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NUMERICAL SOLUTION OF *N*-ORDER FUZZY DIFFERENTIAL EQUATIONS BY RUNGE-KUTTA METHOD

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Abstract - In this paper we study a numerical method for *n*-th order fuzzy differential equations based on Seikkala derivative with initial value conditions. The Runge-Kutta method is used for the numerical solution of this problem and the convergence and stability of the method is proved. By this method, we can obtain strong fuzzy solution. This method is illustrated by solving some examples.

Keywords - Fuzzy numbers, *n*-th order fuzzy differential equations, Runge-Kutta method, Lipschitz condition.

1.INTRODUCTION

The topic of fuzzy differential equations (FDEs) have been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [1], it was followed up by Dubois and Prade [2] by using the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [3] and Goetschel and Voxman [4]. Kandel and Byatt [5] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [6,7], Seikkala [8], He and Yi [9], Kloeden [10] and by other researchers (see [11,15]). The numerical methods for solving fuzzy differential equations are introduced in [16-18].

Buckley and Feuring [20] introduced two analytical methods for solving *n*-th-order linear differential equations with fuzzy initial value conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, they first solved the fuzzy initial value problem and the checked to see if it defined a fuzzy function.

In this paper, a numerical method to solve *n*-th-order linear differential equations with fuzzy initial conditions is presented. The structure of the paper is organized as follows: In Section 2, we give some basic results on fuzzy numbers and define a fuzzy derivative and a fuzzy integral. Then the fuzzy initial values is treated in Section 3 using the extension principle of Zadeh and the concept of fuzzy derivative. It is shown that the fuzzy initial value problem has a unique fuzzy solution when f satisfies Lipschitz condition which guarantees a unique solution to the deterministic initial value problem. In Section 4, the Runge-Kutta method of order 4 for solving *n*-th order fuzzy differential equations is introduced. In Section 5 convergence and stability are illustrated. In Section 6 the proposed method is illustrated by solving several examples, and the conclusion is drawn in Section 7.

2. PRELIMINARIES

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r),\overline{u}(r))$ for all $r \in [0,1]$, which satisfy the following requirements [2]:

$\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,
$\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,
$\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals.

2.1. Lemma [23]

Let $[\underline{v}(\alpha),\overline{v}(\alpha)]$, $\alpha \in (0,1]$ be a given family of non-empty intervals. If

(i) $$[\underline{v}(\alpha),\overline{v}(\alpha)] \supset [\underline{v}(\beta),\overline{v}(\beta)] \quad \textit{for} \quad 0 < \alpha \le \beta,$$

and

(ii) $$[\lim_{k\to\infty}\underline{v}(\alpha_k),\lim_{k\to\infty}\overline{v}(\alpha_k)] = [\underline{v}(\alpha),\overline{v}(\alpha)],$$

whenever (α_k) is a non-decreasing sequence converging to $\alpha \in (0,1]$, then the family $[\underline{v}(\alpha),\overline{v}(\alpha)]$, $\alpha \in (0,1]$, represent the α-level sets of a fuzzy number v in E. Conversely if $[\underline{v}(\alpha),\overline{v}(\alpha)]$, $\alpha \in (0,1]$, are α-level sets of a fuzzy number $v \in E$, then the conditions (i) and (ii) hold true.

2.2. Definition [8]

Let I be a real interval. A mapping $v: I \to E$ is called a fuzzy process and we denoted the α-level set by $[v(t)]_\alpha = [\underline{v}(t,\alpha),\overline{v}(t,\alpha)]$.

The Seikkala derivative $v'(t)$ of v is defined by

$$[v'(t)]_\alpha = [\underline{v}'(t,\alpha),\overline{v}'(t,\alpha)],$$

provided that is a equation defines a fuzzy number $v'(t) \in E$.

2.3. Definition [20]

suppose u and v are fuzzy sets in E. Then their Hausdorff

$$D: E \times E \to R_+ \cup \{0\},$$

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}(\alpha)-\underline{v}(\alpha)|,|\overline{u}(\alpha)-\overline{v}(\alpha)|\},$$

i.e. D(u,v) is maximal distance between α level sets of u and v.

3. FUZZY INITIAL VALUE PROBLEM

Now we consider the initial value problem

$$\begin{cases} x^{(n)}(t) = \psi(t, x, x', ..., x^{(n-1)}) , \\ x(0) = a_1, .., x^{(n-1)}(0) = a_n , \end{cases} \tag{3.1}$$

where ψ is a continuous mapping from $R_+ \times R^n$ into R and a_i $(0 \le i \le n)$ are fuzzy numbers in *E*. The mentioned *n*-th order fuzzy differential equation by changing variables

$$y_1(t) = x(t),\ y_2(t) = x'(t), ..., y_n(t) = x^{(n-1)}(t) ,$$

converts to the following fuzzy system

$$\begin{cases} y_1' = f_1(t, y_1, ..., y_n) , \\ \quad\vdots \\ y_n' = f_n(t, y_1, ..., y_n) , \\ y_1(0) = y_1^{[0]} = a_1, ..., y_n(0) = y_n^{[0]} = a_n , \end{cases} \tag{3.2}$$

where f_i $(1 \le i \le n)$ are continuous mapping from $R_+ \times R^n$ into R and $y_i^{[0]}$ are fuzzy numbers in *E* with α -level intervals

$$[y_i^{[0]}]_\alpha = [\underline{y}_i^{[0]}(\alpha), \overline{y}_i^{[0]}(\alpha)] \quad \text{for i=1,...}n \text{ and } 0 < \alpha \le 1 .$$

We call $\boldsymbol{y} = (y_1, ..., y_n)^t$ is a fuzzy solution of (3.2) on an interval *I*, if

$$\underline{y}_i'(t\ ,\alpha) = \min\{ f_i(t, u_1, ..., u_n); u_j \in [\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)]\} = \underline{f}_i(t, \boldsymbol{y}(t,\alpha)) , \tag{3.3}$$

$$\overline{y}_i'(t,\ \alpha) = \max\{ f_i(t, u_1, ..., u_n); u_j \in [\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)]\} = \overline{f}_i(t, \boldsymbol{y}(t,\alpha)) , \tag{3.4}$$

and

$$\underline{y}_i(0,\alpha) = \underline{y}_i^{[0]}(\alpha),\ \overline{y}_i(0,\alpha) = \overline{y}_i^{[0]}(\alpha) .$$

Thus for fixed α we have a system of initial value problem in R^{2n}. If we can solve it (uniquely), we have only to verify that the intervals, $[\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)]$ define a fuzzy number $y_i(t) \in E$. Now let $\underline{\boldsymbol{y}}^{[0]}(\alpha) = (\underline{y}_1^{[0]}(\alpha), ..., \underline{y}_n^{[0]}(\alpha))^t$ and $\overline{\boldsymbol{y}}^{[0]}(\alpha) = (\overline{y}_1^{[0]}(\alpha), ..., \overline{y}_n^{[0]}(\alpha))^t$, with respect to the above mentioned indicators, system (3.2) can be writhen as with assumption

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{F}(t, \boldsymbol{y}(t)) , \\ \boldsymbol{y}(0) = \boldsymbol{y}^{[0]} \in E^n . \end{cases} \tag{3.5}$$

With assumption $\boldsymbol{y}(t,\alpha) = [\underline{\boldsymbol{y}}(t,\alpha), \overline{\boldsymbol{y}}(t,\alpha)]$ and $\boldsymbol{y}'(t,\alpha) = [\underline{\boldsymbol{y}'}(t,\alpha), \overline{\boldsymbol{y}'}(t,\alpha)]$ where

$$\underline{\boldsymbol{y}}(t,\alpha) = (\underline{y}(t,\alpha), ..., \underline{y}(t,\alpha))^t , \tag{3.6}$$

$$\overline{\boldsymbol{y}}(t,\alpha) = (\overline{y}(t,\alpha), ..., \overline{y}(t,\alpha))^t , \tag{3.7}$$

$$\underline{\boldsymbol{y}'}(t,\alpha) = (\underline{y'}(t,\alpha), ..., \underline{y'}(t,\alpha))^t , \tag{3.8}$$

$$\overline{\boldsymbol{y}'}(t,\alpha) = (\overline{y'}(t,\alpha), ..., \overline{y'}(t,\alpha))^t , \tag{3.9}$$

and with assumption $\boldsymbol{F}(t,\boldsymbol{y}(t,\alpha))=[\underline{\boldsymbol{F}}(t,\boldsymbol{y}(t,\alpha)),\overline{\boldsymbol{F}}(t,\boldsymbol{y}(t,\alpha))]$,where

$$\underline{\boldsymbol{F}}(t,\boldsymbol{y}(t,\alpha))=(\underline{f_1}(t,\boldsymbol{y}(t,\alpha)),...,\underline{f_n}(t,\boldsymbol{y}(t,\alpha)))^t , \quad (3.10)$$

$$\overline{\boldsymbol{F}}(t,\boldsymbol{y}(t,\alpha))=(\overline{f_1}(t,\boldsymbol{y}(t,\alpha)),...,\overline{f_n}(t,\boldsymbol{y}(t,\alpha)))^t , \quad (3.11)$$

$\boldsymbol{y}$ (t) is a fuzzy solution of (3.5) on an interval I for all $\alpha\in(0,1]$, if

$$\begin{cases}\underline{\boldsymbol{y}}'(t,\alpha)=\underline{\boldsymbol{F}}(t,\boldsymbol{y}(t,\alpha));\\ \overline{\boldsymbol{y}}'(t,\alpha)=\overline{\boldsymbol{F}}(t,\boldsymbol{y}(t,\alpha))\\ \underline{\boldsymbol{y}}(0,\alpha)=\underline{\boldsymbol{y}}^{[0]}(\alpha)\ ,\ \overline{\boldsymbol{y}}(0,\alpha)=\overline{\boldsymbol{y}}^{[0]}(\alpha)\end{cases} \quad (3.12)$$

or

$$\begin{cases}\boldsymbol{y}'(t,\alpha)=\boldsymbol{F}(t,\boldsymbol{y}(t,\alpha)),\\ \boldsymbol{y}(0,\alpha)=\boldsymbol{y}^{[0]}(\alpha)\,.\end{cases} \quad (3.13)$$

Now we show that under the assumptions for functions f_i, for i=1,…,n how we can obtain a unique fuzzy solution for system (3.2).

3.1. Theorem

If $f_i(t,u_1,...,u_n)$ for i=1,…,n are continuous function of t and satisfies the Lipschitz condition in $\boldsymbol{u}=(u_1,...,u_n)^t$ in the region $D=\{(t,\boldsymbol{u})\,|\,t\in I=[0,1],-\infty<u_i<\infty\ \mathit{for}\ i=1,...,n\}$ with constant L_i then the initial value problem (3.2) has a unique fuzzy solution in each case.

Proof. Denote $\boldsymbol{G}=(\underline{\boldsymbol{F}},\overline{\boldsymbol{F}})^t=(\underline{f_1},...,\underline{f_n},\overline{f_1},...,\overline{f_n})^t$ where

$$\underline{f_i}(t,\boldsymbol{u})=\min\{f_i(t,u_1,...,u_n);\ u_j\in[\underline{y_j},\overline{y_j}],\ \ \mathit{for}\ \ j=1,...,n\}, \quad (3.14)$$

$$\overline{f_i}(t,\boldsymbol{u})=\max\{f_i(t,u_1,...,u_n);\ u_j\in[\underline{y_j},\ \overline{y_j}],\ \ \mathit{for}\ \ j=1,...,n\}, \quad (3.15)$$

and $\boldsymbol{y}=(\underline{\boldsymbol{y}},\overline{\boldsymbol{y}})^t=(\underline{y_1},...,\underline{y_n},\overline{y_1},...,\overline{y_n})^t\in R^{2n}$. It can be shown that Lipschitz condition of functions f_i imply

$$\|\mathrm{F}(t,\mathrm{z})-\mathrm{F}(t,\mathrm{z}^*)\|\le L\,\|\mathrm{z}-\mathrm{z}^*\|\,.$$

This guarantees the existence and uniqueness solution of

$$\begin{cases}\mathrm{y}'(t)=\mathrm{F}(t,\mathrm{y}(t)),\\ \mathrm{y}(0)=\mathrm{y}^{[0]}=(\underline{\mathrm{y}}^{[0]},\overline{\mathrm{y}}^{[0]})^{2t}\in R^{2n}.\end{cases} \quad (3.16)$$

Also for any continuous function $\boldsymbol{y}^{[1]}:R_+\to R^{2n}$ the successive approximations

$$\boldsymbol{y}^{[m+1]}(t)=\boldsymbol{y}^{[0]}+\int_0^t\boldsymbol{F}(s,\boldsymbol{y}^{[m]}(s))ds\,,\quad t\ge 0\,,\quad m=1,2,... \quad (3.17)$$

converge uniformly on closed subintervals of R_+ to the solution of (3.16). In other word we have the following successive approximations

$$\underline{\mathrm{y}_\mathrm{i}}^{[m+1]}(t)=\underline{\mathrm{y}_\mathrm{i}}^{[0]}+\int_0^t\underline{\mathrm{f}_\mathrm{i}}(s,\mathrm{y}^{[m]}(s))ds\,,\quad \mathit{for}\ i=1,...,n, \quad (3.18)$$

$$\overline{\mathrm{y}_i}^{[m+1]}(t)=\overline{\mathrm{y}_i}^{[0]}+\int_0^t\overline{\mathrm{f}}_i(s,\mathrm{y}^{[m]}(s))ds\,,\quad \mathit{for}\ i=1,...,n. \quad (3.19)$$

By choosing $\boldsymbol{y}^{[0]} = (\underline{\boldsymbol{y}}^{[0]}(\alpha), \overline{\boldsymbol{y}}^{[0]}(\alpha))$ in (3.16) we get a unique solution $\boldsymbol{y}^{\alpha}(t) = (\underline{\boldsymbol{y}}(t,\alpha), \overline{\boldsymbol{y}}(t,\alpha))$ to (3.3) and (3.4) for each $\alpha \in (0,1]$.

Next we will show that $\boldsymbol{y}(t,\alpha) = ([\underline{\boldsymbol{y}}(t,\alpha), \overline{\boldsymbol{y}}(t,\alpha)]$, defines a fuzzy number in E^n for each $0 \le t \le T$, i.e. that $\boldsymbol{y} = (y_1,..., y_n)^t$ is a fuzzy solution to (3.14) and (3.15). Thus we will show that the intervals $[\underline{y_i}(t,\alpha), \overline{y_i}(t,\alpha)]$,

for i=1,…,n satisfy the conditions of Lemma (2.1). The successive approximations $\boldsymbol{y}^{[1]} = \boldsymbol{y}^{[0]} \in E^n$,

$$\boldsymbol{y}^{[m+1]}(t) = \boldsymbol{y}^{[0]} + \int_0^t \boldsymbol{F}(s, \boldsymbol{y}^{[m]}(s))ds, \quad t \ge 0, \quad m = 1,2,... \tag{3.20}$$

where the integrals are the fuzzy integrals, define a sequence of fuzzy numbers $\boldsymbol{y}^{[m]}(t) = (y_1^{[m]}(t),..., y_n^{[m]}(t))^t$ for each $0 \le t \le T$. Hence

$$[y_i^{[m]}(t)]_\beta \subset [y_i^{[m]}(t)]_\alpha, \quad if\ 0 < \alpha \le \beta \le 1,$$

which implies that

$$[\underline{y_i}^{[m]}(t,\beta), \overline{y_i}^{[m]}(t,\beta)] \subset [\underline{y_i}^{[m]}(t,\alpha), \overline{y_i}^{[m]}(t,\alpha)], \quad (0 < \alpha \le \beta \le 1),$$

since, by the convergence of sequences (3.16) and (3.19) , the end points of $[y_i^{[m]}(t)]_\alpha$ converge to $\underline{y_i}(t,\alpha)$ and $\overline{y_i}(t,\alpha)$ that means

$$\underline{y_i}^{[m]}(t,\alpha) \to \underline{y_i}(t,\alpha) \text{ and } \overline{y_i}^{[m]}(t,\alpha) \to \overline{y_i}(t,\alpha). \tag{3.21}$$

Thus the inclusion property (i) of Lemma (2.1) holds for the intervals $[\underline{y_i}(t,\alpha), \overline{y_i}(t,\alpha)]$, for $0 < \alpha \le 1$. For the proof of the property (ii) of Lemma (2.1), let (α_p) be a non-decreasing sequence in (0,1] converging to α. Then $\underline{\boldsymbol{y}}^{[0]}(\alpha_p) \to \underline{\boldsymbol{y}}^{[0]}(\alpha)$ and $\overline{\boldsymbol{y}}^{[0]}(\alpha_p) \to \overline{\mathrm{y}}^{[0]}(\alpha)$, because of $\boldsymbol{y}^{[0]} \in E^n$. But by the continuous dependence on the initial value of the solution (3.16), $\underline{\boldsymbol{y}}(t,\alpha_p) \to \underline{\boldsymbol{y}}(t,\alpha)$ and $\overline{\boldsymbol{y}}(\alpha_p) \to \overline{\boldsymbol{y}}(\alpha)$, this means (ii) holds for the intervals $([\underline{\boldsymbol{y}}(t,\alpha), \overline{\boldsymbol{y}}(t,\alpha)]$, for $0 < \alpha \le 1$.. Hence by Lemma (2.1), $\boldsymbol{y}(t) \in E^n$ and so $\boldsymbol{y}$ is a fuzzy solution of (3.1). The uniqueness follows from the uniqueness of the solution of (3.16).

4. THE RUNGE- KUTTA METHOD OF ORDER 4

With before assumptions, the initial values problem (3.2) has a unique solution, such as $\boldsymbol{y} = (y_1,..., y_n)^t \in E^n$. for found an approximate solution for (3.2) with the Runge-Kutta method of order 4, we first define

$$\underline{k_{i1}}(t, \boldsymbol{y}(t,\alpha)) = \min\{ f_i(t, s_1,..., s_n);\ s_j \in [\underline{y_j}(t,\alpha), \overline{y_j}(t,\alpha)]\}, \qquad (1 \le i, j \le n)$$

$$\overline{k_{i1}}(t, \boldsymbol{y}(t,\alpha)) = \max\{ f_i(t, s_1,..., s_n);\ s_j \in [\underline{y_j}(t,\alpha), \overline{y_j}(t,\alpha)]\},$$

$$\underline{k}_{i2}(t,\boldsymbol{y}(t,\alpha))=\min\{f_i(t+\frac{h}{2},s_1,...,s_n);\ s_j\in[\underline{z}_{j1}(t,\boldsymbol{y}(t,\alpha),h),\overline{z}_{j1}(t,\boldsymbol{y}(t,\alpha),h)]\},$$
$$\overline{k}_{i2}(t,\boldsymbol{y}(t,\alpha))=\min\{f_i(t+\frac{h}{2},s_1,...,s_n);\ s_j\in[\underline{z}_{j1}(t,\boldsymbol{y}(t,\alpha),h),\overline{z}_{j1}(t,\boldsymbol{y}(t,\alpha),h)]\},$$
$$\underline{k}_{i3}(t,\boldsymbol{y}(t,\alpha))=\min\{f_i(t+\frac{h}{2},s_1,...,s_n);\ s_j\in[\underline{z}_{j2}(t,\boldsymbol{y}(t,\alpha),h),\overline{z}_{j2}(t,\boldsymbol{y}(t,\alpha),h)]\},$$
$$\overline{k}_{i3}(t,\boldsymbol{y}(t,\alpha))=\min\{f_i(t+\frac{h}{2},s_1,...,s_n);\ s_j\in[\underline{z}_{j2}(t,\boldsymbol{y}(t,\alpha),h),\overline{z}_{j2}(t,\boldsymbol{y}(t,\alpha),h)]\},$$
$$\underline{k}_{i4}(t,\boldsymbol{y}(t,\alpha))=\min\{f_i(t+h,s_1,...,s_n);\ s_j\in[\underline{z}_{j3}(t,\boldsymbol{y}(t,\alpha),h),\overline{z}_{j3}(t,\boldsymbol{y}(t,\alpha),h)]\},$$
$$\overline{k}_{i4}(t,\boldsymbol{y}(t,\alpha))=\min\{f_i(t+h,s_1,...,s_n);\ s_j\in[\underline{z}_{j3}(t,\boldsymbol{y}(t,\alpha),h),\overline{z}_{j3}(t,\boldsymbol{y}(t,\alpha),h)]\},$$

such that

$$\underline{z}_{j1}(t,\boldsymbol{y}(t,\alpha),h)=\underline{y}_j(t,\alpha)+\frac{h}{2}\underline{k}_{j1}(t,\boldsymbol{y}(t,\alpha)),$$
$$\overline{z}_{j1}(t,\boldsymbol{y}(t,\alpha),h)=\overline{y}_j(t,\alpha)+\frac{h}{2}\overline{k}_{j1}(t,\boldsymbol{y}(t,\alpha)),$$
$$\underline{z}_{j2}(t,\boldsymbol{y}(t,\alpha),h)=\underline{y}_j(t,\alpha)+\frac{h}{2}\underline{k}_{j2}(t,\boldsymbol{y}(t,\alpha),h),$$
$$\overline{z}_{j2}(t,\boldsymbol{y}(t,\alpha),h)=\overline{y}_j(t,\alpha)+\frac{h}{2}\overline{k}_{j2}(t,\boldsymbol{y}(t,\alpha),h),$$
$$\underline{z}_{j3}(t,\boldsymbol{y}(t,\alpha),h)=\underline{y}_j(t,\alpha)+h\underline{k}_{j3}(t,\boldsymbol{y}(t,\alpha),h),$$
$$\overline{z}_{j3}(t,\boldsymbol{y}(t,\alpha),h)=\overline{y}_j(t,\alpha)+h\overline{k}_{j3}(t,\boldsymbol{y}(t,\alpha),h),$$

now we consider the following relations

$$F_i(t,\boldsymbol{y}(t,\alpha),h)=\underline{k}_{i1}(t,\boldsymbol{y}(t,\alpha))+2\underline{k}_{i2}(t,\boldsymbol{y}(t,\alpha),h)+2\underline{k}_{i3}(t,\boldsymbol{y}(t,\alpha),h)+\underline{k}_{i4}(t,\boldsymbol{y}(t,\alpha),h),$$
$$G_i(t,\boldsymbol{y}(t,\alpha),h)=\overline{k}_{i1}(t,\boldsymbol{y}(t,\alpha))+2\overline{k}_{i2}(t,\boldsymbol{y}(t,\alpha),h)+2\overline{k}_{i3}(t,\boldsymbol{y}(t,\alpha),h)+\overline{k}_{i4}(t,\boldsymbol{y}(t,\alpha),h),$$

and suppose that the discrete equally spaced grid points $\{t_0=0,t_1,...,t_N=T\}$ is a partition for interval $[0,T]$. If the exact and approximate solution in the i-th α cut at $t_m, 0\le m\le N$ are denoted by $[\underline{y}_i^{[m]}(\alpha),\overline{y}_i^{[m]}(\alpha)]$ and $[\underline{w}_i^{[m]}(\alpha),\overline{w}_i^{[m]}(\alpha)]$ respectively, then the numerical method for solution approximation in the i-th coordinate α cut, with the Runge-Kutta method is

$$\underline{w}_i^{[m+1]}(\alpha)=\underline{w}_i^{[m]}(\alpha)+\frac{h}{6}F_i(t_m,\boldsymbol{w}^m(\alpha),h),\qquad \underline{w}_i^{[0]}(\alpha)=\underline{y}_i^{[0]}(\alpha),$$
$$\overline{w}_i^{[m+1]}(\alpha)=\overline{w}_i^{[m]}(\alpha)+\frac{h}{6}G_i(t_m,\boldsymbol{w}^m(\alpha),h),\qquad \overline{w}_i^{[0]}(\alpha)=\overline{y}_i^{[0]}(\alpha),$$

where $[w_i(t)]_\alpha=[\underline{w}_i(t,\alpha),\overline{w}_i(t,\alpha)]$, $\boldsymbol{w}^{[m]}(\alpha)=[\underline{\boldsymbol{w}}^{[m]}(\alpha),\overline{\boldsymbol{w}}^{[m]}(\alpha)]$

$\underline{\boldsymbol{w}}^{[m]}(\alpha)=(\underline{w}_1^{[m]}(\alpha),...,\underline{w}_n^{[m]}(\alpha))^t$, and $\overline{\boldsymbol{w}}^{[m]}(\alpha)=(\overline{w}_1^{[m]}(\alpha),...,\overline{w}_n^{[m]}(\alpha))^t$.

Now we input

$$\boldsymbol{F}^*(t,\boldsymbol{w}^{[m]}(\alpha),h)=\frac{1}{6}(F_1(t,\boldsymbol{w}^{[m]}(\alpha),h),...,F_n(t,\boldsymbol{w}^{[m]}(\alpha),h))^t, \tag{4.22}$$

$$\mathrm{G}^*(t,\mathrm{w}^{[\mathrm{m}]}(\alpha),h)=\frac{1}{6}(G_1(t,\mathrm{w}^{[\mathrm{m}]}(\alpha),h),...,G_n(t,\mathrm{w}^{[\mathrm{m}]}(\alpha),h))^t. \tag{4.23}$$

The Runge-Kutta method for solutions approximation α -cut of differential equation (3.13) is as follow

$$\mathrm{w}^{[\mathrm{m}+1]}(\alpha)=\mathrm{w}^{[\mathrm{m}]}(\alpha)+h\mathrm{H}(t_m,\mathrm{w}^{[\mathrm{m}]}(\alpha),h), \quad \mathrm{w}^{[0]}(\alpha)=\mathrm{y}^{[0]}(\alpha) \tag{4.24}$$

where

$$\boldsymbol{H}(t_m,\boldsymbol{w}^{[m]}(\alpha),h)=[\boldsymbol{F}^*(t_m,\boldsymbol{w}^{[m]}(\alpha),h),\boldsymbol{G}^*(t_m,\boldsymbol{w}^{[m]}(\alpha),h)],$$

and

$$\boldsymbol{F}^*(t_m,\boldsymbol{w}^{[m]}(\alpha),h)=\frac{1}{6}[\underline{\boldsymbol{k}_1}(t_m,\boldsymbol{w}^{[m]}(\alpha))+2\underline{\boldsymbol{k}_2}(t_m,\boldsymbol{w}^{[m]}(\alpha),h)+2\underline{\boldsymbol{k}_3}(t_m,\boldsymbol{w}(\alpha),h)+\underline{\boldsymbol{k}_4}(t_m,\boldsymbol{w}(t,\alpha),h)\,] \tag{4.25}$$

$$\boldsymbol{G}^*(t_m,\boldsymbol{w}^{[m]}(\alpha),h)=\frac{1}{6}[\overline{\boldsymbol{k}}_1(t_m,\boldsymbol{w}^{[m]}(\alpha))+2\overline{\boldsymbol{k}}_2(t_m,\boldsymbol{w}^{[m]}(\alpha),h)+2\overline{\boldsymbol{k}}_3(t_m,\boldsymbol{w}(\alpha),h)+\overline{\boldsymbol{k}}_4(t_m,\boldsymbol{w}(t,\alpha),h] \tag{4.26}$$

and also

$$\underline{\boldsymbol{k}_j}(t,\boldsymbol{w}^{[m]}(\alpha),h)=(\underline{k_{1j}}(t,\boldsymbol{w}^{[m]}(\alpha),h),...,\underline{k_{nj}}(t,\boldsymbol{w}^{[m]}(\alpha),h))^t,$$

$$\overline{\mathrm{k}}_j(t,\mathrm{w}^{[\mathrm{m}]}(\alpha),h)=\overline{k_{1j}}(t,\mathrm{w}^{[\mathrm{m}]}(\alpha),h),...,\overline{k_{nj}}(t,\mathrm{w}^{[\mathrm{m}]}(\alpha),h))^t. \qquad (j=1,2,3,4)$$

5. CONVERGENCE AND STABILITY

5.1. Definition [24]

A one-step method for approximating the solution of a differential equation

$$\begin{cases}\mathrm{y}'(t)=\mathrm{F}(t,\mathrm{y}(t)),\\ \mathrm{y}(0)=\mathrm{y}^{[0]}\in R^n,\end{cases} \tag{5.27}$$

which F is a n-th ordered as follow $\boldsymbol{f}=(f_1,...,f_n)^t$ *and* $f_i:R_+\times R^n\to R\,(1\le i\le n)$, is a method which can be written in the form

$$\boldsymbol{w}^{[n+1]}=\boldsymbol{w}^{[n]}+h\boldsymbol{\psi}(t_n,\boldsymbol{w}^{[n]},h), \tag{5.28}$$

where the increment function $\boldsymbol{\psi}$ is determined by $\boldsymbol{F}$ and is a function of t_n, $w^{[n]}$ and h only.

5.2 . Theorem

If $\boldsymbol{\psi}(t,\boldsymbol{y},h)$ satisfies a Lipschitz condition in $\boldsymbol{y}$, then the method given by (5.28) is stable.

5.3 . Theorem

In relation (3.5), if $\boldsymbol{F}(t,\boldsymbol{y})$ satisfies a Lipschitz condition in $\boldsymbol{y}$, then the method given by (4.24) is stable.

5.4. Theorem

If

$$\boldsymbol{w}^{[m+1]}(\alpha)=\boldsymbol{w}^{[m]}(\alpha)+h\boldsymbol{\psi}(t_m,\boldsymbol{w}^{[m]}(\alpha),h), \qquad \boldsymbol{w}^{[0]}(\alpha)=\boldsymbol{y}^{[m]}(\alpha) \tag{5.29}$$

where

$\boldsymbol{\psi}(t_m, \boldsymbol{w}^{[m]}(\alpha), h) = [\boldsymbol{\psi}_1(t_m, \boldsymbol{w}^{[m]}(\alpha), h), \boldsymbol{\psi}_2(t_m, \boldsymbol{w}^{[m]}(\alpha), h)]$ is a numerical method for approximation of differential equation (3.13), and $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are continuous in t, $\boldsymbol{y}$, h for $0 \le t \le T, 0 \le h \le h_0$ and all $\boldsymbol{y}$, and if they satisfy a Lipschitz condition in the region $D = \{(t, \boldsymbol{u}, \boldsymbol{v}, h) \,|\, 0 \le t \le T,\ -\infty < u_i \le v_i,\ -\infty < v_i \le +\infty,\ 0 \le h \le h_0 \ \ i = 1,\dots,n\}$, necessary and sufficient conditions for convergence above mentioned method is

$$\boldsymbol{\psi}(t, \boldsymbol{y}(t,\alpha), h) = \mathrm{F}(t, \mathrm{y}(t,\alpha)). \tag{5.30}$$

Proof: Suppose that $\boldsymbol{\psi}(t, \boldsymbol{y}(t,\alpha), 0) = \boldsymbol{F}(t, \boldsymbol{y}(t,\alpha))$, since, $\boldsymbol{F}(t, \boldsymbol{y}(t,\alpha))$ satisfying the conditions of theory (3.1), then the following equation

$$\begin{cases} \mathrm{y}'(t) = \mathrm{F}(t, \mathrm{y}(t)), \\ \mathrm{y}(0) = \mathrm{y}^{[0]}(\alpha), \end{cases} \tag{5.31}$$

has a unique solution such as

$\boldsymbol{y}(t,\alpha) = ([\underline{\boldsymbol{y}}(t,\alpha), \overline{\boldsymbol{y}}(t,\alpha)]$, where $\underline{\boldsymbol{y}}(t,\alpha) = (\underline{y}(t,\alpha),\dots,\underline{y}(t,\alpha))^t$ and $\overline{\boldsymbol{y}}(t,\alpha) = (\overline{y}(t,\alpha),\dots,\overline{y}(t,\alpha))^t$. We will show that the numerical solutions given by (5.29) convergent to the $\boldsymbol{y}(t)$. By the mean value theorem,

$$\underline{y}_i^{[m+1]} = \underline{y}_i^{[m]} + h\,\underline{f}_i(t_m + \underline{\theta}_i h, \boldsymbol{y}(t_m + \underline{\theta}_i h)), \qquad \textit{for } 0 < \underline{\theta}_i < 1, \tag{5.32}$$

$$\overline{y}_i^{[m+1]} = \overline{y}_i^{[m]} + h\,\overline{f}_i(t_m + \overline{\theta}_i h, \boldsymbol{y}(t_m + \overline{\theta}_i h)), \qquad \textit{for } 0 < \overline{\theta}_i < 1, \tag{5.33}$$

with assumption $\underline{\boldsymbol{\psi}}(= (\underline{\psi}_1,\dots,\underline{\psi}_n)^t$ *and* $\overline{\boldsymbol{\psi}} = (\overline{\psi}_1,\dots,\overline{\psi}_n)^t$. From equation (5.29) obtain the following relations

$$\underline{w}_i^{[m+1]}(\alpha) = \underline{w}_i^{[m]}(\alpha) + h\underline{\psi}_i(t_m, \boldsymbol{w}^{[m]}(\alpha), h), \tag{5.34}$$

$$\overline{w}_i^{[m+1]}(\alpha) = \overline{w}_i^{[m]}(\alpha) + h\overline{\psi}_i(t_m, \boldsymbol{w}^{[m]}(\alpha), h), \tag{5.35}$$

and subtracting (5.32) , (5.33) from (5.34) , (5.35) respectively, and setting

$$\boldsymbol{e}^{[m]}(\alpha) = ([\underline{\boldsymbol{e}}^{[m]}(\alpha), \overline{\boldsymbol{e}}^{[m]}(\alpha)],$$

where

$$\underline{\boldsymbol{e}}^{[m]}(\alpha) = \underline{\boldsymbol{e}}(t_m,\alpha) = \underline{\boldsymbol{w}}^{[m]}(\alpha) - \underline{\boldsymbol{y}}^{[m]}(\alpha) \textit{ and } \overline{\boldsymbol{e}}^{[m]}(\alpha) = \overline{\boldsymbol{e}}(t_m,\alpha) = \overline{\boldsymbol{w}}^{[m]}(\alpha) - \overline{\boldsymbol{y}}^{[m]}(\alpha),$$

we get

$$\underline{e}_i^{[m+1]}(\alpha) = \underline{e}_i^{[m]}(\alpha) + h\{\underline{\psi}_i(t_m, \boldsymbol{w}^{[m]}(\alpha), h)) - \underline{\psi}_i(t_m, \boldsymbol{y}^{[m]}(\alpha), h) + \underline{\psi}_i(t_m, \boldsymbol{y}^{[m]}(\alpha), h)$$
$$- \underline{\psi}_i(t_m, \boldsymbol{y}^{[m]}(\alpha), 0) + \underline{\psi}_i(t_m, \boldsymbol{y}^{[m]}(\alpha), 0) - \underline{f}_i(t_m + \underline{\theta}_i h, \boldsymbol{y}(t_m + \underline{\theta}_i h))\}$$

on the other way, with respect to the relation of $\underline{\psi}_i(t_m, \boldsymbol{y}^{[m]}(\alpha), 0) - \underline{f}_i(t_m, \boldsymbol{y}^{[m]}(\alpha))$ we can write

$$|\underline{\psi}_i(t_m, \boldsymbol{y}^{[m]}(\alpha), 0) - \underline{f}_i(t_m + \underline{\theta}_i h, \boldsymbol{y}(t_m + \underline{\theta}_i h)|$$
$$\le hL_1\underline{\theta}_i + L_1 \sum_{i=1}^{n} |\underline{y}_i(t_m + \underline{\theta}_i h) - \underline{y}_i(t_m)| + L_1 \sum_{i=1}^{n} |\overline{y}_i(t_m + \overline{\theta}_i h) - \overline{y}_i(t_m)|$$
$$= hL_1\underline{\theta}_i + L_1 \sum_{i=1}^{n} |\underline{y}_i'(t_m + \underline{\xi}_i \underline{\theta}_i h)\underline{\theta}_i h| + L_1 \sum_{i=1}^{n} |\overline{y}_i'(t_m + \overline{\xi}_i \underline{\theta}_i h)\underline{\theta}_i h| = hL_2,$$

then

$$|\underline{e_i}^{[m+1]}(\alpha)| \le |\underline{e_i}^{[m]}(\alpha)| + hL_1\{\sum_{j=1}^{n}|\underline{e_j}^{[m]}(\alpha)| + |\overline{e_j}^{[m]}(\alpha)|\} + h^2L_1 + h^2L_2$$

$$\le |\underline{e_i}^{[m]}(\alpha)| + nhL_1 \max_{1\le j\le n}\{|\underline{e_j}^{[m]}(\alpha)| + nhL_1 \max_{1\le j\le n}|\overline{e_j}^{[m]}(\alpha)| + h^2(L_1+L_2).$$

On the other hand

$$\max_{1\le j\le n}\{|\underline{e_j}^{[m]}(\alpha)|\} = k_i\,|\underline{e_i}^{[m]}(\alpha)| \quad , \quad \max_{1\le j\le n}\{|\overline{e_j}^{[m]}(\alpha)|\} = k_i'\,|\overline{e_i}^{[m]}(\alpha)|$$

with assumption $k_1 = \max_{1\le i\le n}\{k_i, k_i'\}$ *and* $M = L_1 + L_2$, we can write

$$\begin{aligned} |\underline{e_i}^{[m+1]}(\alpha)| &\le |\underline{e_i}^{[m]}(\alpha)| + nhk_1L_1\{|\underline{e_i}^{[m]}(\alpha)| + |\overline{e_i}^{[m]}(\alpha)|\} + M_1h^2 \\ &\le |\underline{e_i}^{[m]}(\alpha)| + 2nhk_1L_1\max\{|\underline{e_i}^{[m]}(\alpha)|, |\overline{e_i}^{[m]}(\alpha)|\} + M_1h^2, \end{aligned} \tag{5.36}$$

similarly, we can obtain the following relation

$$|\overline{e_i}^{[m+1]}(\alpha)| \le |\overline{e_i}^{[m]}(\alpha)| + 2nhk_2L_1'\max\{|\underline{e_i}^{[m]}(\alpha)|, |\overline{e_i}^{[m]}(\alpha)|\} + M_2h^2. \tag{5.37}$$

Now, we input $L = \max\{L_1, L_1'\}$ *and* $M = \max\{M_1, M_2\}$ so the relations (5.36) and (5.37) can be written as follow

$$|\underline{e_i}^{[m+1]}(\alpha)| \le |\underline{e_i}^{[m]}(\alpha)| + 2nhkL\max\{|\underline{e_i}^{[m]}(\alpha)|, |\overline{e_i}^{[m]}(\alpha)|\} + Mh^2,$$

$$|\overline{e_i}^{[m+1]}(\alpha)| \le |\overline{e_i}^{[m]}(\alpha)| + 2nhkL\max\{|\underline{e_i}^{[m]}(\alpha)|, |\overline{e_i}^{[m]}(\alpha)|\} + Mh^2.$$

Denote $e_i^{[m]} = |\underline{e_i}^{[m]}(\alpha)| + |\overline{e_i}^{[m]}(\alpha)|$. Then By virtue of lemma (5.7)

$$e_i^{[m]}(\alpha) \le (1+4nhkL)^m e_i^{[0]}(\alpha) + 2Mh^2\,\frac{(1+4nhkL)^m - 1}{4nhkL},$$

where $e_i^{[0]} = |\underline{e_i}^{[0]}(\alpha)| + |\overline{e_i}^{[0]}(\alpha)|$. Then

$$|\underline{e_i}^{[m]}(\alpha)| \le e^{4mnkh} \times e_i^{[0]} + M\,\frac{e^{4mnklh}}{2nhkL}h \quad and \quad |\overline{e_i}^{[m]}(\alpha)| \le e^{4mnkh} \times e_i^{[0]} + M\,\frac{e^{4mnklh}}{2nhkL}h.$$

In particular

$$|\underline{e_i}^{[N]}(\alpha)| \le e^{4Nnkh} \times e_i^{[0]} + M\,\frac{e^{4Nnklh}}{2nhkL}h \quad and \quad |\overline{e_i}^{[N]}(\alpha)| \le e^{4Nnkh} \times e_i^{[0]} + M\,\frac{e^{4Nnklh}}{2nhkL}h.$$

Since $\underline{e_i}^{[0]}(\alpha) = \overline{e_i}^{[0]}(\alpha) = 0$, and $h = \dfrac{T}{N}$ we obtain

$\|e_1^{[N]}(\alpha)\| \le M\,\dfrac{e^{4Nnklh}}{2nhkL}h$ *and* $\|e_2^{[N]}(\alpha)\| \le M\,\dfrac{e^{4Nnklh}}{2nhkL}h$. Then $\|\mathrm{e}^{[N]}(\alpha)\| \le 2M\,\dfrac{e^{4Nnklh}}{2nhkL}h$,

if $h \to 0$ we get $\|\boldsymbol{e}^{[N]}(\alpha)\| \to 0$, so the numerical solution (5.29) converge to the solutions (5.31). Conversely, suppose that the numerical method (5.29) convergent to the solution of the system (5.31). With absurd hypothesis we suppose that (5.30) is not correct. Then $\boldsymbol{\psi}(t, \boldsymbol{y}(t,\alpha), 0) = \boldsymbol{g}(t, \boldsymbol{y}(t,\alpha)) \ne \boldsymbol{F}(t, \boldsymbol{y}(t,\alpha))$. Similarly, we can proof that the numerical method of (5.29) is convergent to the solution of following system

$$\begin{cases} \mathrm{u}'(t) = \mathrm{g}(t, \mathrm{y}(t)), \\ \mathrm{u}(0) = \mathrm{y}^{[0]}(\alpha), \end{cases} \tag{5.38}$$

then $\boldsymbol{y}(t,\alpha) = \boldsymbol{u}(t)$. Since $\boldsymbol{g}(t, \boldsymbol{y}(t,\alpha)) \neq \boldsymbol{F}(t, \boldsymbol{y}(t,\alpha))$, suppose that $\boldsymbol{F}$ and $\boldsymbol{g}$ differ at some point $(t_a, \boldsymbol{y}(t_a,\alpha))$. If we consider the initial values problem (5.31) and (5.38) starting from $(t_a, \boldsymbol{y}(t_a,\alpha))$ we have

$$\boldsymbol{y}'(t_a,\alpha) = \boldsymbol{F}(t_a, \boldsymbol{y}(t_a,\alpha)) \neq \boldsymbol{g}(t_a, \boldsymbol{y}(t_a,\alpha)) = \boldsymbol{g}(t_a, \boldsymbol{u}(t_a)) = \mathrm{u}'(t_a),$$

which is a contradiction.

5.5. Corollary

The Runge-Kutta proposed method by (4.24) and is convergent to the solution of the system (3.13) respectively.

6. EXAMPLE

6.1. Example. Consider the following fuzzy differential equation with fuzzy initial value

$$\begin{cases} y'''(t) = 2y''(t) + 3y'(t) \qquad (0 \le t \le 1), \\ y(0) = (3+\alpha, 5-\alpha) \\ y'(0) = (-3+\alpha, -1-\alpha) \\ y''(0) = (8+\alpha, 10-\alpha) \end{cases}$$

the eigenvalue-eigenvector solution is as follows:

$$y(t,r) = (-\frac{1}{3} + \frac{7}{12}e^{3t} + (\frac{11}{4}+r)e^{-t}, -\frac{1}{3} + \frac{7}{12}e^{3t} + (\frac{19}{4}-r)e^{-t}).$$

The Runge-Kutta solution is as follows and Figures 1 and Table 1 show the obtained results:

$$\underline{w_1}^{[m+1]} = \underline{w_1}^{[m]} + (h + \frac{h^3}{6} + \frac{h^4}{4})\underline{w_2}^{[m]} + (\frac{h^2}{2} + \frac{h^3}{3} + \frac{7h^4}{24})\underline{w_3}^{[m]},$$

$$\overline{w_1}^{[m+1]} = \overline{w_1}^{[m]} + (h + \frac{h^3}{6} + \frac{h^4}{4})\overline{w_2}^{[m+1]} + (\frac{h^2}{2} + \frac{h^3}{3} + \frac{7h^4}{24})\overline{w_3}^{[m]},$$

$$\underline{w_2}^{[m+1]} = \underline{w_2}^{[m]} + (h + h^2 + \frac{7h^3}{6} + \frac{5h^4}{6})\underline{w_3}^{[m]} + (\frac{3h^2}{2} + h^3 + \frac{7h^4}{8})\underline{w_2}^{[m]},$$

$$\overline{w_2}^{[m+1]} = \overline{w_2}^{[m]} + (h + h^2 + \frac{7h^3}{6} + \frac{5h^4}{6})\overline{w_2}^{[m]} + (\frac{3h^2}{2} + h^3 + \frac{7h^4}{8})\overline{w_2}^{[m]},$$

$$\underline{w_3}^{[m+1]} = \underline{w_3}^{[m]} + (3h + 3h^2 + \frac{7h^3}{2} + \frac{5h^4}{2})\underline{w_3}^{[m]} + (2h + \frac{7h^2}{2} + \frac{10h^3}{3} + \frac{61h^4}{24})\underline{w_2}^{[m]},$$

$$\overline{w_3}^{[m+1]} = \overline{w_3}^{[m]} + (3h + 3h^2 + \frac{7h^3}{2} + \frac{5h^4}{2})\overline{w_3}^{[m]} + (2h + \frac{7h^2}{2} + \frac{10h^3}{3} + \frac{61h^4}{24})\overline{w_3}^{[m]}.$$

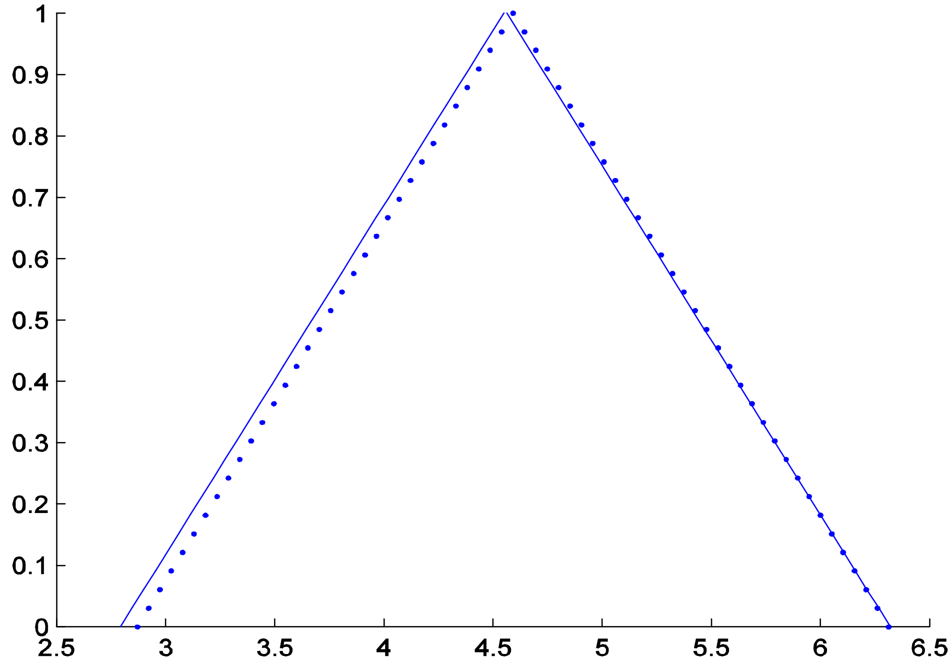

Fig. 3: Comparing obtained solution for $h=\frac{1}{4}$, $h=\frac{1}{2}$, in t=0.5,

$$h=\frac{1}{4}(solid),\ h=\frac{1}{2}(dashed)$$

Table 1.

t	$D(w_1^{[1000]}(t), w_1^{[500]}(t))$	$D(w_1^{[500]}(t), w_1^{[250]}(t))$	$D(w_1^{[250]}(t), w_1^{[125]}(t))$
0.2	4.2994e-007	1.7281e-006	6.9784e-006
0.4	6.6132e-007	2.6164e-006	1.0237e-005
0.6	2.3060e-006	9.1653e-006	3.6200e-005
0.8	5.6995e-006	2.2687e-005	8.9883e-005
1	1.2208e-005	4.8627e-005	1.9292e-004

7. CONCLUTION

In this paper an numerical method for solving n-th order fuzzy linear differential equations with fuzzy initial conditions is presented. In this method a n-th order fuzzy linear differential equation is converted to a fuzzy system which will be solved with the Runge-Kutta method of order 4.

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