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Review

Notes on the Transversality Method for Iterated Function Systems—A Survey †

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† Dedicated to Károly Simon.

Abstract: This is a brief survey of selected results obtained using the "transversality method" developed for studying parametrized families of fractal sets and measures. We mostly focus on the early development of the theory, restricting ourselves to self-similar and self-conformal iterated function systems.

Keywords: iterated function system; self-similar; transversality

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1. Introduction

"Transversality" is a geometric and analytic property which comes up in many areas of mathematics. Here we are concerned with transversality in a narrow sense: a technique to obtain "almost sure" results for parametrized families of fractals. Although there were parallel developments in the theory of random fractals, here we focus on deterministic families. We start with a "historical" exposition, reflecting the author's subjective viewpoint, and, with a few exceptions, focus on relatively "old" results. We do not attempt to give a comprehensive account of the literature on transversality techniques, which is vast. Moreover, we mostly focus on self-similar and conformal systems, only mentioning self-affine ones in passing. In the last section we present a "generalized projection scheme for convolutions", which may be new. Some parts of this article will be incorporated (in a modified form) into the upcoming book [1].

2. Origins of the Method I: Projection Theorems

Here we recall the classical results on the Hausdorff dimension of orthogonal projections by Marstrand [2] and Kaufman [3], and their extensions by Mattila [4].

For $\theta \in [0, \pi)$ let P_θ be the orthogonal projection from the plane $\mathbb{R}^2$ to the line ℓ_θ making the angle θ with the positive x-axis. We write $\mathcal{L}^m$ for the m-dimensional Lebesgue measure and $\dim_H$ for the Hausdorff dimension.

Theorem 1 (Marstrand)**.** *Let $A \subset \mathbb{R}^2$ be a Borel set. Then the following holds:*

(i) $\dim_H(P_\theta(A)) = \min\{1, \dim_H(A)\}$ *for $\mathcal{L}^1$-a.e. $\theta \in [0, \pi)$;*

(ii) *If $\dim_H(A) > 1$, then $\mathcal{L}^1(P_\theta(A)) > 0$ for $\mathcal{L}^1$-a.e. $\theta \in [0, \pi)$.*

Proof sketch. Although the proofs appear in many books, we sketch them here, since they provide a kind of a "template" for the transversality method. Rather than the original, geometric method of Marstrand, we use the method of Kaufman [3] for part (i). The upper bound for the dimension of the projection trivially holds for all θ, so we only need to verify the lower bound. Fix an arbitrary $0 < \alpha < \min\{\dim_H(A), 1\}$. By the potential-theoretic characterization of the Hausdorff dimension, which goes back to Frostman [5], see [6] (Corollary 6.6), there exists a probability measure μ supported on A such that

$$\mathcal{E}_\alpha(\mu) = \iint |\mathbf{x} - \mathbf{y}|^{-\alpha} \, d\mu(\mathbf{x}) \, d\mu(\mathbf{y}) < \infty. \tag{1}$$

Denote by μ_θ the push-forward measure onto the line ℓ_θ: $\mu_\theta = P_\theta\mu = \mu \circ P_\theta^{-1}$. The idea is to show that

$$\mathcal{J}_1 := \int_0^\pi \iint |\xi - \zeta|^{-\alpha}\, d\mu_\theta(\xi)\, d\mu_\theta(\zeta)\, d\theta < \infty. \tag{2}$$

This would imply that μ_θ, a probability measure supported on $P_\theta(A)$, satisfies $\mathcal{E}_\alpha(\mu_\theta) < \infty$ for a.e. θ, hence $\dim_{\mathrm{H}}(P_\theta(A)) \geq \alpha$ for a.e. θ, by the potential-theoretic characterization of the Hausdorff dimension, which yields the desired claim. Using the definition of the measure μ_θ and reversing the order of integration yields

$$\begin{aligned} \mathcal{J}_1 &= \iint_{A\times A} \int_0^\pi |\langle \mathbf{x} - \mathbf{y}, e_\theta\rangle|^{-\alpha}\, d\theta\, d\mu(\mathbf{x})\, d\mu(\mathbf{y}) \\ &= \iint_{A\times A} |\mathbf{x} - \mathbf{y}|^{-\alpha} \left(\int_0^\pi |\langle \mathbf{w}, e_\theta\rangle|^{-\alpha}\, d\theta \right) d\mu(\mathbf{x})\, d\mu(\mathbf{y}), \end{aligned}$$

where e_θ is the unit vector along ℓ_θ and $\mathbf{w} = \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}$ (note that $\mathbf{x} \neq \mathbf{y}$ $(\mu \times \mu)$-a.e. by (1)). The inner integral $\mathcal{I} = \mathcal{I}_\mathbf{w}$ above does not depend on the unit vector $\mathbf{w}$ by rotational symmetry. It is finite, since $\alpha < 1$, due to the simple geometric fact:

$$\mathcal{L}^1\big(\{\theta \in [0,\pi) :\ |\langle \mathbf{w}, e_\theta\rangle| \leq r\}\big) \leq Cr, \quad \text{for all } r > 0. \tag{3}$$

Thus, $\mathcal{J}_1 = \mathcal{I} \cdot \mathcal{E}_\alpha(\mu) < \infty$, completing the proof.

For part (ii), we sketch the proof of Mattila (following [7]), which uses differentiation of measures. Since $\dim_{\mathrm{H}}(A) > 1$, Frostman's Lemma yields a probability μ supported on A, with $\mathcal{E}_1(\mu) < \infty$. Let μ_θ be the push-forward measure on $P_\theta(A)$, as above, and consider the lower derivative of μ_θ with respect to the Lebesgue measure $\mathcal{L}^1$:

$$\underline{D}(\mu_\theta, \zeta) := \liminf_{r\to 0} \frac{\mu_\theta(B(\zeta, r))}{2r}.$$

We would like to show that

$$\mathcal{J}_2 := \int_0^{2\pi} \int \underline{D}(\mu_\theta, \zeta)\, d\mu_\theta(\zeta)\, d\theta < \infty.$$

This would imply that for $\mathcal{L}^1$-a.e. θ we have $\underline{D}(\mu_\theta, \zeta) < \infty$ for μ_θ-a.e. ζ, and then by a standard differentiation of measures lemma (see [7], p. 36), we would obtain that μ_θ is absolutely continuous with respect to the Lebesgue measure for such θ. Since μ_θ is a probability measure on $P_\theta(A)$, this would mean that $\mathcal{L}^1(P_\theta(A)) > 0$ for a.e. θ, as desired.

In order to estimate $\mathcal{J}_2$, we first use Fatou's Lemma to get

$$\mathcal{J}_2 \leq \liminf_{r\to 0}\, (2r)^{-1} \int_0^\pi \int \mu_\theta(B(\zeta, r))\, d\mu_\theta(\zeta)\, d\theta.$$

Further,

$$\begin{aligned} \int \mu_\theta(B(\zeta, r))\, d\mu_\theta(\zeta) &= \iint \mathbf{1}_{B(\zeta,r)}(\xi)\, d\mu_\theta(\xi)\, d\mu_\theta(\zeta) \\ &= \iint \mathbf{1}_{B(P_\theta(\mathbf{y}),r)}(P_\theta(\mathbf{x}))\, d\mu(\mathbf{x})\, d\mu(\mathbf{y}). \end{aligned}$$

Thus, exchanging the order of integration and integrating the characteristic function with respect to $\mathcal{L}^1$, we obtain

$$\mathcal{J}_2 \leq \liminf_{r\to 0}\, (2r)^{-1} \iint \mathcal{L}^1(\{\theta :\ |\langle \mathbf{x} - \mathbf{y}, \theta\rangle| \leq r\})\, d\mu(\mathbf{x})\, d\mu(\mathbf{y}).$$

By the geometric fact (3), we have $\mathcal{L}^1(\{\theta :\ |\langle \mathbf{x} - \mathbf{y}, \theta\rangle| \leq r\}) \leq Cr|\mathbf{x} - \mathbf{y}|^{-1}$, hence $\mathcal{J}_2 \leq C\mathcal{E}_1(\mu) < \infty$, completing the proof. $\square$

The method of integration over the parameters was also used to obtain estimates on the dimension of exceptions in the above results. In fact, Kaufman [3] proved that if A is a Borel set in $\mathbb{R}^2$ of Hausdorff dimension $\dim_{\mathrm{H}}(A) = s < 1$, then

$$\dim_{\mathrm{H}}\big(\{\theta : \dim_{\mathrm{H}}(P_\theta(A)) < s\}\big) \le s.$$

The proof is similar to the proof of Theorem 1(i), using integration with respect to the appropriate Frostman measure $[0, \pi)$. Falconer [8] showed that if $\dim_{\mathrm{H}}(A) = s \in (1, 2)$, then

$$\dim_{\mathrm{H}}\big(\{\theta : \mathcal{L}^1(P_\theta(A)) = 0\}\big) \le 2 - s.$$

Falconer's proof makes use of the Fourier transform.

In order to state the higher-dimensional generalization of Marstrand's Theorem, let $G(d, n)$ be the Grassmanian manifold of m-dimensional linear subspaces of $\mathbb{R}^d$, and let $\gamma_{d,n}$ be the Haar measure on $G(d, n)$. For $V \in \mathbb{R}^d$ denote by P_V the orthogonal projection from $\mathbb{R}^d$ onto V.

Theorem 2 (Mattila)**.** *Let $A \subset \mathbb{R}^d$ be a Borel set.*

(i) $\dim_{\mathrm{H}}(P_V(A)) = \min\{n, \dim_{\mathrm{H}}(A)\}$ *for $\gamma_{d,m}$-a.e. $V \in G(d, n)$;*
(ii) *If $\dim_{\mathrm{H}}(A) > n$, then $\mathcal{L}^n(P_V(A)) > 0$ for $\gamma_{d,n}$-a.e. $V \in G(d, n)$.*

The proof is essentially the same as for Theorem 1, with the geometric inequality (3) replaced by the more general

$$\gamma_{d,n}\big(\{V \in G(d, n) : \ |P_V \mathbf{x}| \le r\}\big) \le C_d \cdot r^n |\mathbf{x}|^{-n}, \ \text{ for all } \ \mathbf{x} \in \mathbb{R}^d \setminus \{0\} \ \text{ and } \ r > 0. \tag{4}$$

For results on exceptional sets in higher dimensions, see the survey [9].

3. Origins of the Method II: Iterated Function Systems and Dynamics

We start by recalling the background, which is standard, in order to set the notation. Consider an iterated function system (IFS) of uniformly contracting (injective) maps $\mathcal{F} = \{f_1, \ldots, f_m\}$ in $\mathbb{R}^d$. By Hutchinson's Theorem [10], there exists a unique non-empty compact set $\Lambda = \Lambda_{\mathcal{F}}$, called the *attractor* of the IFS, satisfying

$$\Lambda = \bigcup_{j=1}^{d} f_j(\Lambda).$$

Moreover, given a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$, there exists a unique Borel probability measure $\nu = \nu_{\mathbf{p}}(\mathcal{F})$, called the *invariant measure* for the probabilistic IFS, such that

$$\nu_{\mathbf{p}} = \sum_{j=1}^{m} p_j f_j \nu_{\mathbf{p}}.$$

Denote by $\Sigma_m = \mathcal{A}^\infty$ the symbolic space of one-sided infinite sequences $\omega = \omega_0\omega_1\ldots$, with $\mathcal{A} = \{1, \ldots, m\}$, equipped with the product discrete topology. The *natural projection* is the map $\Pi : \Sigma_m \to \mathbb{R}^d$, defined by

$$\Pi(\omega) = \lim_{n\to\infty} f_{\omega_0\ldots\omega_n}(\mathbf{x}_0), \ \ \omega \in \Sigma_m,$$

for an arbitrary $\mathbf{x}_0 \in \mathbb{R}^d$, where $f_{\omega_0\ldots\omega_n} = f_{\omega_0} \circ \cdots \circ f_{\omega_n}$. Then $\Lambda = \Pi(\Sigma_m)$ is the attractor of the IFS, and if $\mathbf{p}$ is a probability vector, then the push-forward $\nu_{\mathbf{p}} = \Pi\mu_{\mathbf{p}}$ is the corresponding invariant measure, where $\mu_{\mathbf{p}} = \mathbf{p}^{\mathbb{N}}$ is the infinite product (Bernoulli) measure on Σ_m.

If f_j are affine contractions, that is, $f_j(\mathbf{x}) = A_j\mathbf{x} + \mathbf{t}_j$ for a contracting non-singular linear map $A_j : \mathbb{R}^d \to \mathbb{R}^d$ and a vector $\mathbf{t}_j \in \mathbb{R}^d$, then $\mathcal{F}$ is called a *self-affine IFS*, and its attractor and invariant measures are called self-affine. If, moreover, A_j is a similarity, i.e., $A_j = r_j\mathcal{O}_j$, for some $r_j \in (0,1)$, and $\mathcal{O}_j$ is an orthogonal transformation, then the attractor and invariant measures are self-similar. The numbers r_j are called the *contraction ratios*.

If $\mathcal{F}$ is a self-similar IFS, the "natural" cover of the attractor $\Lambda = \Lambda_{\mathcal{F}}$ is by the cylinder sets Λ_u, with $u \in \mathcal{A}^n$, where $\Lambda_u = f_u(\Lambda)$. This immediately yields the upper bound for the Hausdorff dimension of the attractor:

$$\dim_{\mathrm{H}}(\Lambda_{\mathcal{F}}) \le s(\mathcal{F}), \;\text{ where }\; s = s(\mathcal{F}) \text{ is such that } \sum_{j=1}^{m} r_j^s = 1. \tag{5}$$

We call $s(\mathcal{F})$ the *similarity dimension* of the self-similar IFS. For the invariant measure $\nu_{\mathbf{p}}$ we have

$$\dim_{\mathrm{H}}(\nu_{\mathbf{p}}) \le s(\mathcal{F}, \mathbf{p}) := \frac{H(\mathbf{p})}{\chi(\mathcal{F}, \mathbf{p})}, \tag{6}$$

where $H(\mathbf{p}) = -\sum_{j=1}^m p_j \log p_j$ is the *entropy* of the probability vector (and of the Bernoulli measure) and $\chi(\mathcal{F}, \mathbf{p}) = -\sum_{j=1}^m p_j \log r_j$ is the Lyapunov exponent of the probabilistic IFS. A fundamental problem in fractal geometry is to determine when the inequalities (5) and (6) are actually equalities. The simplest case is when the "cylinders" of the attractor $\Lambda_j = f_j(\Lambda)$ are all mutually disjoint; then we say that the *Strong Separation Condition* (SSC) holds. If there exists a non-empty open set U such that $f_j(U) \subset U$ and all $f_i(U), f_j(U)$ are mutually disjoint for $i \ne j$, then the *Open Set Condition* (OSC) holds. By a well-known result of Moran [11] and Hutchinson [10], the OSC implies equality in (5) and (6). Difficulties begin when the cylinders "overlap" (although the term itself is somewhat vague). It is generally believed that in the general case, "typically",

$$\dim_{\mathrm{H}}(\Lambda_{\mathcal{F}}) = \min\{d, s(\mathcal{F})\} \;\text{ and }\; \dim_{\mathrm{H}}(\mu_{\mathbf{p}}) = \min\{d, s(\mathcal{F}, \mathbf{p})\}, \tag{7}$$

and if $s(\mathcal{F}, \mathbf{p}) > d$, then "typically" $\mu_{\mathbf{p}}$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$. The transversality method was developed with this goal in mind.

3.1. Early Work on Fractals and Attractors of Overlapping Construction

The pioneering paper [12] by Kenneth Falconer was, perhaps, the first where the question of Hausdorff dimension for self-similar sets with overlaps was studied. He considered the "translation family" of self-similar IFS on the line:

$$\mathcal{F}^{\mathbf{t}} := \{\lambda_1 x + t_1, \ldots, \lambda_d x + t_d\}, \;\; x \in \mathbb{R},\; \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d, \text{ with } |\lambda_j| < 1 \text{ for } j = 1, \ldots, d. \tag{8}$$

Note that the contraction ratios are fixed, so the similarity dimension is the same for all translation parameters: $s(\mathcal{F}^{\mathbf{t}}) = s(\boldsymbol{\lambda}) = s$, where $\sum_{j=1}^m \lambda_j^s = 1$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$. The following result is from the paper [13], but the proof is essentially from [12], extended to the "natural" generality in terms of the range of parameters.

Theorem 3 ([12,13])**.** *Let $\Lambda_{\mathbf{t}}$ be the attractor of the IFS* (8)*. Then*

$$\dim_{\mathrm{H}}(\Lambda_{\mathbf{t}}) = \min\{1, s(\boldsymbol{\lambda})\} \;\textit{for}\; \mathcal{L}^d\textit{-a.e. } \mathbf{t} \in \mathbb{R}^d.$$

In fact, there are two cases. If

$$\max_{i \ne j}(|\lambda_i| + |\lambda_j|) \ge 1,$$

then an elementary argument shows that the attractor $\Lambda_{\mathbf{t}}$ contains an interval, as long as the corresponding fixed points $t_i(1-\lambda_i)^{-1}$ and $t_j(1-\lambda_j)^{-1}$ are distinct. Otherwise, one can "lift" the family of the IFS $\mathcal{F}^{\mathbf{t}}$ to a single self-similar IFS $\widetilde{\mathcal{F}}$ in the space $\mathbb{R}^{d+1}$ of the form $\mathbf{x} \mapsto \lambda_j \mathbf{x} + \boldsymbol{a}_j$, with the same contraction ratios, so that it becomes non-overlapping, in such a way that $\mathcal{F}^{\mathbf{t}}$ is obtained by an orthogonal projection to a line $\ell(\mathbf{t})$ through the origin in $\mathbb{R}^{d+1}$, and sets of zero Lebesgue measure in $\mathbb{R}^d$ correspond to sets of zero measure in the Grassmanian $G(d+1,1)$. Then the dimension claim follows from Mattila's projection Theorem 2.

A similar idea of lifting a system into a higher-dimensional space was used by Falconer [12] to obtain an "almost sure" result for the dimension of the attractor of a "slanting baker's transformation". The map T^{c_1,c_2} from $\mathbb{R} \times [-1,1]$ into itself is defined by

$$T^{c_1,c_2}(x,y) = \begin{cases} T_1^{c_1,c_2}(x,y) = (\lambda_1 x + \mu_1 y + c_1, 2y-1) & \text{if } y \geq 0; \\ T_2^{c_1,c_2}(x,y) = (\lambda_2 x + \mu_2 y + c_2, 2y+1) & \text{if } y < 0. \end{cases}$$

Here $0 < |\lambda_j|, |\mu_j| < 1$, for $j = 1,2$. For a sufficiently large compact interval $K \subset \mathbb{R}$, the transformation T^{c_1,c_2} maps $K \times [-1,1]$ into itself, and there is a well-defined attractor. In Theorem 2 of [12] it is proved that the dimension of the attractor is equal to $1+s$, where $|\lambda_1|^s + |\lambda_2|^s = 1$, for Lebesgue-a.e. (c_1,c_2). The proof proceeds by lifting the system into $\mathbb{R}^3$, in order to remove the "overlapping", and then use Marstrand's projection theorem combined with Marstrand's slicing theorem; see [12] for details.

The next significant event in the "pre-history" of the transversality method was the paper by Károly Simon on non-invertible (possibly non-linear) endomorphims of the unit square [14]. The key new insight was the realization that "overlaps do not matter" in certain regimes, when the unstable manifolds intersect transversely (at non-zero angles). This allowed K. Simon to prove, in particular, that the formula for the dimension of the attractor of Falconer's slanting baker's transformations, holds *for all*, rather than almost all, (c_1,c_2), under the assumption

$$0 < |\lambda_1|, |\lambda_2| < |\mu_1 - \mu_2|,$$

see Corollary 1 of [14]. Moreover, the dimension of the attractor was computed for a sufficiently small non-linear perturbation of the slanting baker's map, as another corollary of a general result, see Theorem 1 of [14]. Of course, in the non-linear case one needs to replace the number s with a solution of the appropriate Bowen's equation. This general theorem even allowed critical points, allowing K. Simon to compute the dimension of the so-called "Yakobson twisted map" (a two-dimensional version of the logistic map, see [15], which was the original motivation).

3.2. On Self-Affine IFSs

As mentioned in the introduction, we do not consider non-conformal (including "genuinely self-affine") systems in this article in detail, but a short discussion is called for. Results of T. Bedford [16] and C. McMullen [17] (independently) on self-affine "carpets", and later, those of Przytycki and Urbański [18], showed that the situation is much more complicated than for self-similar/conformal systems. In particular, even when the Strong Separation Condition holds, it may happen that the Hausdorff dimension of the attractor is strictly smaller than the box-counting dimension (for conformal uniformly contracting IFS these dimensions are always equal [19]). On the other hand, K. Falconer [20] proved that for a "typical" self-affine IFS $\{A_j \mathbf{x} + \mathbf{t}_j\}_{j \leq m}$, these dimensions are both equal to the "singularity" or "affinity" dimension (see [20] for definitions), assuming that the norms of A_j are small enough. "Small enough" was less than $\frac{1}{3}$ in [20] and improved to less than $\frac{1}{2}$ in [21], which is sharp. "Typical", similarly to Theorem 3, meant for a Lebesgue-a.e. family of translation vectors $\mathbf{t}_j$. The paper [20] was very influential for the future development of the transversality method. Very briefly, as in Kaufman's proof of Marstrand's projection theorem above, the lower bound for the Hausdorff dimension for an a.e. parameter is obtained in [20] by proving finiteness of the parameter-dependent energy integral, integrated over an appropriate region of parameters. Significantly, the *geometric orthogonal projection* is replaced here by the *natural projection* from the symbolic space. The measures, appearing in the energy integral, are obtained as push-forwards of a fixed measure on the symbolic space, a kind of a "net measure" adapted to the self-affine IFS under consideration. Much later, this was sharpened and extended to a more general class of self-affine systems by T. Jordan, M. Pollicott, and K. Simon [22] in the framework of "self-affine transversality". Very recently, self-affine transversality was further extended to non-linear non-conformal systems by De-Jun Feng and Károly Simon [23].

4. Transversality Method for Homogeneous Self-Similar IFS on the Line

Although many of the relevant ideas were around earlier, as discussed above, it seems to the author that the real "birth" of this technique should be dated to the 1995 paper by M. Pollicott and K. Simon *"The Hausdorff dimension of λ-expansions with deleted digits"* [24] (a shorter preliminary version was circulated in 1994). It is interesting to review the history of this paper.

4.1. The {1, 2, 3}-Problem and Expansions with Deleted Digits

This problem was first studied by M. Keane and M. Smorodinsky in the early 1990s. They considered the homogeneous self-similar IFS on the line

$$\mathcal{S}^\lambda := \{S_0(x) = \lambda x,\ \ S_1(x) = \lambda x + 1,\ \ S_3(x) = \lambda x + 3\},$$

with a real parameter $\lambda \in (0,1)$ being the contraction coefficient. Denote the attractor of $\mathcal{S}^\lambda$ by Λ^λ. A direct verification shows that

$$\Lambda^\lambda = \left\{x = \sum_{n=0}^{\infty} a_n \lambda^n : \ a_n \in \{0,1,3\}\right\}. \tag{9}$$

Thus Λ^λ is the set of base-λ expansions with digits $\{0,1,3\}$, with 2 being the "deleted digit". The motivation came from a problem of J. Palis (see [25]) who asked whether $\mathcal{C}_\alpha - \mathcal{C}_\beta$, the arithmetic difference of middle-α and middle-β Cantor sets, contains an interval whenever it has positive Lebesgue measure. (We will discuss the problem of differences/sums of Cantor sets and a related problem of convolutions of Cantor measures below, in Section 7). Note that if, for instance, $\gamma = \lambda^\ell$, $\ell \in \mathbb{N}$, where $\lambda = \frac{1-\alpha}{2}$ and $\gamma = \frac{1-\beta}{2}$ are the contraction ratios of the Cantor sets, then the arithmetic difference $\mathcal{C}_\alpha - \mathcal{C}_\beta$ may be expressed in terms of base-λ expansions, with an "unusual" set of digits. The $\{0,1,3\}$-problem was chosen as one of the "simplest non-trivial examples of this kind", according to M. Smorodinsky [personal comminication]. The next lemma is elementary. The symbol $\mathcal{H}^\alpha$ denotes the α-dimensional Hausdorff measure.

Lemma 1. **(i)** *The smallest closed interval containing* Λ^λ *is* $I_\lambda := [0, \frac{3}{1-\lambda}]$.

(ii) *For all* $\lambda \in (0,1)$, $\dim_H(\Lambda^\lambda) \le \min\{1, \frac{\log 3}{|\log \lambda|}\}$.

(iii) *the IFS* $\mathcal{S}^\lambda$ *satisfies the Open Set Condition for all* $\lambda \in (0, \frac{1}{4}]$, *hence*

$$\dim_H(\Lambda^\lambda) = s_\lambda := \frac{\log 3}{|\log \lambda|} \ \ \textit{and}\ \ 0 < \mathcal{H}^{s_\lambda}(\Lambda^\lambda) < \infty,\ \lambda \in (0, 1/4];$$

(iv) $\Lambda^\lambda = I_\lambda$ *(the line segment) for all* $\lambda \in [\frac{2}{5}, 1]$.

The lemma implies that the non-trivial parameter range is $\lambda \in (\frac{1}{4}, \frac{2}{5})$, when two first cylinder intervals of level one intersect, but there is a gap between the second and the third interval, hence the attractor Λ^λ is disconnected. What can be said about the measure-theoretic, fractal-dimensional, and topological properties of Λ^λ? It is natural to distinguish between the *subcritical case*: $\lambda \in (\frac{1}{4}, \frac{1}{3})$, *critical case*: $\lambda = \frac{1}{3}$, and *supercritical case:* $\lambda \in (\frac{1}{3}, \frac{2}{5})$, depending on whether the similarity dimension s_λ is smaller, equal, or greater than one. (The critical case is a classical example of an IFS satisfying the so-called *Weak Separation Condition*.)

For $\lambda < \frac{1}{3}$, we have by Lemma 1(i) that Λ^λ is a Cantor set of dimension less than one. Indeed, Λ^λ has zero Lebesgue measure, hence is totally disconnected, and an attractor of a non-degenerate IFS cannot have isolated points. M. Keane raised a question whether $\lambda \mapsto \dim_H(\Lambda^\lambda)$ is continuous. This was answered negatively by M. Pollicott and K. Simon [24].

Theorem 4 (Pollicott and Simon 1995 [24])**.** *Let Λ^λ be the attractor of the IFS $\mathcal{S}^\lambda$. Then*
(i) $\dim_H(\Lambda^\lambda) = s_\lambda = \frac{\log 3}{|\log \lambda|}$ *for Lebesgue-a.e.* $\lambda \in (\frac{1}{4}, \frac{1}{3})$*;*
(ii) *there is a dense subset $\mathcal{E} \subset (\frac{1}{4}, \frac{1}{3})$ such that $\dim_H(\Lambda^\lambda) < s_\lambda$ for $\lambda \in \mathcal{E}$.*

In fact, [24] considered a more general case of finitely many integer digits. In the supercritical case, after some partial results in [24,26], the following was shown in [27].

Theorem 5. *For Lebesgue-a.e. $\lambda \in (\frac{1}{3}, \frac{2}{5})$ the attractor Λ^λ has positive Lebesgue measure.*

Now much stronger results are known. In particular, it follows from the work of Rapaport and Varjú [28] (Theorem A.1) that the only reason for the "dimension drop" $\dim_H(\Lambda^\lambda) < \min\{1, s_\lambda\}$ could be the presence of "exact overlaps", hence the set of exceptions is countable. Surprisingly, we still do not know whether the sets Λ^λ have non-empty interior, even for a single parameter $\lambda \in (\frac{1}{3}, \frac{2}{5})$.

What about the (fractal) measure of the attractor Λ^λ in its dimension in the parameter range $\lambda \in (\frac{1}{4}, \frac{1}{3})$? It was shown by Y. Peres, K. Simon, and the author [29], using another version of transversality, that for Lebesgue-a.e. λ in this range the Hausdorff measure is zero, whereas the packing measure is positive and finite. See a brief discussion of this issue in Section 5.

Remark 1. *A lot of credit should be given to Michael Keane, who was lecturing widely on this problem and circulating his preprint with Meir Smorodinsky, which later became part of [26]. For K. Simon and for the author this gave a very important impetus to their future work.*

4.2. Bernoulli Convolutions; Definition of Transversality

The classical Bernoulli convolution measure ν_λ with a parameter $\lambda \in (0,1)$ is defined as the distribution of the random series $\sum_{n=0}^\infty \pm\lambda^n$, where the signs are i.i.d., with probabilities $(\frac{1}{2}, \frac{1}{2})$. Equivalently, ν_λ is the invariant measure for the probabilistic IFS $\{\lambda x - 1, \lambda x + 1\}$ on $\mathbb{R}$, with the probability vector $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$. We do not repeat here the long and fascinating (still unfinished) history of the problem *"for which λ is ν_λ absolutely continuous?"*, which may be found in many papers and surveys, see e.g., [30–32]. It is natural to consider them in the more general framework of homogeneous IFS on the line, which includes expansions with deleted digits as well.

Let $D = \{d_1, \ldots, d_m\} \subset \mathbb{R}$ be a finite set of "digits", and let $\mathbf{p} = (p_1, \ldots, p_m)$ be a probability vector. We consider the IFS family

$$\mathcal{F}_D^\lambda = \{f_j : x \mapsto \lambda x + d_j\}_{j \le m}, \quad x \in \mathbb{R}, \tag{10}$$

its attractor Λ_D^λ, and the invariant measure $\nu_\lambda^{D,\mathbf{p}}$. Here $\lambda \in (0,1)$ is a parameter. Note that the similarity dimension is equal to $s_\lambda = \frac{\log m}{-\log \lambda}$. The natural projection from the symbolic space Σ_m to the attractor is given by

$$\Pi^\lambda(\omega) = \sum_{n=0}^\infty \omega_n \lambda^n.$$

Note that the Strong Separation Condition holds on a parameter interval J if and only if the graphs of the functions from $\lambda \mapsto \Pi^\lambda(\omega)$ and $\lambda \mapsto \Pi^\lambda(\tau)$, do not intersect over J for distinct ω and τ in Σ_m. We say that the *transversality condition* holds over J if these graphs may intersect, but only transversally, that is, at a non-zero angle. There are several definitions of transversality in the literature. We start with the one used most frequently.

Definition 1. *We say that the transversality condition holds for the IFS family* (10) *on* $J \subset (0,1)$ *if there exists a constant* $C_{\rm tr}$ *such that for any* $\omega, \tau \in \Sigma_m$*, with* $\omega_0 \neq \tau_0$ *holds*

$$\mathcal{L}^1\Big\{\lambda \in J:\ |\Pi^\lambda(\omega) - \Pi^\lambda(\tau)| \le r\Big\} \le C_{\rm tr}\cdot r \ \ \text{for all } r>0. \tag{11}$$

Notice the resemblance of this condition to (3). The next theorem combines [24,27]; see also [31,33,34].

Theorem 6. *Suppose that the IFS family* (10) *satisfies the transversality condition* (11) *on an interval* $J \subset (0,1)$*. Then the following holds:*

(i) $\dim_{\rm H}(\Lambda^\lambda) = \min\{1, s_\lambda\}$ *for Lebesgue-a.e.* $\lambda \in J$*;*

(ii) $\mathcal{L}^1(\Lambda^\lambda) > 0$ *for Lebesgue-a.e.* $\lambda \in J \cap (\frac{1}{m}, 1)$*;*

(iii) *the measure* $\nu_\lambda^{D,\mathbf{p}}$ *is absolutely continuous, with a density in* L^2 *for Lebesgue-a.e.* $\lambda \in J \cap (\sum_{j=1}^m p_j^2, 1)$.

The proof of Theorem 6 proceeds along the lines of the proofs of Theorems 1 and 2 above, with the double integration over Σ_m and the transversality condition (11) replacing the geometric condition (3). In parts (i) and (ii) the "natural" self-similar measure is used, that is, the push-forward of $\mathbf{p}^{\mathbb{N}}$ under the natural projection Π^λ. Replacing the integration of the energy over the Lebesgue measure on J by integration over the appropriate Frostman measure, one can easily obtain an estimate on the dimension of exceptions in Theorem 6(i), in the region of parameters where $s_\lambda < 1$:

$$\text{for every } \lambda_0 \in J,\ \varepsilon > 0,\ \ \dim_{\rm H}\Big(\{\lambda \in B_\varepsilon(\lambda_0):\ \dim_{\rm H}(\Lambda^\lambda) < s_\lambda\}\Big) \le \sup\{s_\gamma:\ \gamma \in B_\varepsilon(\lambda_0)\}.$$

In the special case of expansions with deleted digits this was pointed out by Pollicott and Simon [24].

Now much stronger results are known, due to Hochman [35] and Shmerkin [36]. In particular, in the claims of the theorem "for Lebesgue-a.e." can be replaced by "for all λ outside a set of zero Hausdorff dimension". Even more importantly, the transversality condition, which is often hard to check, can be replaced in the dimension claim by a mild "exponential separation condition", introduced by Hochman. However, if we do know that transversality holds, Theorem 6 yields a much shorter proof of the "almost every parameter" result.

In order to verify the transversality condition, the next lemma is useful.

Lemma 2. *The following are equivalent on a compact interval* $J \subset (0,1)$*:*

(i) *the transversality condition* (11) *holds;*

(ii)

$$\exists\,\delta>0,\ \forall\,\omega,\tau \in \Sigma_m,\ \omega_0 \neq \tau_0,\ \lambda \in J,\ \Big|\Pi^\lambda(\omega) - \Pi^\lambda(\tau)\Big| \le \delta \implies \Big|\tfrac{d}{d\lambda}(\Pi^\lambda(\omega) - \Pi^\lambda(\tau))\Big| \ge \delta; \tag{12}$$

(iii)

$$\forall\,\omega,\tau \in \Sigma_m,\ \omega \neq \tau,\ \Pi^\lambda(\omega) = \Pi^\lambda(\tau) \implies \tfrac{d}{d\lambda}\big(\Pi^\lambda(\omega) - \Pi^\lambda(\tau)\big) \neq 0. \tag{13}$$

Transversality in the Pollicott–Simon paper appeared in the form (12) in [24] (Lemma 1), and the derivation of (11) is contained in the proof of [24] (Lemma 2). The formulation (13) appeared in the author's paper [27].

Proof sketch. The implication (12) $\implies$ (13) is trivial. The implication (13) $\implies$ (12) is proved by compactness. If (12) does not hold, we can find, for any $n \in \mathbb{N}$, two sequences $\omega^{(n)}, \tau^{(n)} \in \Sigma_m$, with $\omega_0^{(n)} \neq \tau_0^{(n)}$, such that for some $\lambda_n \in J$,

$$\left|\Pi^{\lambda_n}(\omega) - \Pi^{\lambda_n}(\tau)\right| \le \varepsilon_n \text{ and } \left|\tfrac{d}{d\lambda}(\Pi^\lambda(\omega) - \Pi^\lambda(\tau))|_{\lambda=\lambda_n}\right| \le \varepsilon_n,$$

where $\varepsilon_n \to 0$. Passing to a subsequence, using compactness of Σ_m and J, we obtain ω, τ, with $\omega_0 \neq \tau_0$, which violate (13). It is important for this argument that the set of possible digits is discrete, so that $\omega_0 \neq \tau_0$ is guaranteed for the limiting sequences.

Next we explain the derivation (12) $\implies$ (11). Denote

$$f_\lambda(\omega, \tau) := \Pi^\lambda(\omega) - \Pi^\lambda(\tau). \tag{14}$$

Fix $\omega, \tau \in \Sigma_m$ with $\omega_0 \neq \tau_0$. It is enough to consider $0 < r < \delta$. Condition (12) implies that the set $\{\lambda \in J : |f_\lambda(\omega, \tau)| \le r\}$, whose $\mathcal{L}^1$-measure we need to estimate, is a union of finitely many intervals on each of which the function $\lambda \mapsto f_\lambda(\omega, \tau)$, $\lambda \in J$, is monotonic. Each of these intervals has length at most $2r/\delta$ by the lower bound $|\frac{d}{d\lambda} f_\lambda(\omega, \tau)| \ge \delta$ on them. On the other hand, each of these intervals lies in a larger interval of monotonicity, a component of the set $\{\lambda \in J : |f_\lambda(\omega, \tau)| < \delta\}$, and such intervals are disjoint. There is an easy uniform upper bound $|\frac{d}{d\lambda} f_\lambda(\omega, \tau)| \le C$, depending only on D and $\max J < 1$. This implies that every such larger interval has length at least $2\delta/C$, except possibly the first and the last one. Combining everything together yields that

$$\mathcal{L}^1\{\lambda \in J : |f_\lambda(\omega, \tau)| \le r\} \le \frac{2r|J|}{\delta} \cdot \Big(2 + \frac{|J|C}{2\delta}\Big),$$

concluding the proof of (11).

The implication (11) $\implies$ (13) is very easy; we leave it to the reader. □

4.3. Checking Transversality

For a bounded subset $\Gamma \subset \mathbb{R}$ consider the class of power series

$$\mathcal{B}_\Gamma = \Big\{g(x) = \sum_{n=0}^{\infty} a_n x^n : a_n \in \Gamma, \ n \ge 0, a_0 \neq 0\Big\}.$$

It follows from Lemma 2(iii) that transversality for the IFS (10) holds on any interval J on which functions of the form $\lambda \mapsto f_\lambda(\omega, \tau) := \Pi^\lambda(\omega) - \Pi^\lambda(\tau)$ do not have double zeros, that is, if $\omega \neq \tau$, then

$$\nexists\, \lambda \in J : \ f_\lambda(\omega, \tau) = \tfrac{d}{d\lambda} f_\lambda(\omega, \tau) = 0.$$

By the definition of Π^λ, this is equivalent to the absence of double zeros for functions from $\mathcal{B}_{D-D}$. For instance, for Bernoulli convolutions, we get the class $\mathcal{B}_{0,\pm 1}$ (strictly speaking, $\mathcal{B}_{0,\pm 2}$, but we have an obvious equality $\mathcal{B}_{c\Gamma} = c\mathcal{B}_\Gamma$ for $c \neq 0$). For the $\{0,1,3\}$ problem, we get the class $\mathcal{B}_{0,\pm 1,\pm 2,\pm 3}$.

In general, it is very difficult to find precisely the set of double zeros for a class of power series, see [37]. However, for practical purposes it is enough to find "reasonable" intervals where transversality holds. With this in mind, in [27], the method of so-called $(*)$-functions was developed. The main idea is that (a) it is easier to find the transversality interval for a class of power series with a convex set of possible coefficients (except for a_0 which we normalize to be 1); (b) for such classes there is an "optimal" power series having the smallest double zero; moreover, such a power series will have exactly two sign changes and at most one coefficient which is not "extremal". This leads to the following.

Definition 2. *For $\gamma > 0$ let*

$$\mathcal{B}_\gamma = \Big\{ g(x) = 1 + \sum_{n=1}^{\infty} a_n x^n : \ |a_n| \le \gamma, \ n \ge 1 \Big\}.$$

An interval J is said to be a transversality interval for $\mathcal{B}_\gamma$ if $g(x) = 0$ implies that $g'(x) \neq 0$ for $g \in \mathcal{B}_\gamma$ and $x \in J$. A power series $h(x)$ is called a $(*)$-function for $\mathcal{B}_\gamma$ if for some $k \ge 1$ and $a_k \in \mathbb{R}$,

$$h(x) = 1 - \gamma \sum_{i=1}^{k-1} x^i + a_k x^k + \gamma \sum_{i=k+1}^{\infty} x^i.$$

Note that $h \in \mathcal{B}_\gamma$ only when $a_k \in [-\gamma, \gamma]$, but this is not required for the definition.

Lemma 3. *Suppose that $h(x)$ is a $(*)$-function for $\mathcal{B}_\gamma$ and $x_0 \in (0, 1)$ is such that*

$$h(x_0) > \delta \quad \text{and} \quad h'(x_0) < -\delta,$$

for some $\delta > 0$. Then $[0, x_0]$ is a transversality interval for $\mathcal{B}_\gamma$.

Observe that $\mathcal{B}_{D-D} \subset \mathcal{B}_{\gamma(D)}$, where

$$\gamma(D) = \max\left\{ \frac{|d_i - d_j|}{|d_k - d_\ell|} : \ 1 \le i, j, k, \ell \le m, \ k \neq \ell \right\},$$

so that any transversality interval for $\mathcal{B}_{\gamma(D)}$ is also a transversality interval for $\mathcal{B}_{D-D}$.

For the proof of Lemma 3, see [33] (Section 3) or [34] (Section 5) (which were based, in turn, on [27] (Section 3)), or the upcoming book [1]. Obtained with the help of the lemma, the next result can be used in specific cases; see the same references for details. Additional results of this kind may be found in Simon–Tóth [38].

Corollary 1. **(i)** *$\mathcal{B}_1$ satisfies the transversality condition on $[0, 0.649]$;*
(ii) *$\mathcal{B}_2$ satisfies the transversality condition on $[0, 0.5)$, and this is sharp (i.e., fails at 0.5);*
(iii) *$\mathcal{B}_3$ satisfies the transversality condition on $[0, 0.415]$;*
(iv) *$\mathcal{B}_\gamma$ satisfies the transversality condition on $[0, (1 + \sqrt{\gamma})^{-1})$ for all $\gamma \ge 1$, and this is sharp for $\gamma \ge 3 + \sqrt{8}$.*

5. Families of Self-Similar IFS—Further Developments

It is impossible to describe here all the results obtained with the help of the transversality method—there are too many. We mention only a few.

- The results of the last section extend in a rather straightforward way to families of homogeneous self-similar IFS in the complex plane $\{\lambda z + d_j\}_{j \le m}$ for a complex λ, with $0 < |\lambda| < 1$, see [21]. However, even for the simplest case $\{\lambda z, \lambda z + 1\}$ (the IFS of the "complex Bernoulli convolution") there is a new feature: a non-trivial "connectedness locus" (the set of parameters for which the attractor is connected). This leads to a whole separate direction of research, see [37,39–42] and references therein.
- In [29], Peres, Simon, and the author studied families of homogeneous self-similar IFSs with overlaps on the line. It was shown that under the transversality condition in the "subcritical" parameter region, the packing measure of the attractor in its dimension is positive and finite for almost every parameter, whereas the Hausdorff measure is typically zero, for those parameters where the cylinders do overlap. Here transversality was used as well, but in a different way. Zero Hausdorff measure (in the similarity dimension) was deduced with the help of a Bandt–Graf criterion [43] saying that it is equivalent (for a self-similar IFS) to the existence of two ε-relatively close cylinders for any $\varepsilon > 0$. The idea is, roughly speaking, as follows: Transversality

implies that if two particular cylinders of size $\sim r^n$ intersect for a parameter t_0, then there is an exact overlap of these cylinders for some parameter $t_1 \in B(t_0, C_1 r^n)$ and then these cylinders are ε-relatively close for all parameters $t \in B(t_1, c_2 r^n)$, with positive constants C_1, c_2, independent of n.

- Michał Rams [44,45] developed what became known as the *intersection numbers method*, based on a careful counting of the number of intersecting cylinders. This method also used transversality, but in a more combinatorial way. This led to new proofs of "almost every parameter" theorems on the dimension and packing measure of the attractors, as well as several sharper results. For instance, ref. [45] gave a *packing dimension estimate* for the set of exceptional parameters for which there is a certain dimension drop. Rams' setting is, actually, more general: he considered conformal IFS in $\mathbb{R}^d$.
- Peres and Schlag [46] developed a powerful "generalized projection theorem", based on harmonic analysis (Paley–Wiener decomposition, fractional Sobolev spaces), using a version of transversality. They obtained results on the "Sobolev dimension" of projected measures for typical parameters and estimates on the dimension of exceptions for absolute continuity.
- Neunhäuserer [47] and Ngai–Wang [48] obtained results on absolute continuity for a.e. parameters in a certain subset of the super-critical region for specific examples of non-homogeneous self-similar IFS, such as $\{\lambda_1 x, \lambda_2 x + 1\}$, essentially using the transversality methods of [27,33,34].
- Most of the results mentioned in this section have by now been superseded by those of Hochman [35,49], Shmerkin [36,50], Shmerkin–Solomyak [51], Saglietti–Shmerkin–Solomyak [52], Varjú [53,54], and others, with the help of new, additive combinatorics and harmonic analysis methods. These methods are much more difficult and technical, and we do not discuss them here. Moreover, so far these methods have not been extended to nonlinear IFS of any generality, whereas transversality has been widely used for them, as we describe in the next section.

6. Transversality Method for Conformal IFS on $\mathbb{R}$ and for Some Classes of Dynamical Systems

The transversality method was extended to families of $C^{1+\theta}$-smooth uniformly hyperbolic IFSs with overlaps on the line, by K. Simon and the author in 1996; this manuscript remained unpublished. Instead, it was incorporated into a paper [55], on the dimension of non-conformal horseshoes in $\mathbb{R}^3$, and into the papers [56,57], which extended the method to infinite hyperbolic IFS and (finite) parabolic IFS on the line. Here we present only the most basic "sample" result for illustration.

Let $\theta > 0$. An IFS $\mathcal{F} = \{f_1, \ldots, f_m\}$ on a compact interval $I \subset \mathbb{R}$ is called a *$C^{1+\theta}$-smooth hyperbolic IFS* if $f_j \in C^{1+\theta}(I \to I)$ and there exist $0 < \gamma_1 \le \gamma_2 < 1$ such that

$$0 < \gamma_1 \le |f_j'(x)| \le \gamma_2 < 1, \ \text{ for all } j \le m, \ x \in I.$$

Denote by $\Lambda_{\mathcal{F}}$ the attractor of $\mathcal{F}$ and by $\Pi^{\mathcal{F}} : \Sigma_m \to \Lambda_{\mathcal{F}}$ the corresponding natural projection map. If the OSC holds, the formula for the Hausdorff dimension $\dim_{\mathrm{H}}(\Lambda_{\mathcal{F}})$ is well known. It is given by the Bowen's equation [16,58,59]:

$$\dim_{\mathrm{H}}(\Lambda_{\mathcal{F}}) = s(\mathcal{F}) \quad \text{where} \quad P_{\mathcal{F}}(s(\mathcal{F})) = 0. \tag{15}$$

Here $P_{\mathcal{F}}(t)$ is the *pressure function*, defined by

$$P_{\mathcal{F}}(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{|u|=n} \|f_u'\|^t,$$

where $\|\cdot\|$ is the sup norm on I. It is easy to see that for a self-similar IFS, the value of $s(\mathcal{F})$ agrees with the similarity dimension, defined in (5). In general (when the OSC fails or is not known to hold), we have the inequality

$$\dim_{\mathrm{H}}(\Lambda_{\mathcal{F}}) \le \min\{1, s(\mathcal{F})\}.$$

Given an ergodic shift-invariant measure μ on Σ_m, consider the push-forward measure $\nu(\mathcal{F},\mu) = \Pi^{\mathcal{F}}\mu$, which we call the *invariant measure* for the IFS. For the Hausdorff dimension of $\nu(\mathcal{F},\mu)$ we have an inequality, which extends (6):

$$\dim_{\mathrm{H}}(\nu(\mathcal{F},\mu)) \le \min\{1, s(\mathcal{F},\mu)\}, \quad \text{where } s(\mathcal{F},\mu) := \frac{h_\mu}{\chi_\mu(\mathcal{F})}.$$

Here h_μ is the Kolmogorov–Sinai entropy of μ and $\chi_\mu(\mathcal{F})$ is the Lyapunov exponent, defined by

$$\chi_\mu(\mathcal{F}) = -\int_{\Sigma_m} \log|f'_{\omega_1}(\Pi^{\mathcal{F}}(\sigma\omega))|\, d\mu(\omega).$$

Again, assuming the OSC, we get the equality: $\dim_{\mathrm{H}}(\nu(\mathcal{F},\mu)) = s(\mathcal{F},\mu)$.

Now suppose that we have a family of $C^{1+\theta}$-smooth hyperbolic IFS $\mathcal{F}^{\mathbf{t}} = \{f_1^{\mathbf{t}}, \dots, f_m^{\mathbf{t}}\}$ on the interval I depending on a parameter $\mathbf{t} \in U$, for some open set $U \subset \mathbb{R}^d$, with $d \ge 1$. Assume that the functions $\mathbf{t} \mapsto f_j^{\mathbf{t}}$ are continuous in the $C^{1+\theta}$ norm, defined by

$$\|f\|_{C^{1+\theta}} := \|f\|_\infty + \|f'\|_\infty + \sup_{x \ne y \in I} \frac{|f'(x) - f'(y)|}{|x-y|^\theta}.$$

The *transversality condition* is analogous to the one from (1), but we allow a higher-dimensional parameter space here. The transversality condition for the family $\mathcal{F}^{\mathbf{t}}$ is said to hold on $U \subset \mathbb{R}^d$ if there exists $C_{\mathrm{tr}} > 0$ such that for any $\omega, \tau \in \Sigma_m$, with $\omega_0 \ne \tau_0$,

$$\mathcal{L}^d\big(\{\mathbf{t} \in U : \ |\Pi^{\mathbf{t}}(\omega) - \Pi^{\mathbf{t}}(\tau)| \le r\}\big) \le C_{\mathrm{tr}} \cdot r, \quad \text{for all } r > 0, \tag{16}$$

where $\Pi^{\mathbf{t}} = \Pi^{\mathcal{F}^{\mathbf{t}}}$. The following theorem combines results from [56,57] in a special case.

Theorem 7. *Let $\mathcal{F}^{\mathbf{t}}$ be a family of $C^{1+\theta}$-smooth hyperbolic IFS on $I \subset \mathbb{R}$, satisfying the conditions above, including the transversality condition on U. Let $\Lambda^{\mathbf{t}} = \Lambda_{\mathcal{F}^{\mathbf{t}}}$ be the attractor. Further, let μ be an ergodic σ-invariant measure on Σ_m and $\nu^{\mathbf{t}}$ the corresponding invariant measure for the IFS: $\nu^{\mathbf{t}} = \Pi^{\mathbf{t}}\mu$. Then the following holds for $\mathcal{L}^d$-a.e. $\mathbf{t} \in U$:*

(i) $\dim_{\mathrm{H}}(\Lambda^{\mathbf{t}}) = \min\{1, s(\mathcal{F}^{\mathbf{t}})\}$ *and* $\mathcal{L}^1(\Lambda^{\mathbf{t}}) > 0$ *if* $s(\mathcal{F}^{\mathbf{t}}) > 1$, *where* $s(\mathcal{F}^{\mathbf{t}})$ *is the solution of the Bowen equation;*

(ii) $\dim_{\mathrm{H}}(\nu^{\mathbf{t}}) = \min\{1, s(\mathcal{F}^{\mathbf{t}},\mu)\}$ *and* $\nu^{\mathbf{t}} \ll \mathcal{L}^1$ *if* $s(\mathcal{F}^{\mathbf{t}},\mu) > 1$, *where* $s(\mathcal{F}^{\mathbf{t}},\mu) = \frac{h_\mu}{\chi_\mu(\mathcal{F}^{\mathbf{t}})}$.

There are situations when it is natural to assume that the ergodic measure depends on the parameter as well, so that $\nu^{\mathbf{t}} = \Pi^{\mathbf{t}}\mu^{\mathbf{t}}$, for instance, when considering probabilistic IFS with place-dependent probabilities. Recently, results analogous to Theorem 7 were obtained by Bárány, Simon, Śpiewak, and the author [60], under appropriate (stronger) smoothness conditions and assumptions on the parameter dependence for the IFS and for the measure. This required developing novel techniques (a delicate extension of [46]) for the absolute continuity result.

Checking transversality for non-linear systems is difficult, in general. We present here a "sample" result, which concerns the "translation family" of IFS. In some special cases, it can be strengthened; see [56,57,60] for details. In this form the result appears in the recent work of B. Bárány, I. Kolossváry, M. Rams, and K. Simon.

Proposition 1 ([61], Lemma 2.14)**.** *Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a $C^{1+\theta}$-smooth uniformly hyperbolic IFS on a compact interval $I \subset \mathbb{R}$, such that $f_j(I) \subset \mathrm{int}(I)$ for $j \le m$. Let $U = \{\mathbf{t} = (t_1, \ldots, t_m) : f_j(I) + t_j \subset I$ for all $j \le m\}$, and consider the family of IFS $\mathcal{F}^{\mathbf{t}} = \{f_1^{\mathbf{t}}, \ldots, f_m^{\mathbf{t}}\}_{\mathbf{t}\in U}$, where $f_j^{\mathbf{t}}(x) = f_j(x) + t_j$. If*

$$\max_{i \ne j}\left(\|f_i'\|_\infty + \|f_j'\|_\infty\right) < 1,$$

then the transversality condition holds on U. In particular, this holds when $\max_{i\le m}\|f_i'\|_\infty < \frac{1}{2}$.

From the very beginning, the transversality method was not restricted to IFS, but applied to various classes of dynamical systems. In fact, transversality was used already in the work of Simon [14] on the Hausdorff dimension for noninvertible maps, as mentioned above. Here we only list a few papers to indicate the wide variety of settings where it was applied, without giving any details.

- Transversality was used in [55] to obtain dimension and measure results on "nonlinear fat baker's maps" and certain horseshoes in $\mathbb{R}^3$.
- Ledrappier [62] showed how the knowledge of the dimension of distributions of certain random sums yields dimension formulas for fractal graphs. Combined with transversality methods, this allowed us to obtain almost sure results for certain families in [21]. A further development of this method yielded such formulas for the graph of the celebrated Weierstrass function a.e. in some parameter interval, in the work of Barański, Bárány, and Romanowska [63].
- Schmeling and Troubetzkoy [64] considered a class of hyperbolic endomorphisms with singularities and gave a criterion when the map is invertible on a set of full measure with respect to the SBR measure. Schmeling [65] applied this general theory to a specific example: the so-called Belykh family, and obtained results on almost sure invertibility and the dimension of the SBR measure, a.e. in a transversality interval for one of the parameters.
- Mihailescu and Urbański have been using transversality methods in several papers, in particular, in their study of hyperbolic skew-products [66]. Sumi and Urbański [67] obtained dimension and measure results for transversal families of expanding rational semigroups on the Riemann sphere.
- The transversality techniques of [56,57,66] are being applied in the currently active study of "blenders", see, e.g., Biebler [68].
- Bárány, Pollicott, and Simon [69] established absolute continuity of the Furstenberg measure a.e. in some parameter region, using the methods of [57]. Bárany and Rams [70] studied dimension-maximizing measures for planar self-affine systems under the strong separation and the so-called dominated splitting conditions. They introduced the notion of *strong-stable transversality* and obtained Ledrappier–Young formulas for certain Gibbs measures, assuming that it holds.

7. Arithmetic Sums and Differences of Cantor Sets; Convolution of Measures

In the early days of hyperbolic dynamics it was believed that uniform hyperbolicity is a generic property in the space of diffeomorphisms of a manifold. This was disproved by Sheldon Newhouse [71], who discovered what is now known as the *Newhouse phenomenon*: existence of residual sets of C^2-diffeomorphisms of a compact surface with infinitely many attractors. Newhouse's construction is based an a creation of robust homoclinic tangencies. This motivated Jacob Palis to initiate a broad program of studying the phenomena of unfolding homoclinic tangencies. In the study of bifurcations of a generic one-parameter family of surface diffeomorphisms having generic homoclinic tangency at a parameter value, the arithmetic difference of two regular Cantor sets appears in a natural way, see [25]. They arise from the stable and unstable foliations of the basic set. Palis and Takens [72] showed that if the sum of the Hausdorff dimensions of these foliations is less than one, then there is hyperbolicity on a set of parameters of full measure. This corresponds to the simple fact that the arithmetic difference of two Cantor sets, whose Hausdorff dimensions add up to less than one, is of zero Lebesgue measure. In the opposite case, when the sum of the dimensions is greater than one, Palis conjectured that the arithmetic difference has nonempty interior, at least, generically. In dynamics this corresponds to an entire interval of non-hyperbolicity. An important result in this direction was first obtained by Palis and Yoccoz [73]. They proved that, if the sum of the dimensions of the stable and unstable foliations of the basic set is greater than one, then non-hyperbolicity holds on a set of positive measure of parameters. In fact, this paper was an important source of ideas and methods in the development of the transversality method by the author. Palis and Yoccoz write that they were motivated by Kaufman's proof of Marstrand's Theorem (the positive measure part). They explicitly use a transversality hypothesis, and their proof proceeds, in part, by showing that the push-forward of a certain measure μ by a map T_s depending on a parameter s satisfies $\int_J |\widehat{T_s\mu}|^2\, ds < \infty$, which guarantees absolute continuity a.e. This method was used for Bernoulli convolutions in [27].

In a seminal work, Moreira and Yoccoz [74] settled Palis's conjecture, showing that if the sum of the Hausdorff dimensions of two generic Cantor sets is greater than one, then their arithmetic difference has nonempty interior. However, often one is interested in specific families of Cantor sets, with a finite-dimensional set of parameters. In this context the problem of interior is wide open. In particular, Palis and Takens asked what happens for a pair of classical Cantor sets $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$. The Gap Lemma of Newhouse [71] gives a sufficient condition for non-empty interior of $\mathcal{C}_\alpha - \mathcal{C}_\beta$ in terms of their *thickness*, but it is far from sharp from the point of view dimension. As an intermediate step, we address the easier question of when the arithmetic difference (or, equivalently, sum) of the Cantor sets has positive Lebesgue measure.

In this direction it was first proved in [75] that, given $\alpha > 0$, the set $\mathcal{C}_\alpha + \mathcal{C}_\beta$ has positive Lebesgue measure for Lebesgue-a.e. β, such $\dim_H(\mathcal{C}_\alpha) + \dim_H(\mathcal{C}_\beta) > 1$. The proof proceeded by showing absolute continuity of the convolution measure for a.e. parameter and used the transversality method. (Since we can always replace the set $\mathcal{C}_\beta$ by $-\mathcal{C}_\beta$, it is equivalent to study arithmetic sums, which is more convenient from the point of view of convolution measures supported on the set.) A stronger result was obtained by Peres and Solomyak in [34], which we state below.

Consider a family of homogeneous self-similar IFSs $\mathcal{F}^\lambda = \{x \mapsto \lambda x + d_j(\lambda)\}_{j \le m}$, parametrized by $\lambda \in J \subset (0,1)$, with J a compact interval. Denote its attractor by Λ^λ. Assume that $d_j \in C^1(J)$ and the Strong Separation Condition (SSC) holds:

$$(\lambda\Lambda^\lambda + d_i(\lambda)) \cap (\lambda\Lambda^\lambda + d_j(\lambda)) = \emptyset, \ \ \text{for all } i \ne j,\ \lambda \in J.$$

The SSC implies that $s_\lambda := \dim_H(\Lambda^\lambda) = \dfrac{\log m}{\log(1/\lambda)}$ and hence necessarily $J \subset (0, m^{-1})$.

Theorem 8 ([34], Theorem 1.1)**.** *Let $K \subset \mathbb{R}$ be any compact set. Then $\mathcal{L}^1(K + \Lambda^\lambda) > 0$ for Lebesgue-a.e. $\lambda \in J$, such that $s_\lambda + \dim_H(K) > 1$.*

Peres and Schlag [46] obtained an estimate on the dimension of exceptions; in particular, they showed in [46] (Theorem 5.12) that if $d_j \in C^2(J)$, with $J = [\lambda_0, \lambda_1] \subset (0, m^{-1})$, then

$$\dim_H\{\lambda \in J : \mathcal{L}^1(K + \Lambda^\lambda) = 0\} \le 2 - (\dim_H(K) + \dim_H(\Lambda^\lambda)).$$

What is nice about Theorem 8 is that no transversality condition appears explicitly! It is, in fact, "hidden" in the SSC of the IFS, as we show later. Theorem 8 follows from a result on convolution of measures. Let $\Pi^\lambda : \Sigma_m \to \Lambda^\lambda$ be the natural projection corresponding to the IFS $\mathcal{F}^\lambda$. Let μ be a probability Borel measure on Σ_m, such that for some $\gamma_\mu \in (0,1)$,

$$(\mu \times \mu)\{(\omega, \tau) :\ |\omega \wedge \tau| \ge k\} \le C m^{-k\gamma_\mu}, \ \ \text{for all } k \in \mathbb{N}. \tag{17}$$

(Instead, we can assume that the lower correlation dimension of μ with respect to the metric $\varrho(\omega,\tau) = m^{-|\omega\wedge\tau|}$ is greater or equal than $\gamma_\mu \log m$). Consider the family of push-forward measures on Λ^λ:

$$\nu_\lambda = \Pi^\lambda \mu.$$

Further, let η be a compactly supported Borel probability measure on $\mathbb{R}$ satisfying the Frostman condition:

$$\eta(B_r(x)) \le Cr^{\gamma_\eta} \ \text{ for all } x \in \mathbb{R},\ r > 0. \tag{18}$$

Theorem 9 ([34], Theorem 2.1)**.** *Let $\mathcal{F}^\lambda$ be the family of IFS satisfying the SSC, as in Theorem 8 and ν_λ the corresponding family of push-forward measures. Assuming (17) and (18), we have that the measure $\eta * \nu_\lambda$ is absolutely continuous, with a density in $L^2(\mathbb{R})$, for Lebesgue-a.e. $\lambda \in J$, such that*

$$\gamma_\eta + \frac{\gamma_\mu \log m}{\log(1/\lambda)} > 1.$$

Theorem 8 follows, taking η to be the "almost optimal" Frostman measure on K and $\mu = (m^{-1},\dots,m^{-1})^{\mathbb{N}}$, so that $\gamma_\mu = 1$ and ν_λ is the natural self-similar measure on Λ^λ.

The next result deals with a complex-valued generalization. Although formally it may be new, the proof is very similar; we will give a sketch below, as a consequence of a "generalized projection scheme for convolutions".

Let U be an open set in $\mathbb{C}$, such that $\overline{U} \subset \mathbb{D}_* := \{z \in \mathbb{C} :\ 0 < |z| < 1\}$ (the open unit disk with the point at the origin removed). Suppose that b_j are analytic functions on $\overline{U}$ (i.e., on some neighborhood of $\overline{U}$) for $j = 1,\dots,m$, and consider the IFS in the complex plane:

$$\mathcal{G}^\lambda = \{z \mapsto \lambda z + b_j(\lambda)\}_{j\le m}. \tag{19}$$

Assume that $\mathcal{G}$ satisfies the SSC for all $\lambda \in \overline{U}$. Denote by Λ^λ the attractor of the IFS and by $\Pi^\lambda : \Sigma_m \to \Lambda^\lambda$ the natural projection. Let η be a compactly supported Borel probability measure on $\mathbb{C}$ satisfying the Frostman condition:

$$\eta(B_r(z)) \le Cr^{\gamma_\eta} \ \text{ for all } z \in \mathbb{C},\ r > 0. \tag{20}$$

Theorem 10. *Let $\mathcal{G}^\lambda$ be the family of IFS (19), satisfying the SSC, and let ν_λ be a family of push-forward measures on Λ^λ, with μ satisfying (17). Given a measure η satisfying (20), we have that the convolution $\eta * \nu_\lambda$ is absolutely continuous with respect to $\mathcal{L}^2$, with a density in $L^2(\mathbb{R}^2)$, for $\mathcal{L}^2$-a.e. $\lambda \in U$, such that*

$$\gamma_\eta + \frac{\gamma_\mu \log m}{\log(1/\lambda)} > 2.$$

Again, the SSC implies that $\dim_{\mathrm{H}}(\Lambda^\lambda) = s_\lambda = \frac{\log m}{\log(1/\lambda)}$.

Corollary 2. *Let $\mathcal{G}^\lambda$ be the family of IFS (19), satisfying the SSC, with attractor Λ^λ, and let $K \subset \mathbb{C} \cong \mathbb{R}^2$ be any compact set. Then $\mathcal{L}^2(K + \Lambda^\lambda) > 0$ for $\mathcal{L}^2$-a.e. $\lambda \in U$, such that*

$$\dim_{\mathrm{H}}(K) + \dim_{\mathrm{H}}(\Lambda^\lambda) > 2.$$

Pursuing this direction further may be of interest in connection with the investigation of *higher-dimensional Newhouse phenomenon*, which has been very active recently; see especially the work of Berger [76], but also Dujardin [77], Biebler [78], and others. Some of this research uses the framework of iterated function systems and transversality techniques.

Generalized Projection Scheme for Convolutions

This scheme is "modelled" after [79] (Proposition 2.1), extending it from the case of $\mathbb{R}$ to $\mathbb{R}^d$ and allowing $\mathscr{X}$ to be a general compact metric spaces, rather than the symbolic space.

In turn, [79] (Proposition 2.1) was "distilled" and "abstracted" from [34] (Theorem 2.1), which dealt with the case of natural measures ν_λ on homogeneous Cantor sets on $\mathbb{R}$. (A "generalized projection scheme" of a similar flavor, which covered parametrized self-similar self-conformal fractals, as well as geometric projections, was introduced in [21]. The latter should not be confused with the scheme used by Peres and Schlag [46], which was developed with a more specific goal.)

Let $(\mathscr{X}, \rho)$ be a compact metric space (the one we will project) and $\mathscr{U}$ be a set of parameters, a priori also a complete separable metric space, although in all applications it is an open subset of the Euclidean space or a smooth manifold of some dimension. We assume that $\mathscr{U}$ is equipped with a Borel probability measure ϑ. Further, suppose that we are given a family of maps

$$\Phi_\lambda : \mathscr{X} \to \mathbb{R}^d,$$

parametrized by $\lambda \in \mathscr{U}$, having the following properties. For $\omega, \tau \in \mathscr{X}$ we will write

$$\phi_{\omega,\tau}(\lambda) := \Phi_\lambda(\omega) - \Phi_\lambda(\tau), \;\; \lambda \in \mathscr{U}.$$

Hölder continuity: there exist $\alpha > 0$ and $C_1 > 0$ such that

$$|\phi_{\omega,\tau}(\lambda)| \le C_1 \rho(\omega,\tau)^\alpha, \;\; \text{for all } \;\omega, \tau \in \mathscr{X}, \;\; \lambda \in \mathscr{U}, \tag{21}$$

where we write $|\cdot|$ for the Euclidean norm in $\mathbb{R}^d$.

Transversality: there exist $\delta > 0$, $\beta \ge \alpha d$, and $C_2 > 0$, such that for all $r > 0$,

$$\sup_{v \in \mathbb{R}^d} \vartheta\big(\{\lambda \in \mathscr{U} : \; |v + \phi_{\omega,\tau}(\lambda)| \le r\}\big) \le C_2 \rho(\omega,\tau)^{-\beta} r^d, \;\; \forall\, \omega, \tau \in \mathscr{X} : \;\; \omega \ne \tau, \; \rho(\omega,\tau) \le \delta. \tag{22}$$

As we will see, the possibility of choosing $\delta > 0$ to be small is crucial. Next, suppose that μ is a Borel probability measure on $\mathscr{X}$ satisfying the

Measure correlation decay condition on $\mathscr{X}$: there exist $\gamma_\mu < \beta$ and $C_3 \ge 0$ such that

$$(\mu \times \mu)\big\{(\omega,\tau) \in \mathscr{X}^2 : \; \rho(\omega,\tau) \le r\big\} \le C_3 r^{\gamma_\mu} \;\; \text{for all } \omega \in \mathscr{X}, \; r > 0. \tag{23}$$

Observe that the latter follows from the *Frostman condition on $\mathscr{X}$:*

$$\mu(B_r(\omega)) \le C_3\, r^{\gamma_\mu}, \;\; \text{for all } \omega \in \mathscr{X}, \; r > 0. \tag{24}$$

We will consider the family of push-forward measures on $\mathbb{R}^d$:

$$\nu_\lambda = \Phi_\lambda \mu, \;\; \lambda \in \mathscr{U}.$$

Note that $\gamma_\mu \ge \beta$ would be unreasonable, since if, for instance, (24) holds, then ν_λ, which is a measure on the attractor, satisfies the Frostman condition with exponent $\gamma_\mu/\alpha \ge d$, which is impossible under the SSC.

Finally, let η be a compactly supported Borel probability measure on $\mathbb{R}^d$ satisfying the *Frostman condition on $\mathbb{R}^d$:* there exist $\gamma_\eta > 0$ and $C_4 \ge 0$ such that

$$\eta(B_r(x)) \le C_4\, r^{\gamma_\eta}, \;\; \text{for all } x \in \mathbb{R}^d, \; r > 0. \tag{25}$$

Theorem 11. *Under all the above assumptions, if*

$$\gamma_\eta > \frac{\beta - \gamma_\mu}{\alpha}, \tag{26}$$

then $\eta * \nu_\lambda \ll \mathcal{L}^d$ with a density in L^2 for ϑ-a.e. $\lambda \in \mathscr{U}$.

Discussion. The transversality condition (22) is the most difficult one to check. For applications, we assume that $\mathscr{U} \subset \mathbb{R}^d$ and $\vartheta = \mathcal{L}^d$. Further, suppose that $\mathscr{X} = \Sigma_m =$

$\{0,\ldots,m-1\}^{\mathbb{N}}$, with the metric $\rho(\omega,\tau) = m^{-|\omega\wedge\tau|}$. Let $\{f_1^\lambda,\ldots,f_m^\lambda\}$ be a family of smooth uniformly contracting conformal IFSs on $\mathbb{R}^d$, depending on the parameter $\lambda \in \mathscr{U} \subset \mathbb{R}^d$, and let $\Phi_\lambda = \Pi^\lambda$ be the natural projection. Let μ be an ergodic shift-invariant measure on Σ_m, and let $\chi^\lambda = \chi_\mu^\lambda$ be the Lyapunov exponent of the IFS with the measure μ. Theorem 11 should be applied locally, in a small neighborhood of $\lambda_0 \in \mathscr{U}$. Then, assuming that the dependence of the IFS on λ is continuous, and $\lambda \mapsto \chi^\lambda$ is continuous, we can take

$$\alpha = \frac{\chi^{\lambda_0}}{\log m} - \varepsilon \quad \text{and} \quad \beta = \left(\frac{\chi^{\lambda_0}}{\log m} + \varepsilon\right)d,$$

for ε sufficiently small. Thus $\beta > \alpha d$, but they can be made arbitrarily close by shrinking the neighborhood of λ_0. It follows that (26) becomes

$$\gamma_\eta + \frac{\gamma_\mu \log m}{\chi^{\lambda_0}} > d.$$

Comparing with Theorems 9 and 10, we obtain exactly the conditions imposed there, keeping in mind that $\chi^\lambda = \log(1/\lambda)$. Thus, in order to deduce these theorems from Theorem 11, we only need to verify the transversality condition (22), with ϑ the Lebesgue measure.

Proof of Theorem 10, assuming Theorem 11. Recall that we have a family (19) of homogeneous self-similar IFSs on $\mathbb{C} \cong \mathbb{R}^2$, analytically depending on a parameter $\lambda \in \overline{U} \subset \mathbb{D}_*$, which is the complex contraction factor of all the maps of the IFSs. Pick $\lambda_0 \in U$ satisfying $\gamma_\eta + \frac{\gamma_\mu \log m}{\log(1/\lambda_0)} > 2$, and let $\varepsilon > 0$ be such that $\overline{B}_\varepsilon(\lambda_0) \subset U$ and $\gamma_\eta + \frac{\gamma_\mu \log m}{\log(1/\lambda)} > 2$ for all $\lambda \in \overline{B}_\varepsilon(\lambda_0)$. We will apply Theorem 11 with the set of parameters $\mathscr{U} = B_\varepsilon(\lambda_0)$, and let

$$\alpha := \min\left\{\frac{\log(1/\lambda)}{\log m} : \lambda \in \overline{B}_\varepsilon(\lambda_0)\right\}, \quad \beta := 2\max\left\{\frac{\log(1/\lambda)}{\log m} : \lambda \in \overline{B}_\varepsilon(\lambda_0)\right\}.$$

Then (21) clearly holds, and we only need to verify (22).

The natural projection is given by

$$\Pi^\lambda(\omega) = \sum_{n=0}^\infty b_{\omega_n}(\lambda)\lambda^n, \quad \omega \in \Sigma_m,$$

and hence

$$\phi_{\omega,\tau}(\lambda) = \Pi^\lambda(\omega) - \Pi^\lambda(\tau) = \sum_{n=0}^\infty (b_{\omega_n}(\lambda) - b_{\tau_n}(\lambda))\lambda^n, \quad \omega,\tau \in \Sigma_m.$$

Recall that $\rho(\omega,\tau) = m^{-|\omega\wedge\tau|}$, so that $\rho(\omega,\tau) \le \delta$ for $\delta > 0$ sufficiently small is equivalent to $|\omega\wedge\tau| \ge N$ for N sufficiently large. Let $|\omega\wedge\tau| = N$. Then

$$\phi_{\omega,\tau}(\lambda) = \lambda^N(\Pi^\lambda(\sigma^N\omega) - \Pi^\lambda(\sigma^N\tau)),$$

hence

$$\phi'_{\omega,\tau}(\lambda) = \lambda^N\Big[N\lambda^{-1}(\Pi^\lambda(\sigma^N\omega) - \Pi^\lambda(\sigma^N\tau)) + \tfrac{d}{d\lambda}(\Pi^\lambda(\sigma^N\omega) - \Pi^\lambda(\sigma^N\tau))\Big]. \tag{27}$$

Now we use the SSC, which means that the minimal distance between two distinct first order cylinders of Λ^λ is positive, and it has a uniform lower bound for $\lambda \in \overline{U}$, say, $c_0 > 0$. Since $N = |\omega\wedge\tau|$, holds

$$\big|\Pi^\lambda(\sigma^N\omega) - \Pi^\lambda(\sigma^N\tau)\big| \ge c_0.$$

Next, note that $\lambda \mapsto \Pi^\lambda(u)$ is an analytic function on $\mathbb{D}$, given by a power series with analytic coefficients from the list $\{b_j(\lambda)\}_{j\le m}$, for any $u \in \Sigma_m$, which implies that

$$M_k := \sup_{u\in\Sigma_m} \left|\frac{d^k\Pi^\lambda(u)}{d\lambda^k}\right| < \infty, \quad k \ge 0.$$

Thus, assuming $N \ge 4M_1/c_0$, we have from (27):

$$\begin{aligned} |\phi'_{\omega,\tau}(\lambda)| &\ge |\lambda|^N(N\cdot c_0 - 2M_1) \\ &\ge |\lambda|^N \cdot Nc_0/2 \\ &= \rho(\omega,\tau)^{\log(1/\lambda)/\log m}\cdot Nc_0/2 \\ &\ge \rho(\omega,\tau)^{\beta/2}\cdot Nc_0/2. \end{aligned}$$

It remains to use the following standard result for the function $g(z) = v + \phi_{\omega,\tau}(z)$, see, e.g., [21] (Lemma 5.2) for a proof.

Lemma 4. *Let g be an analytic function on a closed disk $F = \overline{B}_R(z_0) \subset \mathbb{C}$, such that*

$$|g'(z)| \ge \delta_0, \quad \textit{for all } z \in F.$$

Then

$$\mathcal{L}^2(\{z \in F : \ |g(z)| \le r\}) \le \mathrm{const}\cdot r^2/\delta_0^2, \quad \textit{for all } r > 0.$$

where the constant depends only on $\sup\{|g''(z)| : z \in F\}$.

This concludes the derivation of Theorem 10 from Theorem 11. □

Before we turn to the (complete) proof of Theorem 11, we state another, far-reaching generalization of Theorem 9, which is a consequence of [79] (Theorem 3.7), see also [80].

Theorem 12. *Let $\{\mathcal{F}^\lambda\}_{\lambda\in J}$ be a C^2-smooth family (in x and in λ) of hyperbolic IFS on $\mathbb{R}$, satisfying the SSC for all $\lambda \in J$, where $J \subset \mathbb{R}$ is a compact interval. Let μ be an ergodic shift-invariant probability measure on Σ_m and $\nu_\lambda = \Pi^\lambda\mu$, where Π^λ is the natural projection corresponding to $\mathcal{F}^\lambda$. Let $\chi^\lambda = \chi^\lambda_\mu$ be the Lyapunov exponent of the probabilistic IFS, and assume that $\frac{d}{d\lambda}(\chi^\lambda)$ has finitely many zeros. Then for any compactly supported measure η on $\mathbb{R}$, such that*

$$\dim_{\mathrm{H}}(\eta) + \dim_{\mathrm{H}}(\nu_\lambda) > 1, \quad \textit{for all } \ \lambda \in J,$$

the measure $\eta * \nu_\lambda$ is absolutely continuous with respect to $\mathcal{L}^1$.

Proof of Theorem 11. Since $\mathscr{X}$ is compact, we can represent it as a finite disjoint union of sets of diameter $\le \delta$:

$$\mathscr{X} = \bigsqcup_{k\ge1} \mathscr{X}_k, \quad \mathrm{diam}(\mathscr{X}_k) \le \delta.$$

We have

$$\nu_\lambda = \sum_{k\ge1} \nu_\lambda^{(k)}, \quad \text{where } \ \nu_\lambda^{(k)} := \nu_\lambda|_{\mathscr{X}_k},$$

thus it is enough to prove that for every k holds $\eta * \nu_\lambda^{(k)} \ll \mathcal{L}^d$ with a density in L^2 for ϑ-a.e. $\lambda \in \mathscr{U}$. Thus we can assume without loss of generality that

$$\mathrm{diam}(\mathscr{X}) \le \delta \le 1.$$

Consider the lower density of $\eta * \nu_\lambda$ with respect to the Lebesgue measure in $\mathbb{R}^d$:

$$\underline{D}(\eta * \nu_\lambda, x) = \liminf_{r\downarrow 0}(2r)^{-d}(\eta * \nu_\lambda)[B_r(x)].$$

As in [7] (Theorem 9.7), if

$$\mathscr{J}_\lambda := \int_{\mathbb{R}^d} \underline{D}(\eta * \nu_\lambda, x)\, d(\eta * \nu_\lambda)(x) < \infty,$$

then $\underline{D}(\eta * \nu_\lambda, x)$ is finite for $(\eta * \nu_\lambda)$-a.e. x, and $\eta * \nu_\lambda$ is absolutely continuous, with a Radon–Nikodym derivative in L^2. Thus, it is enough to show that

$$\mathscr{S} := \int_{\mathscr{U}} \mathscr{J}_\lambda \, d\vartheta(\lambda) < \infty.$$

By Fatou's Lemma,

$$\mathscr{S} \le \mathscr{S}_1 := \liminf_{r\downarrow 0} (2r)^{-d} \int_{\mathscr{U}} \int_{\mathbb{R}^d} (\eta * \nu_\lambda)[B_r(x)]\, d(\eta * \nu_\lambda)(x)\, d\lambda.$$

Using the definition of convolution and making a change of variable, we obtain

$$\mathscr{S}_1 = \liminf_{r\downarrow 0} (2r)^{-d} \int_{\mathscr{U}} \int_{\mathbb{R}^d} \int_{\mathscr{X}} (\eta * \nu_\lambda)[B_r(y + \Phi_\lambda(\omega))]\, d\mu(\omega)\, d\eta(y)\, d\vartheta(\lambda). \tag{28}$$

Next we have

$$\begin{aligned} &(\eta * \nu_\lambda)[B_r(y + \Phi_\lambda(\omega))] \\ &\quad = \int_{\mathbb{R}^d} \mathbb{1}_{B_r(y+\Phi_\lambda(\omega))}(w)\, d\eta * \nu_\lambda(w) \\ &\quad = \int_{\mathbb{R}^d} \mathbb{1}_{\{(z,\tau):\, z+\Phi_\lambda(\tau) \in B_r(y+\Phi_\lambda(\omega))\}}(z,\tau)\, d\mu(\tau)\, d\eta(z). \end{aligned}$$

Substituting this into (28) and reversing the order of integration yields

$$\mathscr{S}_1 = \liminf_{r\downarrow 0} (2r)^{-d} \int_{\mathbb{R}^d} \int_{\mathscr{X}} \int_{\mathbb{R}^d} \int_{\mathscr{X}} \vartheta(\Lambda_r(y,z,\omega,\tau))\, d\mu(\tau)\, d\eta(z)\, d\mu(\omega)\, d\eta(y), \tag{29}$$

where

$$\begin{aligned} \Lambda_r(y,z,\omega,\tau) &:= \{\lambda \in \mathscr{U} :\ |(y + \Phi_\lambda(\omega)) - (z + \Phi_\lambda(\tau))| \le r\} \\ &:= \{\lambda \in \mathscr{U} :\ |y - z + \phi_{\omega,\tau}(\lambda)| \le r\}. \end{aligned} \tag{30}$$

Recall that $\operatorname{diam}(\mathscr{X}) \le \delta$ by our assumption, hence

$$\vartheta\big(\Lambda_r(y,z,\omega,\tau)\big) \le C_2 \min\{1, \rho(\omega,\tau)^{-\beta} r^d\}. \tag{31}$$

by (22). Next we consider the integral in (29), use Fubini's Theorem, and then split it according to the distance between y and z:

$$\int_{\mathbb{R}^d} \int_{\mathscr{X}} \int_{\mathbb{R}^d} \int_{\mathscr{X}} = \iint\limits_{\{|y-z|<2r\}} \iint\limits_{\mathscr{X}\times\mathscr{X}} + \iint\limits_{\{|y-z|\ge 2r\}} \iint\limits_{\mathscr{X}\times\mathscr{X}} =: \mathscr{I}_1 + \mathscr{I}_2. \tag{32}$$

To complete the proof, it suffices to show that $\mathscr{I}_1 \lesssim r^d$ and $\mathscr{I}_2 \lesssim r^d$. (The symbol $\lesssim$ means inequality up to a multiplicative constant independent of r.) In view of (31),

$$\mathscr{I}_1 \lesssim \iint_{\mathscr{X}\times\mathscr{X}} \iint_{\{|y-z|<2r\}} \min\{1, \rho(\omega,\tau)^{-\beta} r^d\}\, d\eta(y)\, d\eta(z)\, d\mu(\omega)\, d\mu(\tau).$$

The integrand does not depend on y, z, and we can estimate, using (25):

$$(\eta \times \eta)\{(y,z) \in \mathbb{R}^{2d} :\ |y - z| < 2r\} \le \int \eta(B_{2r}(y))\, d\eta(y) \lesssim r^{\gamma_\eta}.$$

Thus,

$$\begin{aligned}
\mathscr{I}_1 &\lesssim r^{\gamma_\eta} \iint\limits_{\mathcal{X}^2} \min\{1, \rho(\omega,\tau)^{-\beta} r^d\}\, d\mu(\omega)\, d\mu(\tau) \\
&\lesssim r^{\gamma_\eta} \Big[(\mu \times \mu)\{(\omega,\tau) : \rho(\omega,\tau) \le r^{d/\beta}\} + r^d \iint\limits_{\rho(\omega,\tau) > r^{d/\beta}} \rho(\omega,\tau)^{-\beta}\, d\mu(\omega)\, d\mu(\tau) \Big] \\
&\lesssim r^{\gamma_\eta} \Big[r^{\gamma_\mu d/\beta} + r^d \int_1^{r^{-d}} (\mu \times \mu)\{(\omega,\tau) : \rho(\omega,\tau) \le t^{-1/\beta}\}\, dt \Big] \\
&\lesssim r^{\gamma_\eta} \Big[r^{\gamma_\mu d/\beta} + r^d \int_1^{r^{-d}} t^{-\gamma_\mu/\beta}\, dt \Big] \\
&\lesssim r^{\gamma_\eta + \gamma_\mu \frac{d}{\beta}} \le r^d,
\end{aligned}$$

as desired, in view of (23), (25), and (26).

It remains to estimate $\mathscr{I}_2$, see (32). If $C_1 \rho(\omega,\tau)^\alpha < \frac{1}{2}|y - z|$, then $|\phi_{\omega,\tau}(\lambda)| < \frac{|y-z|}{2}$ by (21), and $|y - z + \phi_{\omega,\tau}(\lambda)| > \frac{|y-z|}{2} \ge r$ in $\mathscr{I}_2$, whence $\Lambda_r(y,z,\omega,\tau) = \emptyset$, see (30). Thus,

$$\mathscr{I}_2 \le r^d \iint\limits_{\mathbb{R}^{2d}} \iint\limits_{\{(\omega,\tau):\, \rho(\omega,\tau) \ge (|y-z|/2C_1)^{1/\alpha}\}} \rho(\omega,\tau)^{-\beta}\, d\mu(\omega)\, d\mu(\tau)\, d\eta(y)\, d\eta(z). \tag{33}$$

Denote $\widetilde{c}_1 = (2C_1)^{-\beta/\alpha}$. Then

$$\begin{aligned}
\iint\limits_{\{(\omega,\tau):\, \rho(\omega,\tau) \ge (|y-z|/2C_1)^{1/\alpha}\}} \rho(\omega,\tau)^{-\beta}\, d\mu(\omega)\, d\mu(\tau) &= \int_1^{\widetilde{c}_1 |y-z|^{-\beta/\alpha}} (\mu \times \mu)\{(\omega,\tau) : \rho(\omega,\tau) \le t^{-1/\beta}\}\, dt \\
&\le \int_1^{\widetilde{c}_1 |y-z|^{-\beta/\alpha}} t^{-\gamma_\mu/\beta}\, dt \\
&\lesssim |y - z|^{-\frac{\beta - \gamma_\mu}{\alpha}} \quad (\text{recall that } \gamma_\mu < \beta),
\end{aligned}$$

where we used (23). Finally,

$$\iint_{\mathbb{R}^{2d}} |y - z|^{-\frac{\beta-\gamma_\mu}{\alpha}}\, d\eta(y)\, d\eta(z) < \infty,$$

in view of (26), and so $\mathscr{I}_2 \lesssim r^d$ by (33), as desired. □

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