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# On the Fluctuations of Internal DLA on the Sierpinski Gasket Graph

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**Abstract:** Internal diffusion limited aggregation (IDLA) is a random aggregation model on a graph  $G$ , whose clusters are formed by random walks started in the origin (some fixed vertex) and stopped upon visiting a previously unvisited site. On the Sierpinski gasket graph, the asymptotic shape is known to be a ball in the graph metric. In this paper, we improve the sublinear bounds for the fluctuations known from its known asymptotic shape result by establishing bounds for the odometer function for a divisible sandpile model.

**Keywords:** internal DLA; Sierpinski gasket graph; fluctuations; growth model; Green function

## 1. Introduction

The *internal diffusion limited aggregation model (IDLA)*, which was introduced by Diaconis and Fulton in [1], is a stochastic aggregation model grown by consecutively started particles, where a particle is added upon exiting the current cluster. Let  $G$  be an infinite but locally finite connected graph with a specified vertex  $\circ$  acting as the origin of these particles. Let  $(X^1(t))_{t \geq 0}, (X^2(t))_{t \geq 0}, \dots$  be a sequence of independent simple random walks on  $G$  started in  $\circ$ , representing the particles. The IDLA cluster  $\mathcal{I}(i)$  after  $i \geq 0$  particles is now iteratively defined as

$$\mathcal{I}(0) = \emptyset, \quad \mathcal{I}(i) = \mathcal{I}(i-1) \cup \{X^i(\sigma^i)\}$$

where  $\sigma^i = \inf\{t \geq 0 | X^i(t) \notin \mathcal{I}(i-1)\}$  is the first time that particle  $i$  leaves the existing cluster. So, the first particle stops immediately at  $\circ$  and  $\mathcal{I}(1) = \{\circ\}$ . After  $i$  particles, the cluster contains exactly  $i$  vertices, i.e.,  $|\mathcal{I}(i)| = i$ . Notice that  $(\mathcal{I}(i))_{i \geq 0}$  is a Markov chain on the connected subsets of  $G$ . An important question concerning IDLA is the typical shape of the random set  $\mathcal{I}(i)$  for large  $i$ . On  $\mathbb{Z}^d$ , Lawler, Bramson and Griffeath [2] identified the limit shape as a Euclidean ball, and later in [3], Lawler improved the bounds for the fluctuations. Finally, later in [4,5], these bounds were further improved to sublogarithmic bounds for  $d \geq 3$  and logarithmic ones for  $d = 2$  in [6,7] via two different approaches. Convergence to a scaling limit, that is, rescaling the graph of  $\mathbb{Z}^d$  and taking the limit of clusters started from a particle distribution rather than a single source, is shown in [8] for IDLA and two other aggregation models. For finitely generated groups having exponential growth in [9], the authors prove a shape theorem with a suitable metric. For the special case of homogeneous trees, the authors also give lower bounds for the fluctuations from this basic shape. In [10], the author proves an inner bound for IDLA on supercritical percolation clusters, whereas in [11], a corresponding outer bound (depending on the inner bound) is established. For the comb, i.e.,  $\mathbb{Z}^2$  without all horizontal edges except the  $x$ -axis, [12] show a basic shape result and in [13], fluctuations are established. On the cylinder graph, that is,  $\mathbb{Z}_N \times \mathbb{Z}$ , where  $\mathbb{Z}_N$  is the cyclic group with  $N$  elements, a basic shape result, the fluctuations, and the existence of a coupling between two IDLA chains are introduced in [14]. See also [15] for a survey on IDLA and its counterpart: external DLA, where random walks start from outside the cluster.



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Recently, some interest has developed in the study of aggregation models on fractals, namely on the double-sided Sierpinski gasket graph  $SG$ , which is defined as follows. Let

$$V_0 := \{(0,0), (1,0), (1/2, \sqrt{3}/2)\},$$

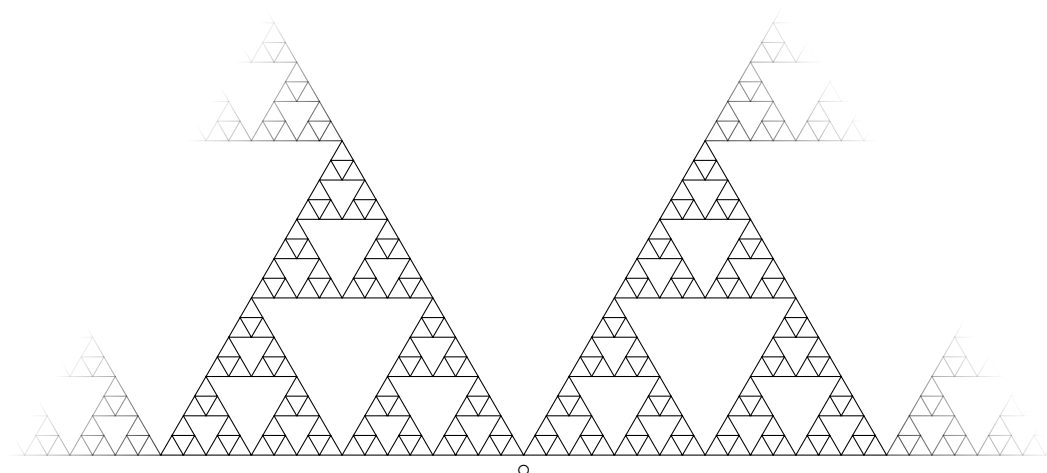
$$E_0 := \left\{ \left( (0,0), (1,0) \right), \left( (1,0), (1/2, \sqrt{3}/2) \right), \left( (0,0), (1/2, \sqrt{3}/2) \right) \right\}$$

and then recursively

$$V_{n+1} = V_n \cup \left( (2^n, 0) + V_n \right) \cup \left( (2^{n-1}, 2^{n-1}\sqrt{3}) + V_n \right),$$

$$E_{n+1} = E_n \cup \left( (2^n, 0) + E_n \right) \cup \left( (2^{n-1}, 2^{n-1}\sqrt{3}) + E_n \right),$$

where  $(x, y) + S = \{(x, y) + z | z \in S\}$ . The Sierpinski gasket graph is defined by  $(V_\infty = \bigcup_{n \in \mathbb{N}} V_n, E_\infty = \bigcup_{n \in \mathbb{N}} E_n)$  and from this, we obtain the double-sided Sierpinski gasket graph  $SG$  by adding a copy of the one-sided version reflected along the  $y$ -axis, namely,  $SG = (-V_\infty \cup V_\infty, -E_\infty \cup E_\infty)$ . See Figure 1 for an illustration of  $SG$ .



**Figure 1.** The double sided Sierpinski gasket graph as illustrated in Figure 1 of [16].

Set  $o = (0,0)$  as the origin, from which random walks are launched successively. We denote by  $B_x(n)$  the closed graph metric balls of radius  $n$  centered at vertex  $x$ , and by  $b_n := |B_o(n)|$  the cardinality of the ball  $B_o(n)$ . In [16], Chen, Huss, Sava-Huss and Teplyaev prove a basic shape result on  $SG$ .

**Theorem 1** (Theorem 1.1 of [16]). *On  $SG$ , the IDLA cluster after  $b_n = |B_o(n)|$  particles occupies a set of sites close to ball of radius  $n$ . That is, for all  $\varepsilon > 0$  and  $n$  large enough, we have*

$$B_o(n(1 - \varepsilon)) \subseteq \mathcal{I}(b_n) \subseteq B_o(n(1 + \varepsilon)) \text{ with probability } 1.$$

For the rotor router aggregation, which is a deterministic counterpart of IDLA starting rotor router walks instead of simple random walks, the shape of the cluster is exactly known to be a ball up to the fluctuations of  $\pm 1$  from the radius [17]. In [18], the authors determine the shape of the divisible sandpile model, also a ball with respect to the graph metric. This yields that the universal shape conjecture—all three aggregation clusters share the same limit shape—holds true for  $SG$ . Furthermore,  $SG$  is the only known non-trivial graph (other than  $\mathbb{Z}$ ), where even a fourth model, the Abelian sandpile model, also shares this basic shape [17].

The Sierpinski gasket graph is originated as an approximation of the Sierpinski gasket, where the latter is a well-known nested fractal of Hausdorff dimension  $\alpha = \ln(3)/\ln(2)$ . In [19], the authors define a Brownian motion on the fractal as a limit of rescaled simple random walks on the Sierpinski gasket graph. The finite generation of the Sierpinski gasket graph can also be used to define a Dirichlet form and a Laplacian on the fractal; see the standard monographs [20,21] for an introduction to the analysis on the Sierpinski gasket. In [22], the author uses the Sierpinski gasket graph to define and study the spectral properties of the Laplacian on the infinite Sierpinski gasket. Additionally, the Sierpinski gasket graph itself found interest in research. From a probabilistic standpoint, the various properties of the random walk on SG [23–25], the loop erased random walk [26,27], and the uniform spanning tree [27] have all been well studied. Combinatorial results, such as the average distances of vertices [28] and the number of spanning trees [27,29,30], have been established. An extension of the Sierpinski gasket to higher dimensions has also been studied in multiple aspects. In [31], the author calculates the exact measures of the gasket in any dimension  $d \geq 2$ . For dimension  $d = 3$  in [32], the authors investigate the loop erased random walk on the corresponding approximating graph and its continuum-limit process on the gasket. Again, on the graph, approximately for the three- and two-dimensional gasket, there exist geometric criteria to decide whether the shortest path between two vertices in neighboring triangles uses the common vertex of those triangles; see [33].

In this work, we will use the ideas from [3] to improve the sublinear fluctuations bounds of the IDLA cluster known from the basic shape Theorem 1.

**Theorem 2.** *On SG, the IDLA cluster after  $b_n = |B_o(n)|$  particles satisfies, with probability 1 for any  $\kappa > 0$ , some constant  $c > 0$  and  $n$  that is large enough:*

$$B_o\left(n - cn^{1/2} \ln(n)^{(1+\kappa)/2\alpha}\right) \subseteq \mathcal{I}(b_n) \subseteq B_o\left(n + cn^{1/2+1/2\alpha} \ln(n)^{(1-1/\alpha)(1+\kappa)/2\alpha}\right),$$

where  $\alpha = \ln(3)/\ln(2)$  is the Hausdorff dimension of the Sierpinski gasket.

The main improvement from the existing result is based on the analysis of the aforementioned divisible sandpile model, which was first introduced in [34]. In contrast to the Abelian sandpile model, it distributes non-integer mass to its neighbors during the toppling procedure, making it more tractable. We develop lower bounds for the odometer function for vertices with a certain distance to the boundary of the cluster in Section 3. We then use these bounds to prove the inner fluctuation of Theorem 2 in Section 4. The bounds for the outer bound in Section 5 work analogously to [11,16].

## 2. Preliminaries

Let  $G = (V, E)$  be an infinite locally finite connected graph. For convenience, we write  $x \in G$  instead of  $x \in V$  and  $x \sim y$  if  $(x, y) \in E$ . Furthermore, we write  $d(x, y)$  for the usual graph distance between vertices  $x, y \in G$ , and for  $x \in G$  and  $A \subseteq G$ , denote

$$\begin{aligned} \deg(x) &:= \{z \in G \mid x \sim z\} \text{ the vertex degree of } x, \\ B_x(n) &:= \{z \in G \mid d(x, z) \leq n\} \text{ the ball of radius } n \text{ with center } x, \\ \partial_I A &:= \{y \in A \mid \exists_{z \notin A} y \sim z\} \text{ the inner boundary of } A. \end{aligned}$$

Let  $\mathbb{P}_x$  be the probability law of the simple random walk  $X = (X(t))_{t \geq 0}$  on  $G$  started in vertex  $x \in G$ , and let  $\mathbb{E}_x$  be the expectation with respect to  $\mathbb{P}_x$ . For this simple random walk, we define the stopping times

$$\tau_x(n) := \inf\{t \geq 0 \mid X(t) \in \partial_I B_x(n) \cup B_x^c(n)\}, \quad \tau_o(n) := \tau_o(n)$$

and the stopped Green function

$$g_n(x, y) := \mathbb{E}_x \left( \sum_{n=0}^{\tau(n)-1} \mathbb{1}_{\{X(t)=y\}} \right),$$

which plays a major role in the analysis of bounds of IDLA. For any function  $f : G \rightarrow \mathbb{R}$ , we define the (probabilistic) graph Laplacian as

$$\Delta f(x) := \frac{1}{\deg(x)} \sum_{y \sim x} h(y) - h(x)$$

and call any  $h : G \rightarrow \mathbb{R}$  satisfying  $\Delta h(x) = 0 \forall x \in S$  a harmonic function on  $S \subseteq G$ . It is easy to show that the aforementioned stopped Green function satisfies  $\Delta g_n(o, x) = \delta_o(x)$  for all  $x \in B_o(n) \setminus \partial_I B_o(n)$  and, therefore, is harmonic on  $B_o(n) \setminus (\{o\} \cup \partial_I B_o(n))$ .

Now, due to its special structure, one can derive three major properties of  $SG$ , all of which can be proven using coverings of proper triangles.

**Lemma 1.** *The  $SG$  satisfies the following properties:*

(EHI) *The elliptic Harnack inequality: there exists a positive constant  $C$  such that for all  $x \in G$ ,  $n > 0$ , and functions  $h \geq 0$  that are harmonic on  $B_x(2n)$ , the following holds:*

$$\sup_{y \in B_x(n)} h(y) \leq C \inf_{y \in B_x(n)} h(y).$$

( $V_\alpha$ ) *The uniform volume growth condition: there exist constants  $c, C > 0$  such that for all  $x \in G$ ,  $n > 0$ , the following holds:*

$$cn^\alpha \leq |B_x(n)| \leq Cn^\alpha.$$

( $E_\beta$ ) *The uniform exit time growth condition: there exist constants  $c, C > 0$  such that for all  $x \in G$ ,  $n > 0$ , the following holds:*

$$cn^\beta \leq \mathbb{E}_x(\tau_x(n)) \leq Cn^\beta.$$

The respective constants in the exponents are  $\alpha := \ln(3)/\ln(2)$ , the Hausdorff dimension, and  $\beta := \ln(5)/\ln(2)$ , the walk dimension of the Sierpinski gasket.

**Proof.** See Chapter 2.2 of [35] for the proofs of ( $V_\alpha$ ) and ( $E_\beta$ ). The proof of (EHI) is then given in Theorem 2.6 of [35].  $\square$

For our proof of the inner bound in Theorem 2, we need some slightly sharper bounds than the ones used in Lemma 2.8, 2.10 of [16]. In fact, the proofs of such lemmas can be easily improved to obtain the following.

**Lemma 2.** *There exist constants  $c_1, c_2 > 0$  such that for every  $x \in SG$  and  $n > 0$ , we have*

$$\begin{aligned} \mathbb{E}_x(\tau_o(n)) &\geq c_1 d(x, \partial_I B_o(n))^\beta \text{ and} \\ g_n(x, x) &\leq c_2 d(x, \partial_I B_o(n))^{\beta-\alpha}. \end{aligned}$$

In the following sections, occurring constants will always be denoted as  $c \in \mathbb{R}$  and may differ from line to line.

### 3. The Divisible Sandpile

In this section, we will establish a lower bound in Lemma 7 needed for our arguments in Section 4. We derive this from the results in [18], which are briefly outlined here.

We call a function  $\mu : SG \rightarrow [0, \infty)$  with finite support  $|\text{supp}(\mu)| < \infty$  a sand distribution, and we call any  $x \in SG$  unstable if  $\mu(x) > 1$ . Any unstable vertex can be toppled such that the excess mass  $\mu(x) - 1$  is split evenly among its neighboring vertices. The resulting distribution is then given by

$$T_x \mu := \mu + \max\{\mu(x) - 1, 0\} \Delta \delta_x,$$

where  $\delta_x(y)$  equals 1 if  $x = y$  and 0 otherwise. We call  $T_x$  the toppling operator at vertex  $x$ . Note that there is no need for  $x$  to be an unstable vertex since, otherwise,  $T_x = \text{Id}$ . We start with an initial sand distribution  $\mu_0$ , and let  $(x_k)_{k \geq 1}$  be a sequence of vertices containing each vertex of  $SG$  infinitely often. We call such sequences toppling sequences and define the sand distribution after  $k$  topples as

$$\mu_k(y) := T_{x_k} \mu_{k-1}(y) = T_{x_k} \dots T_{x_1} \mu_0(y)$$

as well as the odometer function  $u_k$ , which counts the mass emitted by a vertex up to  $k$  topples

$$u_k(y) := \sum_{j \in \{i \leq k | x_i = y\}} \mu_j(y) - \mu_{j+1}(y) = \sum_{j \in \{i \leq k | x_i = y\}} \max\{\mu_j(y) - 1, 0\}.$$

Intuitively, toppling many vertices should spread the mass out such that the mass is covered by more and more vertices until there is not enough mass left to cover any new vertices. This intuition turns out to be the case as the next lemma states. Note here that [34] is dedicated to the case of  $\mathbb{Z}^d$ , but the proof works for any graph.

**Lemma 3** (Lemma 3.1 of [34]). *As  $k \rightarrow \infty$ ,  $\mu_k$  converges to a sandpile distribution  $\mu$  and  $u_k$  converges from below to a limit function  $u$ . Moreover, these limits satisfy*

$$\mu(z) = \mu_0(z) + \Delta u(z) \text{ and } \mu(z) \leq 1, \text{ for any } z \in SG. \quad (1)$$

The least action principle Lemma 3.2 of [8] states that the odometer function  $u$  in Lemma 3 is the smallest function satisfying Equation (1). At first sight, the limits  $u$  and  $\mu$  seem to depend on the choice of the toppling sequence selected for the toppling procedure. This, however, turns out not to be the case as the Abelian property of the divisible sandpile model states.

**Lemma 4** (Lemma 3.6 of [18]). *The limiting odometer function  $u$  is independent of the choice of the toppling sequence.*

Therefore, we call  $\mu$  the sand distribution and  $u$  the odometer function according to the starting distribution  $\mu_0$ ; its sandpile cluster is defined as  $\mathcal{S} := \{z \in SG \mid \mu(z) = 1\}$ . In our case, we are particularly interested in the limiting functions according to the starting distribution  $\mu_0 = |B_\circ(n)| \delta_\circ$  since the odometer function then satisfies  $\Delta u = 1 - |B_\circ(n)| \delta_\circ$  on the sandpile cluster. This will help us in the analysis of the stopped Green function  $g_n$ . The next lemma gives the solution to this problem and is a direct consequence of a result from Huss and Sava-Huss in Theorem 4.2 of [18].

**Lemma 5.** *For any  $n \geq 1$ , the sandpile distribution and therefore the sandpile cluster according to the starting distribution  $\mu_0 = |B_\circ(n)| \delta_\circ$  on  $SG$  are given by*

$$\mu(z) = \mathbb{1}_{B_\circ(n)}(z) = \begin{cases} 1 & \text{for } z \in B_\circ(n) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{S} = B_\circ(n).$$

Note that, from this, we also know that the odometer function on  $z \in \partial_I B_\circ(n)$  equals  $u(z) = 0$  since, otherwise, there would be mass outside of the cluster. For our analysis, we

need a lower bound for the odometer function depending on the distance to the boundary  $\partial_I B_\circ(n)$ , which we derive from the calculations in Section 5 of [18].

Let  $\tilde{u} : SG^+ \rightarrow \mathbb{R}$  be the function on  $SG^+ = SG \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}) = (V_\infty, E_\infty)$  defined by

$$\begin{aligned}\tilde{u}(x, 0) &= 0 \text{ for all } x \geq 0, \\ \tilde{u}(x, \sqrt{3}/2) &= 2, \text{ for all } x \geq 0, \text{ s.t. } (x, \sqrt{3}/2) \in SG^+, \end{aligned}$$

and  $\Delta \tilde{u}(x, y) = 1$  for all  $(x, y) \in SG^+$ .

**Lemma 6** (Theorem 5.6 of [18]). *Let  $k \in \mathbb{N}$ , then for  $z = (2^{k-1}, 2^{k-1}\sqrt{3})$ , the upper boundary point of  $V_k$ , which is a triangle of side length  $2^k$ , holds  $\tilde{u}(z) = 2 \cdot 5^k$ .*

Note that Ref. [18] uses a different parametrization of  $SG$ , giving a slight different expression. Let  $\psi_k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $\psi_k(x, y) = (-x/2 - \sqrt{3}y/2 + 2^k, \sqrt{3}x/2 - y/2)$ , which rotates  $V_k$  by  $120^\circ$  around its big lacuna. With this and the function  $\tilde{u}$ , one can calculate the odometer function for specific starting masses.

**Theorem 3** (Theorem 5.12 of [18]). *Let  $u^{(k)} : SG \rightarrow \mathbb{R}$  be the odometer function of the divisible sandpile with initial mass distribution  $\mu_0 = 3^{k+1}\delta_\circ = (|B_\circ(2^k)| - 2)\delta_\circ$ . Then, for all  $k \geq 0$*

$$u^{(k)}(x, y) = \begin{cases} \tilde{u} \circ \psi_k(|x|, y) & \text{if } (x, y) \in B_\circ(2^k), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, together with the previous lemma, we obtain  $u^{(k)}(\circ) = 2 \cdot 5^k$ . Using this and the fact that the odometer grows when adding the missing mass of 2, we can derive a lower bound for the odometer function.

**Lemma 7.** *Let  $n, \delta \in \mathbb{N}$  such that  $\delta \leq n/2$ , and  $u : SG \rightarrow \mathbb{R}$ , the odometer function of the sandpile cluster according to the starting distribution  $\mu_0 = |B_\circ(n)|\delta_\circ$ . Then, for all  $z \in B_\circ(n - 3\delta)$ , the following holds*

$$u(z) \geq c \delta^\beta,$$

for some  $c > 0$ .

**Proof.** Let  $m \in \mathbb{N}$  such that  $2^{m+1} > \delta \geq 2^m$ . Then, there are triangles  $\{\triangle_i\}$  of size  $2^{m-1}$  inside the annulus  $B_\circ(n) \setminus B_\circ(n - \delta)$ . The removal of all triangles  $\triangle_i$  leaves  $\circ$  in a finite component of a union of larger triangles. Now let  $z_i \in \triangle_i$  be the boundary point closest to  $\circ$ . For all such boundary points  $z_i, z_j$ ,  $u(z_i) = u(z_j)$  holds due to symmetry. Now, any lower bound for  $u(z_i)$  also holds on  $u(z)$  for all  $z \in B_\circ(n - \delta)$  since the odometer is decreasing in distance to  $\circ$  as can be seen by the generalized  $1/5 - 2/5$  rule Theorem A.1 of [18]. Let  $u_\triangle$  be the odometer when starting with just enough mass in  $\circ$  to fill up all the triangles  $\triangle_i$ , then we obviously have  $u_\triangle \leq u$  since after that we only add more mass into the system, which has to be distributed to the outer boundary. Now,  $u_\triangle(z_i)$  equals the odometer at vertex  $\circ$  of the sandpile cluster with starting mass  $|\triangle_i|$ , and from Theorem 3, we can deduce

$$u(z_i) \geq u_\triangle(z_i) \geq u^{(m-1)}(\circ) = 2 \cdot 5^{m-1} \geq c \delta^\beta,$$

for some  $c > 0$ .  $\square$

#### 4. The Inner Bound for IDLA

Similar to [16], we use the standard approach for bounds of the IDLA cluster from [3], which heavily depends on the analysis of the stopped Green functions.

We consider fixed  $z \in B_\circ(n - c n^{\alpha/2} \ln(n)^{(1+\kappa)/2})$  and for  $i = 1, \dots, b_n$ , let  $(X^i(t))_{t \geq 0}$  be the random walks generating the IDLA cluster. Now let

$$M := |\{1 \leq i \leq b_n \mid \tau_z^i < \tau^i(n)\}| = \sum_{i=1}^{b_n} \mathbb{1}_{\tau_z^i < \tau^i(n)}$$

$$L := |\{1 \leq i \leq b_n \mid \sigma^i < \tau_z^i < \tau^i(n)\}| = \sum_{i=1}^{b_n} \mathbb{1}_{\sigma^i < \tau_z^i < \tau^i(n)}$$

where  $\sigma^i := \inf\{t \geq 0 \mid X^i(t) \notin \mathcal{I}(i-1)\}$  and  $\tau_z^i := \inf\{t \geq 0 \mid X^i(t) = z\}$ . So  $M$  equals the number of random walks visiting vertex  $z$  before hitting the boundary  $\partial_I B_\circ(n)$ , whereas  $L$  equals the number of those walks that visit  $z$  after the according particle is already added to the cluster. Therefore, we have

$$\{z \notin \mathcal{I}(b_n)\} \subseteq \{M = L\}.$$

Now for any  $a \geq 0$ , we have

$$\mathbb{P}(z \notin \mathcal{I}(b_n)) \leq \mathbb{P}(M = L) \leq \mathbb{P}(\{M \leq a\} \cup \{L \geq a\}) \leq \mathbb{P}(M \leq a) + \mathbb{P}(L \geq a).$$

We will look for a specific  $a$ , giving us bounds that vanish fast enough. Now for  $M$ , the summands are obviously independent and identically distributed, whereas the summands of  $L$  are not. Therefore, we will instead consider  $\tilde{L}$  satisfying  $\mathbb{P}(\tilde{L} \geq a) \geq \mathbb{P}(L \geq a)$  for all  $a \in \mathbb{R}$  as follows. For every  $y \in B_\circ(n)$ , there is at most one  $i$ , for which  $X^i(\sigma^i) = y$  since  $y$  is already inside the cluster for the following indices,  $j > i$ . Additionally, after the time  $\sigma^i$ , the random walk  $X^i$  has the same distribution as a random walk started from  $y = X^i(\sigma^i)$ . With the random variable  $\mathbb{1}_{\tau_z < \tau(n)}^y$  denoting the indicator function of the event that a random walk started in  $y$  visits vertex  $z$  before hitting the boundary  $\partial_I B_\circ(n)$ , we have

$$L \stackrel{\mathcal{D}}{=} \sum_{i=0}^{b_n} \sum_{y \in B_\circ(n)} \mathbb{1}_{\{X^i(\sigma^i)=y\}} \mathbb{1}_{\tau_z < \tau(n)}^y \leq \sum_{y \in B_\circ(n)} \mathbb{1}_{\tau_z < \tau(n)}^y =: \tilde{L}.$$

Now the summands in  $\tilde{L}$  are independent and we can easily calculate

$$\mathbb{E}(\tilde{L}) = \sum_{y \in B_\circ(n)} P_y(\tau_z < \tau(n)) = \frac{1}{g_n(z, z)} \sum_{y \in B_\circ(n)} g_n(y, z) = \frac{1}{g_n(z, z)} \mathbb{E}_z(\tau(n)),$$

$$\mathbb{E}(M) = b_n P_\circ(\tau_z < \tau(n)) = b_n \frac{g_n(\circ, z)}{g_n(z, z)}.$$

With the help of the following lemma, we are able to calculate  $\mathbb{E}(M) - \mathbb{E}(\tilde{L})$ .

**Lemma 8.** Let  $u$  be the odometer function of the divisible sandpile on SG for the starting distribution  $\mu_0 = |B_\circ(n)|\delta_\circ$ , then it holds for all  $x \in G$

$$|B_\circ(n)| g_n(\circ, x) - \sum_{y \in B_\circ(n)} g_n(y, x) = u(x).$$

**Proof.** By Lemmas 3 and 5, the odometer function solves the following Dirichlet problem:

$$\begin{cases} \Delta u(x) = 1 - |B_\circ(n)|\delta_\circ(x) & \text{if } x \in B_\circ(n) \setminus \partial_I B_\circ(n) \\ u(x) = 0 & \text{if } x \in \partial_I B_\circ(n) \end{cases}$$



Now, let  $h_n(z) = |B_\circ(n)| g_n(\circ, z) - \sum_{y \in B_\circ(n)} g_n(y, z)$ , then we have for  $z \in B_\circ(n) \setminus \partial_I B_\circ(n)$

$$\begin{aligned}\Delta h_n(x) &= \sum_{y \in B_\circ(n)} \delta_y(x) - |B_\circ(n)| \delta_\circ(x) \\ &= 1 - |B_\circ(n)| \delta_\circ(x)\end{aligned}$$

and for  $x \in \partial_I B_\circ(n)$ ,  $y \in SG$ , we have  $g_n(y, x) = 0$  by definition. So  $h_n$  and  $u$  solve the same Dirichlet problem and by the uniqueness principle, we have  $h_n = u$ .  $\square$

Now since by Lemma 2  $\mathbb{E}_z(\tau(n)) \geq c_1 d(z, \partial_I B_\circ(n))^\beta$ ,  $g_n(z, z) \leq c_2 d(z, \partial_I B_\circ(n))^{\beta-\alpha}$ , we have the two following major inequalities:

$$\begin{aligned}c n^\alpha &\geq \mathbb{E}(M) \geq \mathbb{E}(\tilde{L}) \geq \frac{c}{g_n(z, z)} d(z, \partial_I B_\circ(n))^\beta \geq c d(z, \partial_I B_\circ(n))^\alpha, \\ \mathbb{E}(M) - \mathbb{E}(\tilde{L}) &= \frac{1}{g_n(z, z)} \left( |B_\circ(n)| g_n(\circ, z) - \sum_{y \in B_\circ(n)} g_n(y, z) \right) \\ &\geq c d(z, \partial_I B_\circ(n))^{-(\beta-\alpha)} u(z) \geq c d(z, \partial_I B_\circ(n))^\alpha\end{aligned}$$

where in the last inequality, we used the bound  $u(z) \geq c d(z, \partial_I B_\circ(n))^\beta$  from Lemma 7.

**Lemma 9** (Lemma 4 of [2]). *Let  $S$  be a finite sum of independent indicator functions and  $\mathbb{E}(S) = \mu$ . Then, for all sufficiently large  $n$  and  $0 < \gamma < \frac{1}{2}$*

$$\mathbb{P}(|S - \mu| \geq \mu^{1/2+\gamma}) \leq 2 \exp\left(-\frac{1}{4} \mu^{2\gamma}\right).$$

Choose  $\gamma, \gamma'$  such that  $\mathbb{E}(M)^{2\gamma} = \ln(n)^{1+\kappa}$  and  $\mathbb{E}(L)^{2\gamma'} = \ln(n)^{1+\kappa}$ , then we have for  $p_n := c n^{-\ln(n)}$  declining faster to 0 than any polynomial:

$$\begin{aligned}p_n &\geq \mathbb{P}(|M - \mathbb{E}(M)| \geq \mathbb{E}(M)^{1/2} \ln(n)^{(1+\kappa)/2}) + \mathbb{P}(|L - \mathbb{E}(L)| \geq \mathbb{E}(L)^{1/2} \ln(n)^{(1+\kappa)/2}) \\ &\geq \mathbb{P}(M \leq \mathbb{E}(M) - \mathbb{E}(M)^{1/2} \ln(n)^{(1+\kappa)/2}) + \mathbb{P}(L \geq \mathbb{E}(L) + \mathbb{E}(L)^{1/2} \ln(n)^{(1+\kappa)/2}) \\ &\geq \mathbb{P}(M \leq a) + \mathbb{P}(L \geq a),\end{aligned}$$

if  $a \in [\mathbb{E}(L) + \mathbb{E}(L)^{1/2} \ln(n)^{(1+\kappa)/2}, \mathbb{E}(M) - \mathbb{E}(M)^{1/2} \ln(n)^{(1+\kappa)/2}] =: I$ .

Now, since  $\mathbb{E}(M) - \mathbb{E}(\tilde{L}) \geq c d(z, \partial_I B_\circ(n))^\alpha$  and  $\mathbb{E}(M) \leq c n^\alpha$ , the above interval  $I$  is nonempty if  $d(z, \partial_I B_\circ(n))^\alpha \geq c n^{\alpha/2} \ln(n)^{(1+\kappa)/2}$ . To finish the proof one uses the upper bound of  $p_n$  with Borel–Cantelli.  $\square$

The bottleneck of the proof is obviously the very large upper bound of  $\mathbb{E}(M) \leq c n^\alpha$ . Sadly, this poor bound cannot be improved without considering the specific position in the graph: Consider the boundary points of proper triangles  $z$  at distance  $2^m$  from  $\circ$  which satisfy  $\mathbb{P}_\circ(\tau_z < \tau(n)) \geq 1/2$  for any  $n > 2^m$  and therefore  $\mathbb{E}(M) \geq c n^\alpha$ .

## 5. The Outer Bound

The improved outer bound is a direct consequence from our inner bound: with very little adjustments, one can deduce our bound from the results in Section 3.2 of [16]. Since our result can be applied to any inner bound, and for the sake of completeness, we will quickly sketch this here.

The idea of the proof is to consecutively stop the particles generating the cluster when leaving balls of growing radius. For this, we need a little more general notation than we introduced in Section 1. For an existing cluster  $S \subseteq SG$ , we define the IDLA cluster after starting a particle in vertex  $x$  and stopping upon leaving the set  $A$  as

$$\mathcal{I}(S; x \rightarrow A) := S \cup \{X(\min(\sigma_S, \sigma_A - 1))\},$$



where  $\sigma_A := \inf\{t \geq 0 | X(t) \notin A\}$  and the paused particles as

$$\mathcal{P}(S; x \rightarrow A) := \begin{cases} X(\sigma_A) & \text{if } \sigma_A \leq \sigma_S \\ \perp & \text{otherwise,} \end{cases}$$

where  $\perp$  connotes that the random walk was not paused and the particle has settled in the cluster. For starting consecutive particles from vertices  $x_1, \dots, x_n$ , we write  $\mathcal{I}(S; x_1, \dots, x_n \rightarrow A)$  for the resulting cluster and  $\mathcal{P}(S; x_1, \dots, x_n \rightarrow A)$  for the sequence of paused particles. Due to the Abelian property of the IDLA cluster (see [1]), that is

$$\mathcal{I}\left(\mathcal{I}(S; x_1, \dots, x_n \rightarrow A); \mathcal{P}(S; x_1, \dots, x_n \rightarrow A)\right) \stackrel{\mathcal{D}}{=} \mathcal{I}(S; x_1, \dots, x_n \rightarrow SG),$$

it is possible to work on consecutively stopped clusters instead of the unstopped one. For ease of notation, we denote

$$\mathcal{I}_n(x \rightarrow r) := \mathcal{I}(\emptyset; \underbrace{\circ, \dots, \circ}_{n \text{ times}} \rightarrow B_o(r)), \quad \mathcal{P}_n(x \rightarrow r) := \mathcal{P}(\emptyset; \underbrace{\circ, \dots, \circ}_{n \text{ times}} \rightarrow B_o(r)).$$

The next lemma will be essential for the proof and claims that with high probability at least a proportion  $\delta$  of started particles will settle before reaching the next radius, at which they will be stopped again.

**Lemma 10** (Lemma 3.7 of [16]). *There are  $\delta > 0$  and  $p < 1$  such that for all  $n$  large enough with  $n^{1/(\alpha+1)} < k < n^\alpha$  we have for  $x_1, \dots, x_k \in B_o(n)$  and  $S \subseteq B_o(n)$*

$$\mathbb{P}\left(|\mathcal{I}(S; x_1, \dots, x_k \rightarrow B_o(n + k^{1/\alpha})) \setminus S| \leq \delta k\right) \leq p^k.$$

With this, we are now able to establish an outer bound depending on the already proved inner bound in Section 4.

**Theorem 4** (Outer bound in dependence of the inner bound). *Suppose we already established an inner bound  $D_I(n)$  on SG such that*

$$\sum_{n \in \mathbb{N}} \mathbb{P}\left(B_o(n - D_I(n)) \not\subseteq \mathcal{I}_{b_n}(\circ \rightarrow n)\right) < \infty$$

*Then for any  $D \geq D_I^{1-1/\alpha} n^{1/\alpha}$ , it holds with probability 1 for  $n$  large enough*

$$\mathcal{I}(b_n) \subseteq B_o(n + D).$$

**Proof.** We will show that the probability of the event  $\{\mathcal{I}(b_n) \not\subseteq B_o(n + D)\}$  is summable. The claim then follows directly from the Borel–Cantelli lemma. For this, let

$$\begin{aligned} n_0 &:= n, & n_{j+1} &:= \begin{cases} n_j + k_j^{1/\alpha} & \text{if } k_j > n_j^{1/(\alpha+1)} \\ \infty & \text{otherwise,} \end{cases} \\ \mathcal{I}_0 &:= \mathcal{I}_{b_n}(0 \rightarrow n), & \mathcal{I}_{j+1} &:= \mathcal{I}(\mathcal{I}_j; \mathcal{P}_j \rightarrow B_o(n_{j+1})), \\ \mathcal{P}_0 &:= \mathcal{P}_{b_n}(0 \rightarrow n), & \mathcal{P}_{j+1} &:= \mathcal{P}(\mathcal{I}_j; \mathcal{P}_j \rightarrow B_o(n_{j+1})), \\ k_0 &:= |\mathcal{P}_0| & k_{j+1} &:= |\mathcal{P}_{j+1}| \end{aligned}$$

be ascending clusters, whose union form the IDLA cluster. That is, we stop the particles on the boundary of balls of growing radii as long as there are enough unstopped particles. This iterative construction will stop once we do not have enough stopped particles in  $\mathcal{P}_j$ .

Let  $J := \min\{j | k_j \leq n_j^{1/(\alpha+1)}\}$  be the time, after which we let all particles in  $\mathcal{P}_j$  evolve until settlement. Now for  $j > J$ , we have  $\mathcal{I}_j = \mathcal{I}_{j+1}$  and by the Abelian property of the

IDLA cluster,  $\mathcal{I}_{J+1}$  and  $\mathcal{I}(b_n)$  have the same distribution. By construction,  $\mathcal{I}_J \subseteq B_\circ(n_J)$  and since  $k_J \leq n_J^{1/(\alpha+1)}$ , we can deduce  $\mathcal{I}_{J+1} \subseteq B_\circ(n_J + n_J^{1/(\alpha+1)})$ . Therefore, we have

$$\begin{aligned} \mathbb{P}(\mathcal{I}(b_n) \not\subseteq B_\circ(n+D)) &= \mathbb{P}(\mathcal{I}_{J+1} \not\subseteq B_\circ(n+D)) \\ &\leq \mathbb{P}(B_\circ(n_J + n_J^{1/(\alpha+1)}) \not\subseteq B_\circ(n+D)) \\ &= \mathbb{P}(n_J + n_J^{1/(\alpha+1)} > n+D). \end{aligned}$$

For SG, there is an annulus growth bound shown in Lemma 3.8 of [16]: for all  $\varepsilon > 0$ , it holds  $b_n - b_{n(1-\varepsilon)} \leq 4\varepsilon^{\alpha-1}b_n$  for  $n$  large enough. Furthermore, by assumption and Borel–Cantelli, we have, with probability 1 for  $n$  large enough,  $B_\circ(n - D_I) \subseteq \mathcal{I}_{b_n}(0 \rightarrow n)$ . From this, we can deduce  $k_0 \leq b_n - |\mathcal{I}_{b_n}(0 \rightarrow n)| \leq b_n - b_{n-D_I} \leq cD_I^{\alpha-1}n < n^\alpha$  almost surely for  $n$  large enough. So the amount of paused particles  $k_{j-1}$  satisfies  $n_j^{1/(\alpha+1)} < k_{j-1} < n^\alpha < n_j^\alpha$  almost surely for  $j \leq J$ , and we can apply Lemma 10. Accordingly, there are  $\delta < 1$  and  $p < 1$  such that

$$\mathbb{P}(\mathcal{I}(S; \mathcal{P}_{j-1} \rightarrow B_\circ(n_{j-1} + l^{1/\alpha})) \setminus S) \leq (1-\delta)l \leq p^l.$$

Notice that  $k_j - k_{j-1}$  are exactly the number of settled particles in wave  $j$ . Conditioning on  $S$ , the total law of probability yields

$$\mathbb{P}(k_j \geq (1-\delta)k_{j-1} | n_{j-1} = i, k_{j-1} = l) \leq \sum_{S \subseteq B_\circ(i)} p^l \mathbb{P}(\mathcal{I}_{j-1} = S | n_{j-1} = i, k_{j-1} = l) = p^l.$$

With this at hand, by applying the total law of probability twice, we can deduce

$$\begin{aligned} &\mathbb{P}(k_j \geq (1-\delta)k_{j-1} \cap j \leq J) \\ &\leq \sum_{i=n}^{\infty} \mathbb{P}(n_{j-1} = i) \sum_{l=i^{1/(\alpha+1)}}^{n^\alpha} p^l \mathbb{P}(k_{j-1} = l \cap j \leq J | n_{j-1} = i) \leq p^{n^{1/(\alpha+1)}} \end{aligned}$$

and using this bound finally gives

$$\mathbb{P}(\exists_{1 \leq j \leq J} k_j \geq (1-\delta)k_{j-1}) \leq \sum_{j=1}^{n^\alpha} \mathbb{P}(k_j \geq (1-\delta)k_{j-1} \cap j \leq J) \leq n^\alpha p^{n^{1/(\alpha+1)}}.$$

Since  $\{\exists_{1 \leq j \leq J} k_j \geq (1-\delta)k_{j-1}\} \supseteq \{k_l \geq (1-\delta)l k_0\}$ , we have

$$\mathbb{P}(\exists_{1 \leq l \leq J} k_l \geq (1-\delta)l k_0) \leq n^\alpha p^{n^{1/(\alpha+1)}}.$$

On the complementary event  $\{\forall_{1 \leq j \leq J} k_j < (1-\delta)j k_0\}$ , the following inequality holds:

$$n_J = n + \sum_{j=0}^{J-1} k_j^{1/\alpha} < n + k_0^{1/\alpha} \sum_{j=0}^{J-1} ((1-\delta)^{1/\alpha})^j < n + k_0^{1/\alpha} \frac{1}{1 - (1-\delta)^{1/\alpha}}.$$

If additionally  $n_J + n_J^{1/(\alpha+1)} > n + D$  holds, taking  $c' = \left(1 + \frac{1}{1-(1-\delta)^{1/\alpha}}\right)^{1/(\alpha+1)}$  gives that for  $n$  large enough,

$$\begin{aligned} &n + k_0^{1/\alpha} \frac{1}{1 - (1-\delta)^{1/\alpha}} + c' n^{1/(\alpha+1)} \\ &> n_J + n_J^{1/(\alpha+1)} > n + D. \end{aligned}$$

From this, we can easily deduce

$$k_0^{1/\alpha} > (D - c'n^{1/(\alpha+1)})(1 - (1 - \delta)^{1/\alpha}) > c_1 D \text{ for } n \text{ large enough}$$

By conditioning on the event  $n_J < n + k_0^{1/\alpha}(1 - (1 - \delta)^{1/\alpha})^{-1}$ , we obtain

$$\begin{aligned} \mathbb{P}(\mathcal{I}(b_n) \not\subseteq B_\circ(n + D)) &\leq \mathbb{P}\left(n_J + n_J^{1/(\alpha+1)} > n + D\right) \\ &= \mathbb{P}\left(n_J + n_J^{1/(\alpha+1)} > n + D \mid n_J < n + k_0^{1/\alpha} \frac{1}{1 - (1 - \delta)^{1/\alpha}}\right) \mathbb{P}\left(n_J < n + k_0^{1/\alpha} \frac{1}{1 - (1 - \delta)^{1/\alpha}}\right) \\ &\quad + \mathbb{P}\left(n_J + n_J^{1/(\alpha+1)} > n + D \mid n_J \geq n + k_0^{1/\alpha} \frac{1}{1 - (1 - \delta)^{1/\alpha}}\right) \mathbb{P}\left(n_J \geq n + k_0^{1/\alpha} \frac{1}{1 - (1 - \delta)^{1/\alpha}}\right) \\ &\leq \mathbb{P}(k_0 > cD^\alpha) + n^\alpha p^{n^{1/(\alpha+1)}}. \end{aligned}$$

Since, by assumption,  $\mathbb{P}(k_0 > cD_I^{\alpha-1}n) \leq \mathbb{P}(B_\circ(n - D_I) \not\subseteq \mathcal{I}(b_n))$  is summable, the above probability for any  $D \geq D_I^{1-1/\alpha}n^{1/\alpha}$  is summable as well. Now applying Borel–Cantelli yields

$$\mathcal{I}(b_n) \subseteq B_\circ(n + D) \text{ almost surely for } n \text{ large enough.}$$

□

Note that in Section 4, we actually show an inner bound for the stopped cluster  $\mathcal{I}_{b_n}(\circ \rightarrow n)$  as is needed here. This is because the considered  $M, L$  only depends on the behavior of the random walks before leaving the ball of radius  $n$ . Therefore,  $\{z \in \mathcal{I}_{b_n}(\circ \rightarrow n)\} \subseteq \{M = L\}$  holds for any  $z \in B_\circ(n)$  as well. Applying the theorem to the established inner bound  $D_I(n) = cn^{1/2} \ln(n)^{(1+\kappa)/2\alpha}$  yields the desired outer bound

$$D = D_I^{1-1/\alpha}n^{1/\alpha} = c n^{1/2+1/2\alpha} \ln(n)^{(1+\kappa)(1-1/\alpha)/2\alpha}.$$

## 6. Conclusions

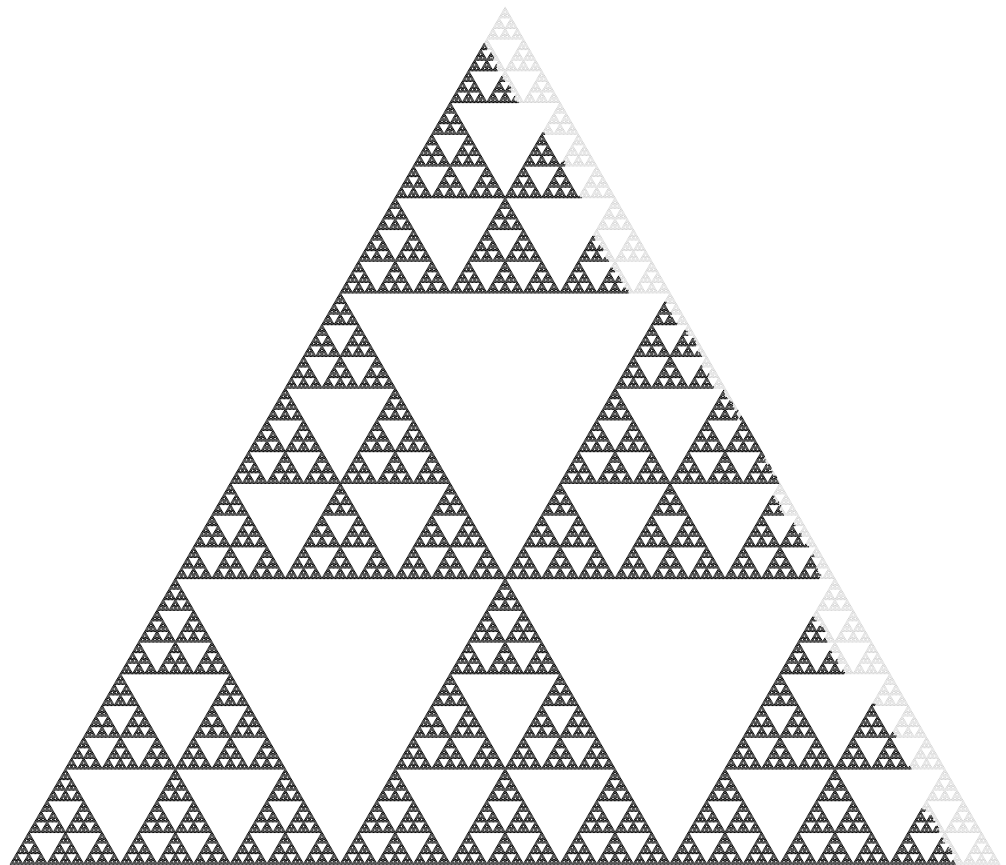
For the inner bound, we used the difference between the expected visits in vertices and visits after settlement of the random walks generating the cluster. Our lower bound for this difference is already pretty sharp since the odometer of the divisible sandpile gives the exact solution for this, and the bound for the odometer function itself is also sharp for some special vertices. Considering the outer bound, our technique rests on stopping at the boundary of growing balls and letting paused particles develop on the annulus to the next bigger ball. In some sense, we consider the balls on whose boundary the particles are stopped to be already completely settled in the subsequent process (since in Lemma 10, there is no further assumption on the sets  $S \subseteq B_\circ(n)$ ). Now, suppose the ball is a proper triangle of size  $n = 2^m$  and suppose we filled the ball perfectly up to  $b_{n-D_I}$ , where  $D_I = 2^k$ , then the remaining particles amount to

$$b_n - b_{n-D_I} = \frac{2^{m-k}(3^{k+1} + 3)}{3^{k+1} + 2} b_n \geq cD_I^{\alpha-1}n.$$

The radius to cover all these particles would be at least  $D \geq cD_I^{1-1/\alpha}n^{1/\alpha}$ . Therefore, this result is also optimal, in the sense that for better bounds one would need to consider different techniques in order to prove these.

Furthermore, this approach can also be used for other self-similar graphs approximating nested fractals. For example, the graphical approximation of the extension of the Sierpinski gasket to higher dimensions  $d \geq 3$  can be extended to a regular graph as we did for SG. We also expect the divisible sandpile on this graph to be an exact ball in the graph metric. Evaluating its odometer function, like we did in Section 3, should lead to fluctuation bounds completely analogous to our approach in Sections 4 and 5. For the Vicsek graph,

the graph approximation of the Vicsek set Example 4.1.5 of [21], we also conjecture our technique to work since the divisible sandpile is also an exact ball. Here, the difficulty lies mainly in the additional technicalities and the exact analysis of a normalized odometer function, which is needed since the graph is not regular anymore. We expect this to also generalize to other nested fractals. However, one may need to consider another metric other than the graph metric to describe the shape of the cluster since the harmonic measure on the usual graph metric balls is not uniform anymore. Take a modified Sierpinski gasket graph, where instead of three copies in each stage of the construction, we take nine copies of the triangle in the previous stage. Here, the cluster will grow much faster on the outer middle triangle than on the other ones. Figure 2 shows a simulation on the 6th generation of such a modified Sierpinski gasket graph, where the fluctuation is of the size of a third-generation triangle. Considering infinitely ramified fractals, our approach seems not too promising, as the generations of the approximating graphs are connected on unbounded many vertices. This makes the analysis of the divisible sandpile and its odometer function very hard. See Figure 2 of [16] for a simulation of IDLA clusters on the Sierpinski carpet graph.



**Figure 2.** Simulation on the modified Sierpinski gasket graph with nine copies.

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