimes $N(z)$ shown as N for simplicity. We need to investigate a joint distribution of the pair (N, S_N). For this purpose we define

$$d_n(I, z) = P\{N = n, S_N \in I\}, \quad \text{if } I \subset B' \text{ and } d_n(I, z) = 0, \quad \text{if } I \subset B, n = 0,1,2,... \tag{4.1}$$

where B' is the complement of B.

From the definition of $N(z)$ it is seen that

$$d_n(I, z) = P\{S_k \in B, k = \overline{1, n-1}, S_n \in B', S_n \in I\}, \quad \text{if } I \subset B'. \tag{4.2}$$

The probabilities $d_n(I, z)$ will be called hitting probabilities. The study of $d_n(I, z)$ is closely connected with the study of the random walk prior to the first entry into B', that is, the random walk is restricted to B. For

$$I \subset B = (-z, \infty)$$

and $n = 1, 2, \cdots$ let

$$a_n(I, z) = P\{S_1 \in B; S_2 \in B; ...; S_n \in B; S_n \in I\}, \tag{4.3}$$

in other words, this is the probability that at epoch n the set $I \subset B$ is visited and up to epoch n no entry into $B' = (-\infty, -z]$ took place. We extend this definition to all sets on the line by letting $a_n(I, z) = 0$ if $I \subset B', z > 0$.

We are concerned with the distribution of the pair (N, S_N). Since N is integral-valued we use generating functions for N and characteristic functions for S_N. Accordingly, we put

$$\tilde{d}^*(s, \theta, z) = \sum_{n=0}^{\infty} s^n \int_{B'} e^{i\theta x} d_n(dx, z), \quad \tilde{a}^*(s, \theta, z) = \sum_{n=0}^{\infty} s^n \int_{B} e^{i\theta x} a_n(dx, z)$$

The zeroth terms of the two series are equal to 0 and 1, respectively. [illegible] for [illegible] (see, for instance, the C. Stein's lemma is given at [illegible]).

For any measurable bounded function $M(x, z)$ define

$$M^*(\theta, z) = \int e^{i\theta x} M(dx, z) \text{ and let's } \varphi(\theta) = E e^{i\theta\eta_1} = \int_{-\infty}^{\infty} e^{i\theta x} dF(x)$$

We have then been able to establish the basic identity

$$1 - \tilde{d}^*(s, \theta, z) = \tilde{a}^*(s, \theta, z)[1 - s\varphi(\theta)], \tag{4.4}$$

which establishes a relation between the probabilities $d_n(I, z)$ and $a_n(I, z)$. Note that much valuable information can be extracted directly from (4.4).

Now, we consider a function $\Psi(s, \theta)$ for a deeper investigation of S_t.

$$\Psi(s, \theta) = \int_0^{\infty} \pi(dz) \int_{x=0}^{\infty} e^{i\theta x} \sum_{n=0}^{\infty} s^n P\{z + S_k > 0, k = \overline{1, n}, z + S_n \in dx\}.$$

Using the basic identity (4.4) we get an important relationship

$$\Psi(s, \theta) = \int_0^{\infty} e^{i\theta z} \frac{1 - \tilde{d}^*(s, \theta, z)}{1 - s\varphi(\theta)} d\pi(z). \tag{4.5}$$

Taking the limit as $s \to 1$ in (4.5), we get

$$\Psi(1, \theta) = \int_0^{\infty} e^{i\theta z} \frac{1 - \tilde{d}^*(1, \theta, z)}{1 - \varphi(\theta)} d\pi(z). \tag{4.6}$$

By the definition of $\varphi(\theta)$ and $\tilde{d}^*(1, \theta, z)$, the formula (4.6) can be rewritten as follows

$$\Psi(1,\theta)=\int_0^\infty e^{i\theta z}\frac{E\{\exp(i\theta S_{N(z)})-1\}}{E\{\exp(i\theta\eta_1)-1\}}d\pi(z). \tag{4.7}$$

Using this relation, it is not difficult to establish a connection between characteristics of ergodic distribution of process $X(t)$ and random variable N.

Note that taking the limit as $\theta\to 0$ in (4.7) and taking into account the Wald's equation, we get an expression for the $\Psi(1,0)$, namely

$$\Psi(1,0)=\lim_{\theta\to 0}\Psi(1,\theta)=\int_0^\infty E(N(z))d\pi(z). \tag{4.8}$$

Let's denote the last integral by EN, that is, $EN=\int_0^\infty EN(z)d\pi(z)$.

Thus the characteristic function of process $X(t)$ as $t\to\infty$ has the form

$$\lim_{t\to\infty}E\exp(i\theta X(t))=\frac{1}{EN}\int_0^\infty e^{i\theta z}\frac{E\{\exp(i\theta S_{N(z)})-1\}}{E\{\exp(i\theta\eta_1)-1\}}d\pi(z). \tag{4.9}$$

Because of the practical importance below, we give only the explicit expression for the first and second order moments of ergodic distribution of $X(t)$.

5. STUDY OF THE FIRST AND SECOND ORDER MOMENTS OF ERGODIC DISTRIBUTION OF PROCESS X(t)

Let us denote $EX^k=\lim_{t\to\infty}EX^k(t)$, $m_k=E\eta_1^k$, k=1,2.

Our aim is to express of EX and EX^2 as the first and second order moments of the random variable $N(z)$. Now we state the following theorem.

THEOREM 5.1. Let the conditions of ergodic Theorem 3.3 and the condition $E|\eta_1|^3<\infty$ be satisfied. Then the first and second order moments of ergodic distribution of process $X(t)$ exist and can be expressed in terms of the $EN(z)$ and $EN^2(z)$ as follows

$$EX=\frac{1}{EN}\int_0^\infty\left\{\frac{m_1}{2}EN^2(z)+zEN(z)\right\}d\pi(z)-\frac{m_1}{2};$$

$$EX^2=\frac{1}{EN}\int_0^\infty\left\{\frac{m_1^2}{3}EN^3(z)+m_1zEN^2(z)+z^2EN(z)+\right.$$
$$\left.+\frac{m_2-m_1^2}{2}EN^2(z)-m_1zEN(z)-\frac{3m_2-4m_1^2}{6}EN(z)\right\}d\pi(z).$$

PROOF. Since $E|\eta_1|^3<\infty$, we can take the first and second order derivatives of both side of the formula (4.9). By taking the limit of the expressions for the derivatives as $\theta\to 0$ and carrying out the necessary calculations, we finally obtain the expressions for EX and EX^2.

This completes the proof.♦

One of the advantages of Theorem 5.1 is that the moments of random variable $N(z)$ can be expressed in terms of the probability characteristics of random variables, known as ladder height and moment. Those probability characteristics are well known in the literature (see, for instance, [4]). Relations between characteristics of $N(z)$ and ladder height and moment are given in the following theorem.

THEOREM 5.2. The first three moments of random variable $N(z)$ can be given as follows

$$E(N(z)) = \mu_1 U_+(z);$$
$$E(N(z))^2 = (\mu_2 - 2\mu_1^2)U_+(z) + 2\mu_1^2 U_+^{*2}(z)$$
$$E(N(z))^3 = 6\mu_1^3 U_+^{*3}(z) + 6\mu_1(\mu_2 - 2\mu_1^2)U_+^{*2}(z) + (\mu_3 - 6\mu_2\mu_1 + \mu_1^3)U_+(z)$$

where

$$\mu_k = E(v_1^+)^k,\ k = 1,2,3 \text{ and } U_+(z) = \sum_{n=0}^{\infty} F_+^{*n}(z),\quad F_+(z) = P\{\chi_1^+ \le z\}.$$

PROOF. From definition of random variable $N(z)$ the following identity is derived

$$P\{N(z) > z\} = P\{-S_n < z\},\ n \ge 0,\ z > 0.$$

Based on this identity it is easy to show that $N(z)$ can be represented by some renewal reward process (see, [15], p.318). In order to rewrite $N(z)$ in the form a renewal reward process, we introduce a sequence of independent and identically distributed pairs of positive valued random variables $(v_i^+, \chi_i^+), (i = 1,2,...)$. The first term in this sequence (v_1^+, χ_1^+) is called the first strict ascending ladder point for random walk $\{-S_n\}$, $(n \ge 0)$ (see, [4], p.391), where

$$v_1^+ = \min\{k \ge 1 : -S_n > 0\},\qquad \chi_1^+ = -S_{v_1^+}.$$

Let $\Gamma(z) = \min\left\{n \ge 1 : \sum_{i=1}^{n} \chi_i^+ \ge z\right\}.$

Note that $\Gamma(z)$ is a renewal process, which is formed by means of the positive valued random variables χ_n^+, $(n \ge 1)$.

It can be shown that using these notations, $N(z)$ may be rewritten in the following form:

$$N(z) = \sum_{i=1}^{\Gamma(z)} \chi_i^+.$$

Furthermore, applying the Wald's identity in this equality and expressing the first three moments of renewal process $\Gamma(z)$ by a renewal function $U_+(z)$, which is formed by means of a distribution function of the random variable χ_1^+, we get the final expressions for $E(N(z))^k$, $k = 1,2,3$. This completes the proof. ♦

Note that, the expressions for $E(\Gamma(z))$ and $E(\Gamma^2(z))$ may be found in a scientific literature (see, [4], p.386), but the expressions for $E(\Gamma^3(z))$ and $E(N^3(z))$ are new results.

CONCLUSIONS

In this study, a process called "A stochastic process with a discrete chance interference" is constructed and the following results are obtained for this process.

- One dimensional distribution functions of the process are expressed by means of the probability characteristics of initial sequence of random triples $\{(\xi_i, \eta_i, \varsigma_i)\}$, $i = 1,2,...$
- The ergodicity of the process is proved under some weak conditions.
- An explicit form of ergodic distribution of process is obtained.
- The Laplace-Fourier double transform of distribution functions of additive functional is calculated.
- A characteristic function of ergodic distribution is expressed by means of the probability characteristics of random variables N and S_N.
- Finally, the first and second moments of ergodic distribution of process are calculated.

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