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ON STRONGLY ALMOST CONVERGENT SEQUENCE SPACES OF FUZZY NUMBERS

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Abstract. In this paper we define some new almost convergent sequence spaces of fuzzy numbers through a non- negative regular matrix and we also examine some topological properties and some inclusion relations for these new sequence spaces.

Key Words: Fuzzy numbers, almost convergence, strongly almost convergence

1. INTRODUCTION

In many branches of mathematics and engineering we often come across different types of sequences and certainly there are situations either the idea of ordinary convergence does not work or the underlying space does not serve our purpose. So to ideal with such situations we have to introduce some new type of measures which can provide a better tool and a suitable frame work. In particular, we are interested to put forward our studies in fuzzy like situations. The concepts of fuzzy sets and fuzzy set operation were first introduced by Zadeh [16] in 1965. Since then a large number of research papers have appeared by using the concept of fuzzy set numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [4], chaos control [6], computer programming [7], nonlinear dynamical systems [8], fuzzy physics [9], etc.
In this paper, we define some new almost convergent sequence spaces of fuzzy numbers through a non- negative regular matrix and we also examine some topological properties.

2. SOME DEFINITIONS

Let D be the set of all bounded intervals $A=\left[\overline{A},\underline{A}\right]$ on the real line $\mathbb{R}$. For $A,B\in D$, define

$$A\le B \text{ if and only if } \underline{A}\le\underline{B} \text{ and } \overline{A}\le\overline{B},$$

$$d(A,B)=\max(\underline{A}-\underline{B},\overline{A}-\overline{B}).$$

Then it can be easily see that d defines a metric on D (see, [5]) and (D,d) is a complete metric space.

A fuzzy number is a fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi-continuous and have compact support, i.e. if $X\in L(\mathbb{R})$ then for any $\alpha\in[0,1]$, X^{α} is compact where

$$X^{\alpha}=\begin{cases} t: X(t)\geq \alpha & \text{if } 0<\alpha\leq 1, \\ t: X(t)\geq 0 & \text{if } \alpha=0 \end{cases}$$

For each $0<\alpha\leq 1$ the α - level set X^{α} is a nonempty compact subset of $\mathbb{R}$. The linear structure of $L(\mathbb{R})$ includes addition $X+Y$ and scalar multiplication λX, (λ a scalar) in terms of α level sets, by

$$[X+Y]^{\alpha}=[X]^{\alpha}+[Y]^{\alpha} \text{ and } [\lambda X]^{\alpha}=\lambda[X]^{\alpha}$$

each for $0\leq \alpha\leq 1$.

Define a map $d(X,Y)=\sup\limits_{0\leq\alpha\leq 1} d(X^{\alpha},Y^{\alpha})$.

For $X,Y\in L(\mathbb{R})$ define $X\leq Y$ if and only if $X^{\alpha}\leq Y^{\alpha}$ for any $\alpha\in[0,1]$. It is known that $(L(\mathbb{R}),\bar{d})$ is complete metric space (see, [11]). We will need the following definitions (see, [15]).

Definition 2.1. *A sequence $X=(X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0; written as $\lim\limits_k X_k = X_0$ if for every $\varepsilon>0$ there exists a positive integer N_0 such that*

$$\bar{d}(X_k, X_0)<\varepsilon \text{ for } k>N_0$$

Let $c(F)$ denote the set of all convergent sequences of fuzzy numbers.

Definition 2.2. *A sequence $X=(X_k)$ of fuzzy numbers is said to be bounded if the set $(X_k : k\in\mathbb{N})$ of fuzzy numbers is bounded. We denote by $l_{\infty}(F)$ the set of all bounded sequences of fuzzy numbers.*

It is straightforward to see that

$$c(F)\subset l_{\infty}(F)\subset w(F).$$

In [15] it was shown that $c(F)$ and $l_{\infty}(F)$ are complete metric spaces.
For further studies we refer [12], [17] and [18].

In this paper we define some new almost sequence spaces of fuzzy numbers through a non-negative regular matrices $A=(a_{nk}),(n,k=1,2,...)$. By the regularity of A we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit invariant (Maddox, [10]).
The famous Silverman-Toeplitz conditions for the regularity of A are as follows:

A is regular if and only if

(i) $\|A\| = \sup_n \sum |a_{nk}| < \infty$
(i) $\lim_n a_{nk} = 0,$ *for each* k
(iii) $\lim_n \sum a_{nk} = 1.$

By a paranorm we mean a function $g = E \to \mathbb{R}$ (where E is a linear space) which satisfies the following conditions:

$(p:1)$ $g(\theta) = 0,$
$(p:2)$ $g(x) \geq 0$ *for all* $x \in E,$
$(p:3)$ $g(-x) = g(x)$ *for all* $x \in E$
$(p:4)$ $g(x+y) \leq g(x) + g(y)$ *for all* $x, y \in E,$
$(p:5)$ If (λ_n) is a sequence of scalars with ¸$\lambda_n \to \lambda$ $(n \to \infty)$ and x_n is a sequence of the elements of E with $g(x_n - x) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty),$

The space E is called the paranormed space with the paranorm g .

Recently E. Savas [17] have defined the following space of sequences of fuzzy numbers.

Definition 2.3. *The sequences* $X = (X_k)$ *of fuzzy numbers is said to be almost convergent to a fuzzy number* L *if*

$$\lim_m \bar{d}\left(t_{mn}(X), L\right) = 0, \text{ uniformly in } n, \tag{1}$$

$$t_{mn}(X) = \frac{1}{m+1} \sum_{i=0}^{m} X_{n+i}$$

This means that for every $\varepsilon > 0$, *there exist a* $m_0 \in \mathbb{N}$ *such that*

$$\lim_m \bar{d}\left(t_{mn}(X),\ L\right) < \varepsilon$$

whenever $m \geq m_0$ *and for all* $n.$

If the limit in (1) exists, then we write

$$\hat{c}(F) - \lim X = L.$$

We are ready to define the following:

Let $A=(a_{nk}),\ (n,k=1,2,...)$ be an infinite regular matrix of non-negative real numbers and let $p=(p_k)$ be a sequence of positive real numbers. We define

$$\left[\hat{A},p\right]_0(F)=\left(X=(X_k):\sum_{k=1}^{\infty}a_{nk}\left[\bar{d}(X_{k+i},0)\right]^{p_k}\to 0,\ (n\to\infty),\ \textit{uniformly in } i\right),$$

$$\left[\hat{A},p\right](F)=\left(X=(X_k):\sum_{k=1}^{\infty}a_{nk}\left[\bar{d}(X_{k+i},X_0)\right]^{p_k}\to 0,\ (n\to\infty),\ \textit{uniformly in } i\right).$$

and call them respectively the spaces of strongly almost *A*- convergent to zero, and strongly almost *A*- convergent to X_0 . We can specialize these spaces as follows.

(i) If $A=(a_{nk})$ is a Cesaro matrix of order 1, i.e.

$$a_{nk}=\begin{cases}\dfrac{1}{n} & k\le n\\ 0, & k>n\end{cases}$$

we have $\left[\hat{A},p\right]_0(F)=[\hat{c},p]_0(F)$, and $\left[\hat{A},p\right]=[\hat{c},p](F)$ which are defined as follows:

$$[\hat{c},p]_0(F)=\left(X=(X_k):n^{-1}\sum_{k=1}^{n}\left[\bar{d}(X_{k+i},0)\right]^{p_k}\to 0,\ (n\to\infty),\ \textit{uniformly in } i\right),$$

$$[\hat{c},p](F)=\left(X=(X_k):n^{-1}\sum_{k=1}^{n}\left[\bar{d}(X_{k+i},X_0)\right]^{p_k}\to 0,\ (n\to\infty),\ \textit{uniformly in } i\right).$$

and further on taking $p_k=1$ for all k , these are reduced to following sequences spaces:

$$[\hat{c}]_0(F)=\left(X=(X_k):n^{-1}\sum_{k=1}^{n}\left[\bar{d}(X_{k+i},0)\right]\to 0,\ (n\to\infty),\ \text{uniformly in } i\right),$$

$$[\hat{c}](F)=\left(X=(X_k):n^{-1}\sum_{k=1}^{n}\left[\bar{d}(X_{k+i},X_0)\right]\to 0,\ (n\to\infty),\ \text{uniformly in } i\right).$$

Strongly almost sequence of fuzzy numbers is discussed in [13].

(ii) If $A=(a_{nk})$ is considered as

$$a_{nk}=\begin{cases}\dfrac{1}{h_r} & k_{r-1}<k\le k_r\\ 0, & \textit{otherwise}\end{cases}$$

where (k_r) is a lacunary sequence, i.e. an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denote by $I_r = (k_{r-1} - k_r]$. We have the following:

$$[\hat{c}]_0^{\theta}(F) = \left(X = (X_k) : h_r^{-1}\sum_{k\in I_r}\left[\bar{d}(X_{k+i},0)\right] \to 0,\ (r\to\infty),\ \textit{uniformly in } i\right),$$

$$[\hat{c}]^{\theta}(F) = \left(X = (X_k) : h_r^{-1}\sum_{k\in I_r}\left[\bar{d}(X_{k+i},X_0)\right] \to 0,\ (r\to\infty),\ \textit{uniformly in } i\right).$$

Strongly almost lacunary sequence of fuzzy numbers is discussed in [3].
A metric $\bar{d}$ on $L(\mathbb{R})$ is said to be a translation invariant if

$$\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y) \quad \textit{for } X,Y,Z \in L(\mathbb{R}).$$

Proposition 2.1. *If* $\bar{d}$ *is a translation invariant metric on* $L(\mathbb{R})$ *then,*
(*i*) $\bar{d}(X+Y,0) \le \bar{d}(X,0) + \bar{d}(Y,0)$
(*ii*) $\bar{d}(\lambda X,0) \le |\lambda|\bar{d}(X,0)$, *where* λ is scalar and $|\lambda|>1$.

Proof (i). By the triangle inequality

$$\bar{d}(X+Y,0) \le \bar{d}(X+Y,Y) + \bar{d}(Y,0) = \bar{d}(X+Y,Y+0) + \bar{d}(Y,0) = \bar{d}(X,0) + \bar{d}(Y,0)$$

since $\bar{d}$ is a translation invariant.

(ii) It follows easily by using *(i)* and induction.
If $\bar{d}$ is a translation invariant, we have the following straightforward result.

Proposition 2.2. *Let* (p_k) *be a bounded sequence of strictly positive real numbers. Then* $[A,p]_0(F)$ *and* $[A,p](F)$ *are linear spaces over the complex field* C.

3. MAIN RESULTS

Theorem 3. 1. $[A,p]_0(F)$ *and* $[A,p](F)$ *are paranormed spaces with the paranorm* g *defined by*

$$g(X) = \sup_n\left(\sum_{k=1}^{n} a_{nk}\left[\bar{d}(X_{k+i},0)\right]^{p_k}\right)^{1/M}$$

where $M = \max(1, \sup_k p_k)$, *where* $\bar{d}$ *is a* translation invariant.

Proof. Clearly $g(\theta)=0, g(-X)=g(X)$. It can also be seen easily that $g(X+Y)\le g(X)+g(Y)$ *for* $X=(X_k)$, $Y=(Y_k)$ *in* $[A,p]_o(F)$ since $\bar{d}$ is a translation invariant.

Now for any λ we have $|\lambda|^{p_k}<\max(1,|\lambda|^H)$, where $H=\sup_k p_k<\infty$, so

$$g(\lambda X)<(\sup_k|\lambda|^{p_k})^{1/M}.g(X) \text{ on } [A,p]_0(F).$$

Hence $\lambda\to 0$, $X\to\theta$ implies $\lambda X\to\theta$ and also $X\to\theta$, λ fixed implies $\lambda X\to\theta.$
Now let $\lambda\to 0$, X fixed. For $|\lambda|<1$, we have

$$\sum_{k=1}^{\infty}a_{nk}\left[\bar{d}\left(\lambda X_{k+i},0\right)\right]^{p_k}<\varepsilon \quad \text{for} \quad n>N(\varepsilon) \text{ and all } i.$$

Also, for $1\le n\le N$, since $\sum_{k=1}^{\infty}a_{nk}\left[\bar{d}\left(X_{k+i},0\right)\right]^{p_k}<\infty$ there exists M such that

$$\sum_{k=M}^{\infty}a_{nk}\left[\bar{d}\left(X_{k+i},0\right)\right]^{p_k}<\varepsilon$$

Taking λ small enough we then have for all n,i

$$\sum_{k=1}^{\infty}a_{nk}\left[\bar{d}\left(X_{k+i},0\right)\right]^{p_k}<2\varepsilon.$$

Hence $g(\lambda X)\to 0$ as $\lambda\to 0$. Therefore g is paranorm on $\left[\bar{A},p\right]_0(F)$. $\left[\hat{A},p\right](F)$ has exactly the same proof.

Theorem 3.2. *Let* $0<p_k\le q_k$ *and* (q_k/p_k) *be bounded. Then* $\left[\hat{A},q\right](F)\subseteq\left[\hat{A},p\right](F)$.

Proof. Let $X=(X_k)\in\left[\hat{A},q\right](F)$. Put $t_{k,i}=\left[\bar{d}\left(X_{k+i},X_0\right)\right]^{q_k}$ and $\lambda_k=(q_k/p_k)$ so that $0<\lambda<\lambda_k\le 1$. Define $u_{k,i}=\begin{cases} t_{k,i} & t_{k,i}\ge 1\\ 0, & t_{k,i}<1\end{cases}$ and $v_{k,i}=\begin{cases} 0, & t_{k,i}\ge 1\\ t_{k,i}, & t_{k,i}<1\end{cases}$. Then we have $t_{k,i}=u_{k,i}+v_{k,i}$ and $t_{k,i}^{\lambda_k}=u_{k,i}^{\lambda_k}+v_{k,i}^{\lambda_k}$ and it follows that $u_{k,i}^{\lambda_k}\le u_{k,i}\le t_{k,i}$ and $v_{k,i}^{\lambda_k}\le v_{k,i}^{\lambda}$. Therefore

$$\sum_{k=1}^{\infty}a_{nk}\left[\bar{d}\left(X_{k+i},X_0\right)\right]^{p_k}=\sum_{k=1}^{\infty}a_{nk}t_{k,i}^{\lambda_k}=\sum_{k=1}^{\infty}a_{nk}\left(u_{k,i}^{\lambda_k}+v_{k,i}^{\lambda_k}\right)$$

$$\le\sum_{k=1}^{\infty}a_{nk}t_{k,i}+\sum_{k=1}^{\infty}a_{nk}v_{k,i}^{\lambda}\to 0 \quad \left(n\to\infty,\ \text{uniformly in } i\right).$$

Since $X\in\left[\hat{A},q\right](F)$, $\sum_{k=1}^{\infty}a_{nk}t_{ki}$ is convergent for all n,i and since $v_{k,i}<1$ and

A is regular, for all *n, i* $\sum_{k=1}^{\infty} a_{nk} v_{ki}^{\lambda}$ is also convergent. Hence $X \in \left[\hat{A}, p\right](F)$, i.e. $\left[\hat{A}, q\right](F) \subseteq \left[\hat{A}, p\right](F)$.

If $X = (X_k)$ is strongly almost A-summable to s we write $X_k \to s\left[\hat{A}, p\right](F)$.

Theorem 3.3. *Suppose* $A = (a_{nk})$ *transforms null sequence into null sequence,i.e.* $A \in \left(c_0(F), c_0(F)\right)$ *and* $p = (p_k)$ *converges to a positive limit. Then* $X_k \to s,\ X_k \to s\left[\hat{A}, p\right](F),\ X_k \to s'\left[\hat{A}, p\right](F)$ *imply* $s = s'$ *if and only if*

$$\sum_k a_{nk} \not\to 0,\ (n \to \infty) \tag{2}$$

Proof. Necessity. Suppose that $A \in \left(c_0(F), c_0(F)\right)$ *and* p_k is bounded. Let $X_k \to s$ imply that $X_k \to s\left[\hat{A}, p\right](F)$ uniquely. we have $e \to 1\left[\hat{A}, p\right](F)$. Therefore condition (2) must hold, for otherwise $e \to 0\left[\hat{A}, p\right](F)$ which contradicts the uniqueness of s.

Sufficiency. Suppose that condition (1) holds. $A \in \left(c_0(F),\ c_0(F)\right)$ *and* $p_k \to r > 0$. Further assume that $X_k \to s$ implies that $X_k \to s\left[\hat{A}, p\right](F)$ and $X_k \to s'\left[\hat{A}, p\right](F)$ where $\bar{d}(s, s') = a > 0$. Then we get

$$\lim_n \sum_{k=1}^{\infty} a_{nk} u_{ik} = 0,\ \textit{uniformly in } i. \tag{3}$$

where

$$u_{ik} = \bar{d}\left(X_{k+i}, s\right)^{p_k} + \bar{d}\left(X_{k+i}, s'\right)^{p_k}.$$

By the assumption $u_{ik} \to a^r$. Since $A \in \left(c_0(F), c_0(F)\right)$, $u_{ik} \to a^r$ implies that

$$\lim_n \sum_{k=1}^{\infty} a_{nk} \bar{d}\left(u_{ik}, a^r\right) = 0,\ \textit{uniformly in } i. \tag{4}$$

But we have

$$a^r \sum_{k=1}^{\infty} a_{nk} \le \sum_{k=1}^{\infty} a_{nk} u_{ik} + \sum_{k=1}^{\infty} a_{nk} \bar{d}\left(u_{ik}, a^r\right) = 0. \tag{5}$$

By (3), (4) and (5) it follows

$$\lim_n \sum_{k=1}^{\infty} a_{nk}.$$

Since this contradicts (2), we must have $s = s'$ and this completes the proof.

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