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INEQUIVALENCE OF CLASSES OF LINEARIZABLE SYSTEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS OBTAINED BY REAL AND COMPLEX SYMMETRY ANALYSIS

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Abstract- Linearizability criteria for systems of two cubically semi-linear second order ordinary differential equations (ODEs) were obtained by geometric means using real symmetry analysis (RSA). Separately, complex symmetry analysis (CSA) was developed to provide means to discuss systems of two ODEs. It was shown that CSA provides a class of linearizable systems of two cubically semi-linear ODEs. Linearizability criteria for this class were also developed. It is proved that the two classes of linearizable systems of two ODEs, provided by CSA and RSA, are inequivalent under point transformations.

Keywords- Linearizability, geometric and complex symmetry analysis.

1. INTRODUCTION

Nonlinear ODEs are notoriously difficult to solve. Using perturbation methods one can approximate the nonlinear ODE by a linear ODE. However these techniques may miss key features of the nonlinearity, which could be important for the phenomenon under discussion. Though these approximations could be improved by an iterative series, its convergence would then need to be proved. Numerical schemes also suffer from the same problems. All the first and linear second order ODEs are equivalent under the change of dependent and independent variables [15]. Lie derived a canonical method for obtaining the *exact* solution of an ODE, or system of ODEs, provided these are invariant under certain transformations [14].

For second order ODEs to be linearizable they must be at most cubic in the first derivative and the coefficients of the terms must satisfy a system of four equations involving two auxiliary functions. Tressé [23] eliminated the auxiliary functions to reduce to two constraint equations involving higher derivatives of the coefficients with respect to the independent and dependent variables. Lie did not extend to systems or higher order ODEs. His work has since been extended to third and fourth order ODEs [6, 7, 8, 9, 12, 13, 21], where explicit linearizability criteria were provided. A different approach [20] was also adopted for the linearization of third order ODEs. Linearizability conditions for systems of a class of second order quadratically semi-linear ODEs were derived in terms of the coefficients of the equations [16]. They provided procedures for constructing the linearizing transformations. This was then extended to obtain linearizability conditions along with the linearizing transformations for a class of systems of cubically semi-linear ODEs [17]. Separately a classification of linearizable systems of quadratically semi-linear ODEs, [18] had been provided.

In the above mentioned paper [17] the linearizability of systems of two semi-linear ODEs of two dependent variables, were worked through explicitly. A linearizable class of systems of semi-linear ODEs appeared in CSA [1, 2], where linearizability criteria for such systems were derived. The question arises whether the two classes are distinct, have some overlap or are identical under point transformations. In this paper we address this issue, and find that they are distinct.

The plan of the paper is as follows. The second section is on preliminaries, where real and complex symmetry approaches are reviewed. Necessary and sufficient conditions for the linearization of the two classes of linearizable systems are stated. In the third section it is proved that there does not exist any point transformation that maps one system to the other. The fourth section contains a summary and discussion on all the classes of linearizable systems obtained by CSA and RSA.

2. PRELIMINARIES

2.1. Geometric linearization

The geometric linearization procedure is based on regarding the system of second order ODEs as a (projective) system of geodesic equations. The system of geodesic equations is

$$x''^{i} + \Gamma^{i}_{jk} x'^{j} x'^{k} = 0, \quad i, j, k = 1, ..., n, \tag{2.1}$$

where the prime refers to total differentiation with respect to the parameter s and Γ^{i}_{jk} are the Christoffel symbols, which depend on x^i and are given in terms of the metric tensor. These are symmetric in the lower indices and the number of coefficients is $n^2(n+1)/2$. A *necessary and sufficient* condition for a system of n second order quadratically semi-linear ODEs for n dependent variables of the form (2.1) to be linearizable via point transformation and admit $sl(n+2, \mathbb{R})$ symmetry algebra is that the Riemann tensor vanishes [17]. Following Aminova and Aminov [5], the system (2.1) can be projected down by one dimension as

$$x^{a''} + A_{bc} x^{a'} x^{b'} x^{c'} + B^{a}_{bc} x^{b'} x^{c'} + C^{a}_{b} x^{b'} + D^{a} = 0, \tag{2.2}$$

where $a = 1, ..., n-1$, and the prime denotes differentiation with respect to the parameter x^n. The coefficients in terms of the Γ^a_{bc}'s are

$$A_{bc} = -\Gamma^1_{bc},\ B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^1_{b)1},\ C^a_b = 2\Gamma^a_{1b} - \delta^a_b\Gamma^1_{11},\ D^a = \Gamma^a_{11}. \tag{2.3}$$

For a concrete comparison of the systems obtained by the two different approaches in the present section, consider the simplest non-trivial case, namely systems of two second order ODEs. This can be done with the help of (2.2) which reads as

$$\begin{aligned}
&f^{''} + \alpha_1 f^{'3} + 2\alpha_2 f^{'2} g^{'} + \alpha_3 f^{'} g^{'2} + \beta_1 f^{'2} + 2\beta_2 f^{'} g^{'} + \beta_3 g^{'2}\\
&+\gamma_1 f^{'} + \gamma_2 g^{'} + \delta_1 = 0,\\
&g^{''} + \alpha_1 f^{'2} g^{'} + 2\alpha_2 f^{'} g^{'2} + \alpha_3 g^{'3} + \beta_4 f^{'2} + 2\beta_5 f^{'} g^{'} + \beta_6 g^{'2}\\
&+\gamma_3 f^{'} - \gamma_4 g^{'} - \delta_2,
\end{aligned} \tag{2.4}$$

where prime denotes differentiation with respect to the independent variable x and the coefficients are in general functions of x, f, g. The above system is linearizable if the coefficient functions satisfy the fifteen conditions given in [17]. These fifteen conditions were derived using the flat space requirement for the corresponding system of three geodesic equations of type (2.2), imposed by means of the vanishing of the Riemann tensor

$$\begin{aligned}
&(\Gamma^i_{j2})_x - (\Gamma^i_{j1})_f + \Gamma^i_{m1}\Gamma^m_{j2} - \Gamma^i_{m2}\Gamma^m_{j1} = 0,\\
&(\Gamma^i_{j3})_x - (\Gamma^i_{j1})_g + \Gamma^i_{m1}\Gamma^m_{j3} - \Gamma^i_{m3}\Gamma^m_{j1} = 0,\\
&(\Gamma^i_{j3})_f - (\Gamma^i_{j2})_g + \Gamma^i_{m2}\Gamma^m_{j3} - \Gamma^i_{m3}\Gamma^m_{j2} = 0.
\end{aligned} \tag{2.5}$$

These equations reduce to the fifteen linearizability conditions for semi-linear systems of two ODEs.

2.2. Complex linearization

Consider a second order ODE in general form

$$u^{''} = w(x, u, u^{'}), \tag{2.6}$$

where $u(x)$ is a complex function of a single real variable x and prime denotes differentiation with respect to x. We can break the complex function into real and imaginary parts

$$u(x) = f(x) + ig(x), \tag{2.7}$$

to obtain a system of two second order ODEs for the two parts

$$f^{''} = w_1(x, f, g, f^{'}, g^{'}),\ \ g^{''} = w_2(x, f, g, f^{'}, g^{'}). \tag{2.8}$$

The nonlinearity of (2.6) leads to a nontrivial system of coupled equations (2.8). The invariance of ODEs of the form (2.6) can be used to study the invariance of the systems of the same order (2.8). Notice that not every system of two ODEs can be obtained from a scalar ODE by treating the dependent variable as a complex function of a real (independent) variable. In [2], the linearizing transformations of ODEs of the form (2.6) were used to linearize systems of two ODEs of the form (2.8). If w is a nonlinear function, i.e if it is cubically semi-linear it will give two real functions, w_1 and w_2, which are also cubically semi-linear. Thus, we can generate some general systems of two second order semi-linear ODEs that can be candidates for linearization. A class of systems of two semi-linear ODEs

$$\begin{aligned} &f^{''} = A_1 f^{'3} - 3A_2 f^{'2} g^{'} - 3A_1 f^{'} g^{'2} + A_2 g^{'3} + B_1 f^{'2} - 2B_2 f^{'} g^{'} - B_1 g^{'2} \\ &+C_1 f^{'} - C_2 g^{'} + D_1, \\ &g^{''} = A_2 f^{'3} + 3A_1 f^{'2} g^{'} - 3A_2 f^{'} g^{'2} - A_1 g^{'3} + B_2 f^{'2} + 2B_1 f^{'} g^{'} - B_2 g^{'2} \\ &+C_2 f^{'} + C_1 g^{'} + D_2, \end{aligned} \tag{2.9}$$

was obtained from a second order ODE of the form

$$u^{''}(x) = A(x,u)u^{'3} + B(x,u)u^{'2} + C(x,u)u^{'} + D(x,u), \tag{2.10}$$

where A, B, C, D are complex valued functions of x and u, so the coefficients in (2.9) A_j, B_j, C_j, and D_j are functions of x, f, g. The system (2.9) is linearizable if and only if the coefficient functions satisfy the conditions

$$\begin{aligned} &12A_{1,xx} + 12C_1A_{1,x} - 12C_2A_{2,x} - 6D_1A_{1,f} - 6D_1A_{2,g} + 6D_2A_{2,f} - \\ &6D_2A_{1,g} + 12A_1C_{1,x} - 12A_2C_{2,x} + C_{1,ff} - C_{1,gg} + 2C_{2,fg} - \\ &12A_1D_{1,f} - 12A_1D_{2,g} + 12A_2D_{2,f} - 12A_2D_{1,g} + 2B_1C_{1,f} + 2B_1C_{2,g} - \\ &2B_2C_{2,f} + 2B_2C_{1,g} - 8B_1B_{1,x} + 8B_2B_{2,x} - 4B_{1,xf} - 4B_{2,xg} = 0, \end{aligned}$$

$$\begin{aligned} &12A_{2,xx} + 12C_2A_{1,x} + 12C_1A_{2,x} - 6D_2A_{1,f} - 6D_2A_{2,g} - 6D_1A_{2,f} + \\ &6D_1A_{1,g} + 12A_2C_{1,x} + 12A_1C_{2,x} + C_{2,ff} - C_{2,gg} - 2C_{1,fg} - \\ &12A_2D_{1,f} - 12A_2D_{2,g} - 12A_1D_{2,f} + 12A_1D_{1,g} + 2B_2C_{1,f} + 2B_2C_{2,g} + \\ &2B_1C_{2,f} - 2B_1C_{1,g} - 8B_2B_{1,x} - 8B_1B_{2,x} - 4B_{2,xf} + 4B_{1,xg} = 0, \end{aligned}$$

$$\begin{aligned} &24D_1A_{1,x} - 24D_2A_{2,x} - 6D_1B_{1,f} - 6D_1B_{2,g} + 6D_2B_{2,f} - 6D_2B_{1,g} + \\ &12A_1D_{1,x} - 12A_2D_{2,x} + 4B_{1,xx} - 4C_{1,xf} - 4C_{2,xg} - 6B_1D_{1,f} - \\ &6B_1D_{2,g} + 6B_2D_{2,g} - 6B_2D_{1,g} + 3D_{1,ff} - 3D_{1,gg} + 6D_{2,fg} + 4C_1C_{1,f} \\ &+4C_1C_{2,g} - 4C_2C_{2,f} + 4C_2C_{1,g} - 4C_1B_{1,x} + 4C_2B_{2,x} = 0, \end{aligned}$$

$$
\begin{aligned}
&24D_2A_{1,x}+24D_1A_{2,x}-6D_2B_{1,f}-6D_2B_{2,g}-6D_1B_{2,f}+6D_1B_{1,g}+\\
&12A_2D_{1,x}+12A_1D_{2,x}+4B_{2,xx}-4C_{2,xf}+4C_{1,xg}-6B_2D_{1,f}-\\
&6B_2D_{2,g}-6B_1D_{2,f}+6B_1D_{1,g}+3D_{2,ff}-3D_{2,gg}-6D_{1,fg}+4C_2C_{1,f}-\\
&4C_2C_{2,g}+4C_1C_{2,f}-4C_1C_{1,g}-4C_2B_{1,x}-4C_1B_{2,x}=0. \qquad (2.11)
\end{aligned}
$$

Notice that the system (2.9) contains 8 distinct coefficients whereas the system (2.4) has 15. It may appear that the system (2.9) is a subcase of the system (2.4). We shall prove that this is not the case. These two classes do not coincide for the cubically semi-linear case.

3. DISJOINT CLASSES OF CUBICALLY SEMI-LINEAR SYSTEMS OF ODES

In order to investigate whether the linearizable classes mentioned above can be related by point transformations or not, we start with a simple case i.e constant linear transformations. We state the following theorem.

Theorem 1.

The linearizable classes of systems of ODEs provided by CSA and RSA are not related by constant linear transformations of the form

$$f=aw+by,\ \ g=cw+dy. \qquad (3.1)$$

Proof.

It is clear that the transformations must be invertible and hence $ad-bc\neq 0$. For convenience in writing set $ad-bc=1$. Using transformations (3.1) and it's derivatives in the system (2.9) yields

$$
\begin{aligned}
w''=&(A_1d-A_2b)[(a^3-3ac^2)w'^3+(3a^2b-3bc^2-6acd)w'^2y'+(3ab^2\\
&-6bcd-3ad^2)w'y'^2+(b^3-3bd^2)y'^3]-(A_2d+A_1b)[(3a^2c-c^3)w'^3\\
&+(3a^2d-3c^2d+6abc)w'^2y'+(3b^2c-3cd^2+6abd)w'y'^2+(3b^2d-d^3)y'^3]\\
&+(B_1(a-c(ab+cd))-B_2(a(ab+cd)+c))w'^2-(B_1d+B_2b)(b^2+d^2)y'^2\\
&-2(B_1c+B_2a)(b^2+d^2)w'y'+(C_1-C_2(ab+cd))w'-(C_2(b^2+d^2)y'\\
&+D_1d-D_2b, \qquad (3.2)
\end{aligned}
$$

$$y^{''} = -(A_1c - A_2a)[(a^3 - 3ac^2)w^{'3} + (3a^2b - 3bc^2 - 6acd)w^{'2}y^{'} + (3ab^2$$
$$-6bcd - 3ad^2)w^{'}y^{'2} + (b^3 - 3bd^2)y^{'3}] + (A_2c + A_1a)[(3a^2c - c^3)w^{'3}$$
$$+(3a^2d - 3c^2d + 6abc)w^{'2}y^{'} + (3b^2c - 3cd^2 + 6abd)w^{'}y^{'2} + (3b^2d - d^3)y^{'3}]$$
$$+(B_1c + B_2a)(a^2 + c^2)w^{'2} + (B_1(b + d(ab + cd)) + B_2(b(ab + cd) - d))y^{'2}$$
$$+2(B_1d + B_2b)(a^2 + c^2)w^{'}y^{'} + C_2(a^2 + c^2)w^{'} + (C_1 + C_2(ab + cd))y^{'}$$
$$+D_2a - D_1c. \tag{3.3}$$

Equating coefficients of the cubic (in the first derivative) terms in the system (3.2), (3.3) and (2.4), the linear independence of A_1, A_2 gives the following equations

$$b^4 + d^4 = 0,\ \ a^4 + c^4 = 0,\ \ bd(b^2 + d^2) = 0,\ \ ac(a^2 + c^2) = 0,$$

$$b^2(ad - bc) = 0,\ \ c^2(ad - bc) = 0,\ \ b^2(ab + cd) = 0,\ \ c^2(ab + cd) = 0. \tag{3.4}$$

These equations are incompatible with invertibility, as they require $a = b = c = d = 0$. Thus there does not exist a constant point transformations (3.1) that maps the linearizable system of two ODEs obtained by a scalar second order complex ODE to a system of two ODEs provided by the geometric method.

The above analysis can now be generalized to those point transformations for which the coefficients are functions of the dependent and independent variables. The following theorem provides inequivalence of the two linearizable classes of systems provided by CSA and RSA, by using general point transformations.

Theorem 2.

The linearizable classes of systems of ODEs provided by CSA and RSA are not related by arbitrary (point) transformations of variables

$$f = a_1w + a_2y,\ \ g = a_3w + a_4y, \tag{3.5}$$

where $a_1, .., a_4,$ are functions of w, y and x.

Proof.

To establish this result, we proceed as we did in Theorem 1. We have

$$f^{'} = a_1^{'}w + a_2^{'}y + a_1w^{'} + a_2y^{'}, \tag{3.6}$$
$$g^{'} = a_3^{'}w + a_4^{'}y + a_3w^{'} + a_4y^{'}, \tag{3.7}$$

where

$$a_i^{'} = a_{i,x} + w^{'}a_{i,w} + y^{'}a_{i,y} \tag{3.8}$$

and

$$a_i^{''} = a_{i,xx} + 2w^{'}a_{i,xw} + 2y^{'}a_{i,xy} + 2w^{'}y^{'}a_{i,wy} + w^{'2}a_{i,ww}$$
$$+y^{'2}a_{i,yy} + w^{''}a_{i,x} + y^{''}a_{i,y}. \tag{3.9}$$

Hence

$$f^{''} = (wa_{1,w} + ya_{2,w} + a_1)w^{''} + (wa_{1,y} + ya_{2,y} + a_2)y^{''} + (wa_{1,ww} + ya_{2,ww}$$
$$+2a_{1,w})w^{'2} + (wa_{1,yy} + ya_{2,yy} + 2a_{2,y})y^{'2} + 2\{(wa_{1,wy} + ya_{2,wy} + 2a_{1,y}$$
$$+2a_{2,w})w^{'}y^{'} + (wa_{1,xw} + ya_{2,xw} + a_{1,x})w^{'} + (wa_{1,xy} + ya_{2,xy} + a_{2,x})y^{'}\}$$
$$+wa_{1,xx} + ya_{2,xx}, \tag{3.10}$$

$$g^{''} = (wa_{3,w} + ya_{4,w} + a_3)w^{''} + (wa_{3,y} + ya_{4,y} + a_4)y^{''} + (wa_{3,ww} + ya_{4,ww}$$
$$+2a_{3,w})w^{'2} + (wa_{3,yy} + ya_{4,yy} + 2a_{4,y})y^{'2} + 2\{(wa_{3,wy} + ya_{4,wy} + a_{3,y}$$
$$+a_{4,w})w^{'}y^{'} + (wa_{3,xw} + ya_{4,xw} + a_{3,x})w^{'} + (wa_{3,xy} + ya_{4,xy} + a_{4,x})y^{'}\}$$
$$+wa_{3,xx} + ya_{4,xx}. \tag{3.11}$$

The coefficients of $w^{''}$ and $y^{''}$ appearing in (3.10) and (3.11) can be denoted by

$$\overline{a}_1 = a_1 + wa_{1,w} + ya_{2,w},\ \ \overline{a}_2 = a_2 + wa_{1,y} + ya_{2,y},$$
$$\overline{a}_3 = a_3 + wa_{3,w} + ya_{4,w},\ \ \overline{a}_4 = a_4 + wa_{3,y} + ya_{4,y}, \tag{3.12}$$

where $\overline{a}_1, .., \overline{a}_4$, are functions of w, y and x. Using (3.12) in (3.10) and (3.11) we obtain

$$w^{''} = \frac{1}{\overline{a}_1\overline{a}_4 - \overline{a}_2\overline{a}_3}[(A_1\overline{a}_4 - A_2\overline{a}_2)((a_1^3 - 3a_1a_3^2)w^{'3} + 3(a_1^2a_2 - a_2a_3^2$$
$$-2a_1a_3a_4)w^{'2}y' + 3(a_1a_2^2 - a_1a_4^2 - 2a_2a_3a_4)w'y^{'2} + (a_2^3 - 3a_2a_4^2)y^{'3})$$
$$+(A_2\overline{a}_4 + A_1\overline{a}_2)((a_3^3 - 3a_1^2a_3)w^{'3} + 3(a_3^2a_4 - a_1^2a_4 - 2a_1a_2a_3)w^{'2}y'$$
$$+3(a_3a_4^2 - a_2^2a_3 - 2a_1a_2a_4)w'y^{'2} + (a_4^3 - 3a_2^2a_4)y^{'3}) + (A_1\overline{a}_4 - A_2\overline{a}_2)\overline{\rho_1}$$
$$+(A_2\overline{a}_4 + A_1\overline{a}_2)\overline{\rho_2} + (B_1\overline{a}_4 - B_2\overline{a}_2)\overline{\rho_3} - 2(B_2\overline{a}_4 + B_1\overline{a}_2)\overline{\rho_4} - \overline{a}_4\overline{\rho_5}$$
$$+\overline{a}_2\overline{\rho_6} + (C_1\overline{a}_4 - C_2\overline{a}_2)f^{'} - (C_2\overline{a}_4 + C_1\overline{a}_2)g^{'} + D_1\overline{a}_4 - D_2\overline{a}_2], \tag{3.13}$$

$$y^{''} = \frac{1}{\overline{a}_2\overline{a}_3 - \overline{a}_1\overline{a}_4}[(A_1\overline{a}_3 - A_2\overline{a}_1)((a_1^3 - 3a_1a_3^2)w^{'3} + 3(a_1^2a_2 - a_2a_3^2$$
$$-2a_1a_3a_4)w^{'2}y' + 3(a_1a_2^2 - a_1a_4^2 - 2a_2a_3a_4)w'y^{'2} + (a_2^3 - 3a_2a_4^2)y^{'3})$$
$$+(A_2\overline{a}_3 + A_1\overline{a}_1)((a_3^3 - 3a_1^2a_3)w^{'3} + 3(a_3^2a_4 - a_1^2a_4 - 2a_1a_2a_3)w^{'2}y'$$
$$+3(a_3a_4^2 - a_2^2a_3 - 2a_1a_2a_4)w'y^{'2} + (a_4^3 - 3a_2^2a_4)y^{'3}) + (A_1\overline{a}_3 - A_2\overline{a}_1)\overline{\rho_1}$$
$$+(A_2\overline{a}_3 + A_1\overline{a}_1)\overline{\rho_2} + (B_1\overline{a}_3 - B_2\overline{a}_1)\overline{\rho_3} - 2(B_2\overline{a}_3 + B_1\overline{a}_1)\overline{\rho_4} - \overline{a}_3\overline{\rho_5}$$
$$+\overline{a}_1\overline{\rho_6} + (C_1\overline{a}_3 - C_2\overline{a}_1)f^{'} - (C_2\overline{a}_3 + C_1\overline{a}_1)g^{'} + D_1\overline{a}_3 - D_2\overline{a}_1]. \tag{3.14}$$

Where $\overline{\rho_1}, ..., \overline{\rho_6}$ in the above equations represent the quadratic, linear (in the first derivative) and constant terms which are not required in the proof. These are given in the appendix for completeness. The requirement here is that $a_1a_4 \neq a_2a_3$, and $\overline{a}_1\overline{a}_4 \neq \overline{a}_2\overline{a}_3$. Now equating the coefficients of the cubic (in the first derivative) terms appearing in the system (2.9) and the system (3.13), (3.14) we obtain the following set of equations

$$
\begin{aligned}
&a_2\overline{a}_4(a_2^2-3a_4^2)=\overline{a}_2a_4(3a_2^2-a_4^2),\;\; a_2\overline{a}_2(a_2^2-3a_4^2)=a_4\overline{a}_4(a_4^2-3a_2^2),\\
&a_1\overline{a}_3(a_1^2-3a_3^2)=\overline{a}_1a_3(3a_1^2-a_3^2),\;\; a_1\overline{a}_1(a_1^2-3a_3^2)=a_3\overline{a}_3(a_3^2-3a_1^2),\\
&a_1\overline{a}_4(a_1^2-3a_3^2)+\overline{a}_2a_3(a_3^2-3a_1^2)-3\overline{a}_3(a_1^2a_2-a_2a_3^2-2a_1a_3a_4)\\
&-3\overline{a}_1(a_3^2a_4-a_1^2a_4-2a_1a_2a_3)=0,\\
&-a_1\overline{a}_2(a_1^2-3a_3^2)+a_3\overline{a}_4(a_3^2-3a_1^2)+3\overline{a}_1(a_1^2a_2-a_2a_3^2-2a_1a_3a_4)\\
&-3\overline{a}_3(a_3^2a_4-a_1^2a_4-2a_1a_2a_3)=0,\\
&\\
&a_2\overline{a}_3(a_2^2-3a_4^2)+\overline{a}_1a_4(a_4^2-3a_2^2)-3\overline{a}_4(a_1a_2^2-a_1a_4^2-2a_2a_3a_4)\\
&-3\overline{a}_2(a_3a_4^2-a_2^2a_3-2a_1a_2a_4)=0,\\
&-\overline{a}_1a_2(a_2^2-3a_4^2)+\overline{a}_3a_4(a_4^2-3a_2^2)+3\overline{a}_2(a_1a_2^2-a_1a_4^2-2a_2a_3a_4)\\
&-3\overline{a}_4(a_3a_4^2-a_2^2a_3-2a_1a_2a_4)=0,\\
&\overline{a}_1(a_3a_4^2-a_2^2a_3-2a_1a_2a_4)-\overline{a}_2(a_3^2a_4-a_1^2a_4-2a_1a_2a_3)\\
&+\overline{a}_3(a_1a_2^2-a_1a_4^2-2a_2a_3a_4)-\overline{a}_4(a_1^2a_2-a_2a_3^2-2a_1a_3a_4)=0,\\
&\overline{a}_1(a_1a_2^2-a_1a_4^2-2a_2a_3a_4)-\overline{a}_2(a_1^2a_2-a_2a_3^2-2a_1a_3a_4)\\
&-\overline{a}_3(a_3a_4^2-a_2^2a_3-2a_1a_2a_4)+\overline{a}_4(a_3^2a_4-a_1^2a_4-2a_1a_2a_3)=0.
\end{aligned}\tag{3.15}
$$

There does not exist a solution for the above set of equations other then $a_i=0$, $i=1,..,4$. Thus, once again, we prove in the general case that the linearizable systems obtained by CSA and RSA can not be related by general (point) transformations of variables.

4. CONCLUSION

Two linearizable classes of systems of cubically semi-linear ODEs were provided by CSA and RSA. The equivalence or inequivalence of these classes was investigated under point transformations. It was shown that they are distinct for systems of genuinely cubically semi-linear ODEs.

There are five linearizable classes of systems of ODEs [19] obtained by group classification having $5, 6, 7, 8$, or 15 symmetry generators . The class of linearizable systems provided by the geometric method has 15 symmetry generators. It needs to be investigated whether CSA provides the remaining four classes mentioned above or not. It is not even clear that CSA is limited to these five classes as it appears that it does not yield a Lie algebra when complex functions of a single variable are involved [3]. These issues, i.e the classification of linearizable systems of ODEs provided by CSA, are being investigated [4].

5. APPENDIX

The extra terms appearing in the equations (3.13) and (3.14) are:

$$\begin{aligned}
\overline{\rho_1} &= 3\{(a_1^2a_1^{'} - a_1^{'}a_3^2 - 2a_1a_3a_3^{'})w + (a_1^2a_2^{'}y - a_2^{'}a_3^2 - 2a_1a_3a_4^{'})y\}w^{'2} \\
&-6\{(a_1^{'}a_3a_4 + a_1a_3^{'}a_4 + a_2a_3a_3^{'} - a_1a_1^{'}a_2)w + (a_2^{'}a_3a_4 + a_1a_4a_4^{'} + a_2a_3a_4^{'} \\
&-a_1a_2a_2^{'})y\}w^{'}y^{'} + 3\{(a_1^{'}a_2^2 - a_1^{'}a_4^2 - 2a_2a_3^{'}a_4)w + (a_2^2a_2^{'} - a_2^{'}a_4^2 - 2a_2a_4a_4^{'})y\}y^{'2} \\
&+3\{(a_1^{'2}a_1 - a_1a_3^{'2} - 2a_1^{'}a_3a_3^{'})w^2 + (a_1a_2^{'2} - a_1a_4^{'2} - 2a_2^{'}a_3a_4^{'})y^2 \\
&+2(a_1a_1^{'}a_2^{'} - a_1^{'}a_3a_4^{'} - a_1a_3^{'}a_4^{'} - a_2^{'}a_3a_3^{'})wy\}w^{'} + 3\{(a_1^{'2}a_2 - a_2a_3^{'2} - 2a_1^{'}a_3^{'}a_4)w^2 \\
&+(a_2a_2^{'2} - a_2a_4^{'2} - 2a_2^{'}a_4a_4^{'})y^2 + 2(a_1^{'}a_2a_2^{'} - a_1^{'}a_4a_4^{'} - a_2^{'}a_3^{'}a_4 - a_2a_3^{'}a_4^{'})wy\}y^{'} \\
&+(a_1^{'3} - 3a_1^{'}a_3^{'2})w^3 + 3(a_1^{'2}a_2^{'} - a_2^{'}a_3^{'2} - 2a_1^{'}a_3^{'}a_4^{'})w^2y + 3(a_1^{'}a_2^{'2} - a_1^{'}a_4^{'2} \\
&-2a_2^{'}a_3^{'}a_4^{'})wy^2 + (a_2^{'3} - 3a_2^{'}a_4^{'2})y^3,
\end{aligned}$$

$$\begin{aligned}
\overline{\rho_2} &= 3\{(a_1^2a_3^{'} + 2a_1a_1^{'}a_3 - a_3^2a_3^{'})w + (a_1^2a_4^{'} + 2a_1a_2^{'}a_3 - a_3^2a_4^{'})y\}w^{'2} \\
&+6\{(a_1a_2a_3^{'} + a_1^{'}a_2a_3 + a_1a_1^{'}a_4 - a_3a_3^{'}a_4)w + (a_1a_2a_4^{'} + a_2a_2^{'}a_3 + a_1a_2^{'}a_4 \\
&-a_3a_4a_4^{'})y\}w^{'}y^{'} + 3\{(a_2^2a_3^{'} + 2a_1^{'}a_2a_4^{'} - a_3^{'}a_4^2)w + (a_2^2a_4^{'} + 2a_2a_2^{'}a_4 - a_4^2a_4^{'})y\}y^{'2} \\
&+3\{(a_1^{'2}a_3 + 2a_1a_1^{'}a_3^{'} - a_3^{'2}a_3)w^2 + (a_2^{'2}a_3 + 2a_1a_2^{'}a_4^{'} - a_3a_4^{'2})y^2 \\
&+2(a_1a_2^{'}a_3^{'} + a_1a_1^{'}a_4^{'} + a_1^{'}a_2^{'}a_3 - a_3a_3^{'}a_4^{'})wy\}w^{'} + 3\{(a_1^{'2}a_4 + 2a_1^{'}a_2a_3^{'} - a_3^{'2}a_4)w^2 \\
&+(a_2^{'2}a_4 + 2a_2a_2^{'}a_4^{'} - a_4a_4^{'2})y^2 + 2(a_1^{'}a_2^{'}a_4 + a_2a_2^{'}a_3^{'} + a_1^{'}a_2a_4^{'} - a_3^{'}a_4a_4^{'})wy\}y^{'} \\
&+(3a_1^{'2}a_3^{'} - a_3^{'3})w^3 + 3(a_1^{'2}a_4^{'} + 2a_1^{'}a_2^{'}a_3^{'} - a_3^{'2}a_4^{'})w^2y \\
&+3(a_2^{'2}a_3^{'} + 2a_1^{'}a_2^{'}a_4^{'} - a_3^{'}a_4^{'2})wy^2 + (3a_2^{'2}a_4^{'} - a_4^{'3})y^3,
\end{aligned}$$

$$\begin{aligned}
\overline{\rho_3} &= (a_1^2 - a_3^2)w^{'2} + 2(a_1a_2 - a_3a_4)w^{'}y^{'} + (a_2^2 - a_4^2)y^{'2} + 2\{(a_1a_1^{'} - a_3a_3^{'})w \\
&+(a_1a_2^{'} - a_3a_4^{'})y\}w^{'} + 2\{(a_1^{'}a_2 - a_3^{'}a_4)w + (a_2a_2^{'} - a_4a_4^{'})y\}y^{'} + (a_1^{'2} - a_3^{'2})w^2 \\
&+2(a_1^{'}a_2^{'} - a_3^{'}a_4^{'})wy + (a_2^{'2} - a_4^{'2})y^2,
\end{aligned}$$

$$\begin{aligned}
\overline{\rho_4} &= a_1a_3w^{'2} + (a_1a_4 + a_2a_3)w^{'}y^{'} + a_2a_4y^{'2} + \{(a_1^{'}a_3 + a_1a_3^{'})w + (a_2^{'}a_3 \\
&+a_1a_4^{'})y\}w^{'} + \{(a_1^{'}a_4 + a_2a_3^{'})w + (a_2^{'}a_4 + a_2a_4^{'})y\}y^{'} + a_1^{'}a_3^{'}w^2 + (a_1^{'}a_4^{'} \\
&+a_2^{'}a_3^{'})wy + a_2^{'}a_4^{'}y^2,
\end{aligned}$$

$$\overline{\rho_5} = (wa_{1,ww} + ya_{2,ww} + 2a_{1,w})w^{'2} + (wa_{1,yy} + ya_{2,yy} + 2a_{2,y})y^{'2}$$
$$+2\{(wa_{1,wy} + ya_{2,wy} + 2a_{1,y} + 2a_{2,w})w^{'}y^{'} + (wa_{1,xw} + ya_{2,xw} + a_{1,x})w^{'}$$
$$+(wa_{1,xy} + ya_{2,xy} + a_{2,x})y^{'}\} + wa_{1,xx} + ya_{2,xx},$$

$$\overline{\rho_6} = (wa_{3,ww} + ya_{4,ww} + 2a_{3,w})w^{'2} + (wa_{3,yy} + ya_{4,yy} + 2a_{4,y})y^{'2}$$
$$+2\{(wa_{3,wy} + ya_{4,wy} + a_{3,y} + a_{4,w})w^{'}y^{'} + (wa_{3,xw} + ya_{4,xw} + a_{3,x})w^{'}$$
$$+(wa_{3,xy} + ya_{4,xy} + a_{4,x})y^{'}\} + wa_{3,xx} + ya_{4,xx}.$$

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