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Short Note

Some Properties of a Function Originating from Geometric Probability for Pairs of Hyperplanes Intersecting with a Convex Body

Feng Qi [1,2,*] and Mansour Mahmoud [3]

1 Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, China
2 College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China
3 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; mansour@mans.edu.eg
* Correspondence: qifeng618@gmail.com; Tel.: +86-22-8395-6502

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Abstract: In the paper, the authors derive an integral representation, present a double inequality, supply an asymptotic formula, find an inequality, and verify complete monotonicity of a function involving the gamma function and originating from geometric probability for pairs of hyperplanes intersecting with a convex body.

Keywords: gamma function; complete monotonicity; inequality; asymptotic formula; integral representation; monotonicity

MSC: Primary 33B15; Secondary 26A48, 26A51, 26D20, 41A60, 44A10

1. Introduction

The problem of studying the increasing property of the sequence

$$p_m = \frac{m-1}{2}\left(\int_0^{\pi/2} \sin^{m-1} t \,\mathrm{d}\, t\right)^2, \quad m \in \mathbb{N}$$

arises from geometric probability for pairs of hyperplanes intersecting with a convex body, see [1]. The sequence p_m was formulated in [2] as

$$q_m = \frac{\pi}{2m}\left[\frac{\Gamma((m+1)/2)}{\Gamma(m/2)}\right]^2, \quad m \in \mathbb{N},$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}\, t, \quad x > 0$$

is the well-known gamma function. Guo and Qi [2] proved the increasing monotonicity of the sequence p_m by considering the sequence

$$Q_m = \frac{1}{m}\left[\frac{\Gamma((m+1)/2)}{\Gamma(m/2)}\right]^2, \quad m \in \mathbb{N}.$$

They presented two indirect proofs with the help of Bustoz and Ismail's results [3] and their own results [4].

In 2015, Qi et al. [5] established an asymptotic formula for the function

$$\phi(x) = 2\left[\ln\Gamma\left(\frac{x+1}{2}\right) - \ln\Gamma\left(\frac{x}{2}\right)\right] - \ln x, \quad x > 0$$

and investigated some properties of the sequence $Q_m = e^{\phi(m)}$ for $m \in \mathbb{N}$. They also posed two problems about the monotonicity of the sequence $\sqrt[m]{\alpha Q_m}$ for $0 < \alpha \leq 2$.

In this paper, we will derive an integral representation, present a double inequality, supply an asymptotic formula, find an inequality, and verify complete monotonicity of the function $\phi(x)$ or $Q(x) = e^{\phi(x)}$. As consequences, the above-mentioned two problems posed in [5] are confirmatively answered.

2. An Integral Representation and a Double Inequality for $\phi(x)$

In this section, we derive an integral representation and a double inequality for the function $\phi(x)$ as follows. As a consequence, the complete monotonicity of the function $-\phi(x) - \ln 2$ is concluded.

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$0 \leq (-1)^{k-1} f^{(k-1)}(x) < \infty$$

for $x \in I$ and $k \in \mathbb{N}$, where $f^{(0)}(x)$ means $f(x)$ and $\mathbb{N}$ is the set of all positive integers. See ([6] Chapter XIII), ([7] Chapter 1), and ([8] Chapter IV). The class of completely monotonic functions may be characterized by the celebrated Bernstein-Widder Theorem ([8] p. 160, Theorem 12a) which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \leq x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \, \mathrm{d}\,\alpha(t), \tag{1}$$

where $\alpha(t)$ is bounded and non-decreasing and the integral converges for $0 \leq x < \infty$. The integral (1) means that $f(x)$ is the Laplace transform of the measure $\alpha(t)$.

Theorem 1. *For $x > 0$ and $n \in \mathbb{N}$, we have the integral representation*

$$\phi(x) = -\ln 2 - \int_0^\infty \frac{\tanh t}{t} e^{-2xt} \, \mathrm{d}\, t \tag{2}$$

and the double inequality

$$-\ln 2 - \sum_{k=1}^{2n} \frac{(2^{2k}-1)B_{2k}}{k(2k-1)} \frac{1}{x^{2k-1}} < \phi(x) < -\ln 2 - \sum_{k=1}^{2n-1} \frac{(2^{2k}-1)B_{2k}}{k(2k-1)} \frac{1}{x^{2k-1}}, \tag{3}$$

where B_{2k} are the Bernoulli numbers which can be generated by

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + \sum_{k=0}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Consequently, the function $-\phi(x) - \ln 2$ is a Laplace transform, or say, completely monotonic on $(0, \infty)$.

Proof. Using Legendre's formula

$$2^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \sqrt{\pi}\,\Gamma(x)$$

and the integral representation

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-xt}}{t} \, \mathrm{d}\, t, \quad x > 0$$

in [9], Slavić [10] obtained the relation

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} = \sqrt{x} \exp\left(\int_0^\infty \frac{\tanh t}{2t} e^{-4xt} \, \mathrm{d}\, t\right), \quad x > 0 \tag{4}$$

and the double inequality

$$\sqrt{x} \exp\left[\sum_{k=1}^{2n} \frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right] < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{x} \exp\left[\sum_{k=1}^{2l-1} \frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right] \tag{5}$$

for $x > 0$, where $l, n \in \mathbb{N}$ and B_{2k} are the Bernoulli numbers. Replacing x by $\frac{x}{2}$ in (4) and (5) and taking the logarithm lead to (2) and (3). Theorem 1 is thus proved. □

Remark 1. The double inequality (1.1) in [5] is a special case of the double inequality (3).

Remark 2. The integral representation (2) or the double inequality (3) means readily that

$$\lim_{x \to \infty} \phi(x) = -\ln 2. \tag{6}$$

3. An Asymptotic Formula for $\phi(x)$

We now supply an asymptotic formula of the function $\phi(x)$, which is of a form different from the one presented in [5].

Theorem 2. *The function $\phi(x)$ satisfies the asymptotic formula*

$$\phi(x) \sim -\ln 2 - \sum_{m=0}^{\infty} \frac{2\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)(2m+1)} \frac{1}{x^{2m+1}}, \quad x \to \infty. \tag{7}$$

Proof. Using the expansion

$$T(t) = \frac{\tanh(t/2)}{t} = \sum_{m=1}^{\infty} \frac{2(2^{2m}-1)B_{2m}}{(2m)!} t^{2m-2}, \quad |t| < \pi$$

and Watson's lemma (see [11,12]), we have

$$\int_0^\infty \frac{\tanh(t/2)}{t} e^{-xt} \, \mathrm{d}\, t \sim \sum_{r=0}^{\infty} \frac{T^{(r)}(0)}{x^{r+1}}, \quad x \to \infty,$$

where

$$T^{(2r+1)}(0) = 0 \quad \text{and} \qquad T^{(2r)}(0) = \frac{2\left(2^{2r+2}-1\right)B_{2r+2}}{(2r+2)(2r+1)}, \quad r \geq 0.$$

Hence

$$\int_0^\infty \frac{\tanh(t/2)}{t} e^{-xt} \, \mathrm{d}\, t \sim \sum_{r=0}^{\infty} \frac{2\left(2^{2r+2}-1\right)B_{2r+2}}{(2r+2)(2r+1)x^{2r+1}}, \quad x \to \infty.$$

The Formula (7) is thus proved. □

4. Monotonicity and Inequalities of $\phi(x)$

In this section, we present an inequality for the function $\phi(x)$, find a necessary and sufficient condition on α such that the function $\frac{\phi(x)+\ln\alpha}{x^r}$ is increasing with respect to $x \in (0,\infty)$, and establish three properties of the function $Q(x)$. As a consequence of a property of $Q(x)$, the above-mentioned two problems are confirmatively answered.

Theorem 3. *The function $\phi(x)$ satisfies the following properties:*

1. *if $p \le q$ and $x \ge q$, then*
$$\frac{\phi(x+p)+\phi(x-q)}{2} < \frac{p}{q}\phi(x); \tag{8}$$
2. *for $x>0$ and $r>0$, the function $\frac{\phi(x)+\ln\alpha}{x^r}$ is strictly increasing if and only if $0<\alpha\le 2$.*

Proof. Using the integral representation (2), we obtain

$$\phi(x+p)+\phi(x-q)-\frac{2p}{q}\phi(x) = -\int_0^\infty \frac{\tanh t}{t}e^{-2xt}\left(e^{-2pt}+e^{2qt}-\frac{2p}{q}\right)\mathrm{d}\,t - \left(1-\frac{p}{q}\right)\ln 4$$

for $x \ge q$. Because

$$e^{-2pt}+e^{2qt}-\frac{2p}{q} = \sum_{n=1}^{\infty}\frac{(2qt)^n}{n!}\left[1+\left(-\frac{p}{q}\right)^n\right]+2\left(1-\frac{p}{q}\right) > 0$$

for $p \le q$, we procure

$$\phi(x+p)+\phi(x-q)-\frac{2p}{q}\phi(x) < 0$$

for $p \le q$ and $x \ge q$.

It is clear that

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\phi(x)+\ln\alpha}{x^r}\right) = \frac{1}{x^r}\int_0^\infty M_{t,r,\alpha}(x)e^{-xt}\,\mathrm{d}\,t,$$

where

$$M_{t,r,\alpha}(x) = r\ln\left(\frac{2}{\alpha}\right)+\left(\frac{r}{xt}+1\right)\tanh\frac{t}{2}, \quad t,r,\alpha>0,$$

and $M_{t,r,\alpha}(x)$ is obviously a decreasing function and

$$\lim_{x\to\infty} M_{t,r,\alpha}(x) = r\ln\left(\frac{2}{\alpha}\right)+\tanh\frac{t}{2}.$$

This means that

$$M_{t,r,\alpha}(x) \ge 0 \quad \text{if and only if} \quad \frac{2}{\alpha} \ge \exp\left[-\frac{\tanh(t/2)}{r}\right].$$

Moreover, the function $f_r(t) = \exp\left[-\frac{\tanh(t/2)}{r}\right]$ is decreasing for $r>0$. As a result, it follows that

$$M_{t,r,\alpha}(x) \ge 0 \quad \text{if and only if} \quad \frac{2}{\alpha} \ge \lim_{t\to 0}\exp\left[-\frac{\tanh(t/2)}{r}\right] = 1.$$

The proof of Theorem 3 is complete. □

Theorem 4. *The function*

$$Q(x) = \frac{1}{x}\left[\frac{\Gamma((x+1)/2)}{\Gamma(x/2)}\right]^2, \quad x>0$$

has the following properties:

1. *the limit* $\lim_{x\to\infty} Q(x) = \frac{1}{2}$ *is valid;*
2. *for fixed* $r > 0$*, the function* $[\alpha Q(x)]^{1/x^r}$ *is strictly increasing with respect to* x *if and only if* $0 < \alpha \le 2$*;*
3. *the function* $Q(x)$ *satisfies the Pául type inequality*

$$Q(x+p)Q(x-q) < [Q(x)]^{2p/q}, \quad p \le q, \quad x \ge q; \tag{9}$$

in particular, when $p = q = 1$ *and* $x \in \mathbb{N}$ *in (9), the strictly logarithmic concavity of the sequence* Q_m *follows, that is,*

$$Q_{m+1}Q_{m-1} < Q_m^2, \quad m \in \mathbb{N}.$$

Proof. Using the relation

$$Q(x) = e^{\phi(x)}, \quad x > 0$$

and the limit (6), we obtain $\lim_{x\to\infty} Q(x) = \frac{1}{2}$. From the second property in Theorem 3 and the relation

$$\frac{\mathrm{d}}{\mathrm{d}\,x}[\alpha Q(x)]^{1/x^r} = [\alpha Q(x)]^{1/x^r} \frac{\mathrm{d}}{\mathrm{d}\,x}\left[\frac{\ln \alpha + \phi(x)}{x^r}\right],$$

we obtain that the function $[\alpha Q(x)]^{1/x^r}$ is strictly increasing with respect to $x > 0$ for fixed $r > 0$ if and only if $0 < \alpha \le 2$.

By using the inequality (8), we have

$$\ln Q(x+p) + \ln Q(x-q) - \frac{2p}{q} \ln Q(x) < 0, \quad p \le q, \quad x \ge q$$

which gives us the inequality (9). The proof of Theorem 4 is complete. □

Remark 3. For $x \in \mathbb{N}$, the third conclusion in Theorem 4 was proved in [5] with a different proof.

Remark 4. Using the second conclusion in Theorem 4, for $x \in \mathbb{N}$ and $r = 1$, we can see that the sequence $\sqrt[m]{\alpha Q_m}$ is increasing with respect to $m \in \mathbb{N}$ if and only if $0 < \alpha \le 2$. This gives a solution to two problems posed in [5].

Remark 5. In the papers [13,14], the authors investigated by probabilistic methods and approaches the monotonicity of incomplete gamma functions and their ratios and applied their results to probability and actuarial area.

5. Conclusions

The main results, including an integral representation, a double inequality, an asymptotic formula, an inequality, and complete monotonicity of a function involving the gamma function and originating from geometric probability for pairs of hyperplanes intersecting with a convex body, of this paper are deeper and more extensive researches of the papers [2,5] and references cited therein.

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