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TAYLOR COLLOCATION METHOD FOR SOLVING A CLASS OF THE FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract- In this study, we present a reliable numerical approximation of the some first order nonlinear ordinary differential equations with the mixed condition by the using a new Taylor collocation method. The solution is obtained in the form of a truncated Taylor series with easily determined components. Also, the method can be used to solve Riccati equation. The numerical results show the effectuality of the method for this type of equations. Comparing the methodology with some known techniques shows that the existing approximation is relatively easy and highly accurate.

Key Words- Nonlinear ordinary differential equations, Riccati equation, Taylor polynomials, collocation points.

1. INTRODUCTION

Nonlinear ordinary differential equations are frequently used to model a wide class of problems in many areas of scientific fields; chemical reactions, spring-mass systems bending of beams, resistor-capacitor-inductance circuits, pendulums, the motion of a rotating mass around another body and so forth $[1,2]$. These equations here also demonstrated their usefulness in ecology, economics, biology, astrophysics and engineering. Thus, methods of solution for these equations are of great importance to engineers and scientists $[3,4]$.

In this paper, for our aim we consider the first order nonlinear ordinary differential equation of the form

$$P(x)y(x)+Q(x)y'(x)+R(x)y^2(x)+S(x)y(x)y'(x)+T(x)(y'(x))^2=g(x)\,,\ a\le x\le b \tag{1}$$

under the mixed conditions

$$\alpha y(\mathrm{a})+\beta y(\mathrm{b})=\lambda \tag{2}$$

and look for the approximate solution in the form

$$y(x)=\sum_{n=0}^{N} y_n(x\text{-}c)^n,\qquad y_n=\frac{y^{(n)}(c)}{n!}\quad ,\qquad a\le c\le b \tag{3}$$

which is a Taylor polynomial of degree N at $x=c$, where $y_n\left(n = 0,1,\ldots,N\right)$ are the coefficients to be determined. Here $P(x)$, $Q(x)$, $R(x)$, $S(x)$, $T(x)$ and $g(x)$ are the functions defined on $\mathrm{a}\le\mathrm{x}\le\mathrm{b}$; the real coefficients α, β and λ are appropriate constants. Note that, if $S(x)=T(x)=0$ in Eq. (1), it is a Riccati Equation $[5-8]$.

2. FUNDAMENTAL MATRIX RELATIONS

Our aim is to find the matrix form of each term in the nonlinear equation given by Eq. (1). Firstly, we consider the solution $y(x)$ defined by a truncated series (3) and then we can convert to the matrix form

$$y(x) = \mathbf{X}(x)\mathbf{Y} \tag{4}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & (x-c) & (x-c)^2 & \cdots & (x-c)^N \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_N \end{bmatrix}^T .$$

If we differentiate expression (4) with respect to x, we obtain

$$\begin{aligned} y'(x) &= \mathbf{X}'(x)\mathbf{Y} \\ &= \mathbf{X}(x)\mathbf{B}\mathbf{Y} \end{aligned} \tag{5}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} .$$

On the other hand, the matrix form of expression $y^2(x)$ is obtained as

$$y^2(x) = \begin{bmatrix} 1 & (x-c) & (x-c)^2 & \cdots & (x-c)^N \end{bmatrix} \begin{bmatrix} \mathbf{X}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x) \end{bmatrix} \begin{bmatrix} y_0\mathbf{Y} \\ y_1\mathbf{Y} \\ \vdots \\ y_N\mathbf{Y} \end{bmatrix}$$

or briefly

$$y^2(x) = \mathbf{X}(x)\mathbf{X}^*(x)\mathbf{Y}^* \tag{6}$$

where

$$\mathbf{X}^*(x) = \begin{bmatrix} \mathbf{X}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x) \end{bmatrix} ,$$

$$\mathbf{Y}^* = \begin{bmatrix} y_0\mathbf{Y} & \mathrm{y}_1\mathbf{Y} & \cdots & \mathrm{y}_N\mathbf{Y} \end{bmatrix}^T, \ [8] .$$

By using the expression (4), (5) and (6) we obtain

$$y(x)y'(x) = \mathbf{X}(x)\mathbf{X}^*(x)\mathbf{B}^*\mathbf{Y}^* . \tag{7}$$

Following a similar way to (6), we have

$$(y'(x))^2 = \mathbf{X}(x)\mathbf{B}\mathbf{X}^*(x)\mathbf{B}^*\mathbf{Y}^* \tag{8}$$

where

$$\mathbf{B}^* = \begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}.$$

3. MATRIX RELATIONS BASED ON COLLOCATION POINTS

Let us use the collocation points defined by

$$x_i = a + \frac{b-a}{N} i \quad , \quad i = 0,1,...,N \tag{9}$$

in order to

$$a = x_0 < x_1 < \ldots < x_n = b.$$

By putting the collocation points (9) into Eq. (1), we get the equation

$$P(x_i)y(x_i) + Q(x_i)y'(x_i) + R(x_i)y^2(x_i) + S(x_i)y(x_i)y'(x_i) + T(x_i)(y'(x_i))^2 = g(x_i) \tag{10}$$

$$i = 0,1,\ldots, N \ ; a \le x_i \le b \ .$$

By using the relations (4), (5), (6), (7) and (8); the system (10) can be written in the matrix form

$$\left(\mathbf{PX} + \mathbf{QXB}\right)\mathbf{Y} + \left(\mathbf{RXX}^* + \mathbf{SXX}^*\mathbf{B}^* + \mathbf{TXBX}^*\mathbf{B}^*\right)\mathbf{Y}^* = \mathbf{G}$$

or shortly

$$\mathbf{WY} + \mathbf{VY}^* = \mathbf{G} \tag{11}$$

where

$$\mathbf{P} = \begin{bmatrix} P(x_0) & 0 & \cdots & 0 \\ 0 & P(x_1) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & P(x_N) \end{bmatrix} ; \ \mathbf{Q} = \begin{bmatrix} Q(x_0) & 0 & \cdots & 0 \\ 0 & Q(x_1) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & Q(x_N) \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} R(x_0) & 0 & \cdots & 0 \\ 0 & R(x_1) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & R(x_N) \end{bmatrix} ; \ \mathbf{S} = \begin{bmatrix} S(x_0) & 0 & \cdots & 0 \\ 0 & S(x_1) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & S(x_N) \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} T(x_0) & 0 & \cdots & 0 \\ 0 & T(x_1) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & T(x_N) \end{bmatrix} ; \ \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix}\mathbf{X}(x_0)\\ \mathbf{X}(x_1)\\ \vdots\\ \mathbf{X}(x_N)\end{bmatrix}=\begin{bmatrix}1 & (x_0-c) & (x_0-c)^2 & \cdots & (x_0-c)^N\\ 1 & (x_1-c) & (x_1-c)^2 & \cdots & (x_1-c)^N\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & (x_N-c) & (x_N-c)^2 & \cdots & (x_N-c)^N\end{bmatrix}$$

$$\mathbf{X}^*=\begin{bmatrix}\mathbf{X}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{X}(x_1) & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & \mathbf{X}(x_N)\end{bmatrix}.$$

4. METHOD OF SOLUTION

The fundamental matrix equation (11) corresponding to Eq. (1), can be written as

$$\mathbf{WY}+\mathbf{VY}^*=\mathbf{G}$$

or

$$\left[\mathbf{W};\mathbf{V}:\mathbf{G}\right] \tag{12}$$

where

$$\mathbf{W}=\left[w_{\mathrm{pq}}\right]=\mathbf{PX}+\mathbf{QXB}$$

$$\mathbf{V}=\left[v_{\mathrm{pq}}\right]=\mathbf{RXX}^*+\mathbf{SXX}^*\mathbf{B}^*+\mathbf{TXBX}^*\mathbf{B}^*.$$

We can find the corresponding matrix equation for the condition (2), using the relation (4), as follows:

$$\left\{\alpha\mathbf{X}(a)+\beta\mathbf{X}(b)\right\}\mathbf{Y}=\left[\lambda\right] \tag{13}$$

so that

$$\mathbf{X}(a)=\begin{bmatrix}1 & (a-c) & \cdots & (a-c)^N\end{bmatrix}$$

$$\mathbf{X}(b)=\begin{bmatrix}1 & (b-c) & \cdots & (b-c)^N\end{bmatrix}.$$

We can write the corresponding matrix form (13) for the mixed condition (2) in the augmented matrix form as

$$\left[\mathbf{Z};\ \mathbf{0}:\ \lambda\right] \tag{14}$$

where

$$\mathbf{Z}=[\mathrm{z}_0\ \mathrm{z}_1\ \cdots\ \mathrm{z}_\mathrm{N}]=\alpha\mathbf{X}(\mathrm{a})+\beta\mathbf{X}(\mathrm{b})$$

$$\mathbf{Z}=\left[z_j\right]=\left[1 \quad (a-c) \quad \cdots \quad (a-c)^N\right]+\left[1 \quad (b-c) \quad \cdots \quad (b-c)^N\right]$$

$$\mathbf{0}=[0 \;\; 0 \;\; \cdots \;\; 0\,]_{1x(N+1)}\,.$$

To obtain the approximate solution of Eq. (1) with the mixed condition (2) in the terms of Taylor polynomials, by replacing the row matrix (14) by the last row of the matrix (11), we obtain the required augmented matrix:

$$\left[\overline{\mathbf{W}};\overline{\mathbf{V}}:\overline{\mathbf{G}}\right]=\begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & v_{00} & v_{01} & v_{02} & \cdots & v_{0N} & : & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & v_{10} & v_{11} & v_{12} & \cdots & v_{1N} & : & g(x_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots & ; & \vdots & \vdots & \vdots & \cdots & \vdots & : & \vdots \\ w_{N-1,0} & w_{N-1,1} & w_{N-1,2} & \cdots & w_{N-1,N} & ; & v_{N-1,0} & v_{N-1,1} & v_{N-1,2} & \cdots & v_{N-1,N} & : & g(x_N) \\ z_0 & z_1 & z_2 & \cdots & z_N & ; & 0 & 0 & 0 & \cdots & 0 & : & \lambda \end{bmatrix}$$

or the corresponding matrix equation

$$\overline{\mathbf{W}}\mathbf{Y}+\overline{\mathbf{V}}\mathbf{Y}^*=\overline{\mathbf{G}} \tag{15}$$

where

$$\overline{\mathbf{W}}=\begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} \\ w_{10} & w_{11} & \dots & w_{1N} \\ \vdots & \vdots & & \vdots \\ w_{N-1,0} & w_{N-1,1} & \dots & w_{N-1,1} \\ z_0 & z_1 & \dots & z_N \end{bmatrix}, \quad \overline{\mathbf{V}}=\begin{bmatrix} v_{00} & v_{01} & \dots & v_{0N} \\ v_{10} & v_{11} & \dots & v_{1N} \\ \vdots & \vdots & & \vdots \\ v_{N-1,0} & v_{N\text{-}1,1} & \dots & v_{N-1,N} \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\overline{\mathbf{G}}=\begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_{N-1}) \\ \lambda \end{bmatrix}.$$

The unknown coefficients set $\{y_0, y_1, \ldots, y_N\}$ can be determined from the nonlinear system (15). As a result, we can obtain approximate solution in the truncated series form (3).

5. ACCURACY OF SOLUTION

We can check the accuracy of the solution by following procedure $[9-12]$: The truncated Taylor series in (3) have to be approximately satisfying Eq. (1); that is, for each $x = x_i \in [a,b],\ i = 1,2,\dots,$

$$E(x_i) = \left| P(x_i)y(x_i) + Q(x_i)y'(x_i) + R(x_i)y^2(x_i) + S(x_i)y(x_i)y'(x_i) + T(x_i)(y'(x_i))^2 - g(x_i) \right| \cong 0$$

and $E(x_i) \le 10^{-k_i}$ (k_i is any positive integer).

If $\max(10^{-k_i}) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

6. NUMERICAL EXAMPLES

In this section, two numerical examples are given to illustrate the accuracy and efficiency of the presented method.

Example 6.1. Let us first consider the first-order nonlinear differential equation

$$\left(y'(x)\right)^2 - xy(x)y'(x) + y'(x) - x^2y(x) + \left(y(x)\right)^2 = 1 - x^3 \qquad (16)$$

with condition

$$y(-1) + 2y(1) = 1 \ ,\ -1 \le x \le 1$$

and the approximate solution $y(x)$ by the truncated Taylor polynomial

$$y(x) = \sum_{n=0}^{2} y_n x^n \quad , \quad -1 \le x \le 1$$

where

$$P(x) = -x^2,\ Q(x) = 0,\ R(x) = 1, S(x) = -x, T(x) = 1,\ g(x) = 1 - x^3.$$

For $N = 2$ the collocation points become

$$x_0 = -1,\ x_1 = 0,\ x_2 = 1.$$

From the fundamental matrix equations for the given equation and condition respectively are obtained as

$$\left(\mathbf{PX} + \mathbf{QXB}\right)\mathbf{Y} + \left(\mathbf{RXX}^* + \mathbf{SXX}^*\mathbf{B}^* + \mathbf{TXBX}^*\mathbf{B}^*\right)\mathbf{Y}^* = \mathbf{G}$$

and

$$\left\{\mathbf{X}(-1) + 2\mathbf{X}(1)\right\}\mathbf{Y} = [1]$$

so that

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},\quad \mathbf{Y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix},\quad \mathbf{G} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},\quad \mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{Q}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\ \mathbf{R}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \mathbf{S}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix},\ \mathbf{T}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\mathbf{X}^*=\begin{bmatrix}1&-1&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&1&1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}$$

$$\mathbf{Y}^*=\begin{bmatrix}y_0y_0 & y_0y_1 & y_0y_2 & y_1y_0 & y_1y_1 & y_1y_2 & y_2y_0 & y_2y_1 & y_2y_2\end{bmatrix}^T.$$

The augmented matrix for this fundamental matrix equation is calculated

$$\left[\overline{\mathbf{W}};\overline{\mathbf{V}}:\overline{\mathbf{G}}\right]=\begin{bmatrix}-1&1&-1&;&1&0&-1&-1&0&0&1&0&-1&:&2\\0&0&0&;&1&-1&1&0&1&0&0&0&0&:&1\\3&1&3&;&0&0&0&0&0&0&0&0&0&:&1\end{bmatrix}.$$

From the obtained system, the coefficients y_0, y_1 and y_2 are found as

$$y_0=0\ ,\ y_1=1\ \text{and}\ y_2=0.$$

Hence we have the Taylor polynomial solution

$$y(x)=x.$$

Example 6.2. Consider the following nonlinear differential equation (Riccati Equation) given by

$$xy'-xy+y^2=e^{2x} \tag{17}$$

with the initial condition

$$y(0)=1.$$

So that $a=0, b=1, c=0, P(x)=-x, Q(x)=x, R(x)=1, S(x)=0, T(x)=0, g(x)=e^{2x}$.

The solutions obtained for $N=3,5,7$ are compared with the exact solution is e^x, which are given in Fig 1.We compare the numerical solution and absolute errors for $N=3,5,7$ in Table 1.

Table 1. Comparison of the absolute errors of Example 6.2

		Present method					
		$N=3$		$N=5$		$N=7$	
x_i	Exact Solution	$y(x_i)$	Absolute Errors	$y(x_i)$	Absolute Errors	$y(x_i)$	Absolute Errors
0.0	1.000000	1.00000	0.0	1.00000	0.0	1.00000	0.0
0.1	1.105171	1.10529	1.192369 E-04	1.105184	1.259568 E-05	1.105172	8.648132 E-07
0.2	1.221403	1.221161	2.421382 E-04	1.221451	4.798519 E-05	1.221406	3.468062 E-06
0.3	1.349859	1.347611	2.247413 E-03	1.349952	9.291372 E-05	1.349867	7.789426 E-06
0.4	1.491825	1.484642	7.182218 E-03	1.491936	1.117825 E-04	1.491838	1.363126 E-05
0.5	1.648721	1.632254	1.64674 E-02	1.648755	3.378159 E-05	1.648742	2.025157 E-05
0.6	1.822119	1.790446	3.167322 E-02	1.821857	2.6133 E-04	1.822144	2.564376 E-05
0.7	2.013753	1.959218	5.453511 E-02	2.012794	9.589406 E-04	2.013778	2.532079 E-05
0.8	2.225541	2.13857	8.697101 E-02	2.223214	2.326954 E-03	2.225551	1.044657 E-05
0.9	2.459603	2.328503	1.311006 E-01	2.454868	4.734986 E-03	2.459568	3.485946 E-05
1.0	2.718282	2.529015	1.892663 E-01	2.709606	8.675577 E-03	2.718145	1.372364 E-04

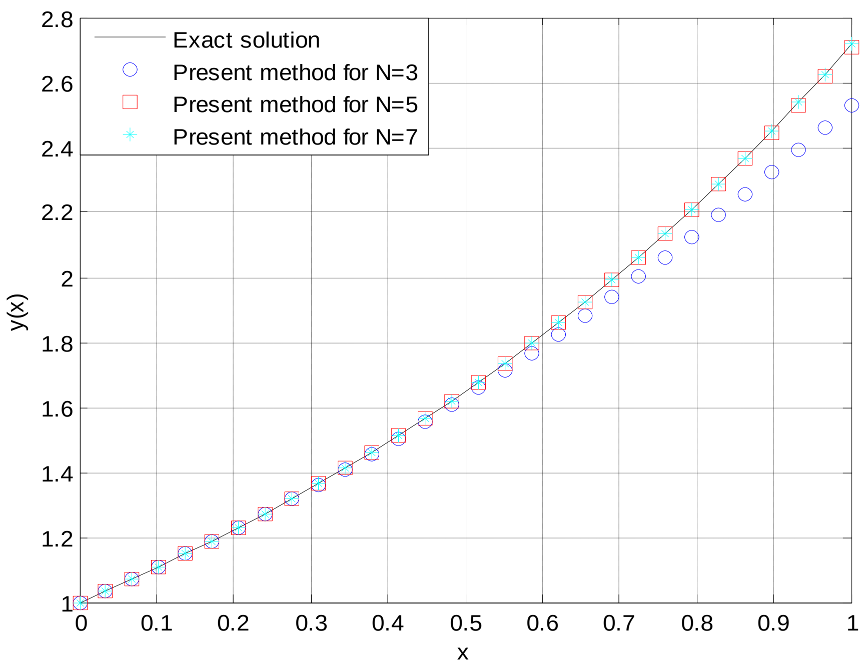

Figure 1. Numerical and exact solution of Example 6.2 for $\mathbf{N=3,5,7}$

7. CONCLUSION

In this study, a new Taylor approximation method for the solution of a class of first order nonlinear differential equations has been presented. The principal advantage of this method, at the around $x = c$, is the capability to succeed in the solution up to all term of Taylor expansion. It is seen from Example 6.2 that Taylor collocation method gives well results for the different values N. Also it is important to note that Taylor coefficients of the solution are found very simply by using the computer programs.

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