Mathematical and Computational Applications, Vol. 16, No. 4, pp. 890-899, 2011.
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A NEW PERTURBATION-ITERATION APPROACH FOR FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract- Two new perturbation-iteration algorithms for solving differential equations of first order are proposed. Variants of the algorithm are developed depending on the differential order of Taylor series expansions. The iteration algorithms are tested on a number of first order equations. Much better solutions than the regular perturbation solutions are achieved.
Keywords- Perturbation Methods**,** Perturbation-Iteration Algorithms, First Order Differential Equations, Numerical Solutions, Convergence.

1. INTRODUCTION

Perturbations methods are widely used for over a century to determine approximate analytical solutions for mathematical models. Algebraic equations, integrals, differential equations, difference equations and integro-differential equations can be solved approximately with these techniques. The direct expansion method (straightforward expansion) does not produce physically valid solutions for most of the cases and depending on the nature of the equation, many different perturbation techniques such as Lindstedt-Poincare technique, Renormalization method, Method of Multiple Scales, Averaging methods, Method of Matched Asymptotic Expansions etc.[1] are developed within time.

One of the deficiencies in applying perturbation methods is that a small parameter is needed in the equations or the small parameter should be introduced artificially to the equations. Nevertheless, the solved problem is a weak nonlinear problem and it becomes hard to obtain a valid approximate solution for strongly nonlinear systems.

While a complete review of the attempts to validate perturbation solutions for strongly nonlinear oscillators is beyond the scope of this work, a partial list will be given. Among the many developed methods, modifications of the Lindstedt-Poincare method with a different frequency expansion [2-5], Linearized perturbation method [6-8], parameter expanding method [9, 10], new time transformations as modifications of Lindstedt-Poincare method [11-13] are some examples. Recently, Multiple Scales and Lindstedt Poincare method are unified [14, 15] with a frequency expansion as in [2-5] to obtain convergent solutions for strongly nonlinear systems.

An alternative attempt in the literature to validate solutions for strongly nonlinear systems is the perturbation-iteration methods (or alternatively named as iteration-perturbation methods) [9, 16-27]. Usually, the equations are cast into an alternative form before applying the iteration procedure. Some of the algorithms developed can only work for specific problems. A general approach valid for all type of equations which do not require non-standard pre-transformations and initial assumptions is lacking in the literature.

The aim in this study is to develop new perturbation iteration algorithms which do not require special transformations and initial assumptions. (Please see reference [28]). Motivated by the results for algebraic equations [29-31] which leaded to a vast number of iteration algorithms, the basic logic will be extended to first order differential equations in this study. It is shown for algebraic equations [29-31] that by taking n correction terms in the perturbation expansions and m'th derivative correction terms in the Taylor series expansions, many iteration algorithms can be developed (n≤m). Similar reasoning will lead to perturbation-iteration algorithms for differential equations also. Two different algorithms with (n,m) equal to (1,1) and (1,2) are developed as examples. The algorithms are tested using a number of first order differential equations and convergent results are achieved in the iterations.

2. PERTURBATION-ITERATION ALGORITHM PIA(1,1)

In this section, a perturbation-iteration algorithm is developed by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor Series expansion, i.e. n=1, m=1. The algorithm is called PIA(1,1). Consider the general first order differential equation

$$F(u,\dot{u},\varepsilon)=0 \tag{1}$$

with u=u(t) and ε the perturbation parameter. Only one correction term is taken in the perturbation expansion

$$u_1=u_0+\varepsilon u_c+\ldots \tag{2}$$

Upon substitution of (2) into (1) and expanding in a Taylor series with first derivatives only yields

$$F(u_0,\dot{u}_0,0)+F_u(u_0,\dot{u}_0,0)\,\varepsilon u_c+F_{\dot{u}}(u_0,\dot{u}_0,0)\,\varepsilon\dot{u}_c+F_\varepsilon(u_0,\dot{u}_0,0)\,\varepsilon=0 \tag{3}$$

where subscripts denote differentiation with respect to the variable. Reorganizing the equation

$$\dot{u}_c+\frac{F_u}{F_{\dot{u}}}u_c=-\frac{F_\varepsilon+F/\varepsilon}{F_{\dot{u}}} \tag{4}$$

and keeping in mind that all derivatives are evaluated at ε=0, it is readily observed that the above equation is a variable coefficient first order differential equation whose solution is

$$u_c=c\exp\left(-\int\frac{F_u}{F_{\dot{u}}}dt\right)-\left(\int\frac{F_\varepsilon+F/\varepsilon}{F_{\dot{u}}}\exp\left(\int\frac{F_u}{F_{\dot{u}}}dt\right)dt\right)\exp\left(-\int\frac{F_u}{F_{\dot{u}}}dt\right) \tag{5}$$

Substitution of (5) into (2) and constructing the iteration scheme yields

$$\begin{aligned}u_{n+1}&=u_n+\varepsilon c_n\exp\left(-\int\frac{F_u(u_n,\dot{u}_n,0)}{F_{\dot{u}}(u_n,\dot{u}_n,0)}dt\right)\\&-\varepsilon\left(\int\frac{F_\varepsilon(u_n,\dot{u}_n,0)+F(u_n,\dot{u}_n,0)/\varepsilon}{F_{\dot{u}}(u_n,\dot{u}_n,0)}\exp\left(\int\frac{F_u(u_n,\dot{u}_n,0)}{F_{\dot{u}}(u_n,\dot{u}_n,0)}dt\right)dt\right)\exp\left(-\int\frac{F_u(u_n,\dot{u}_n,0)}{F_{\dot{u}}(u_n,\dot{u}_n,0)}dt\right)\end{aligned} \tag{6}$$

2.1. Example Problem 1

Consider the differential equation with the condition

$$\dot{u} + \varepsilon u^2 = 0 \qquad u(0)=1 \tag{7}$$

for which the exact solution is

$$u = \frac{1}{1+\varepsilon t} \tag{8}$$

Terms in iteration formula (6) are $F(u_n, \dot{u}_n, 0) = \dot{u}_n$, $F_u(u_n, \dot{u}_n, 0) = 0$, $F_{\dot{u}}(u_n, \dot{u}_n, 0) = 1$, $F_\varepsilon(u_n, \dot{u}_n, 0) = u_n^2$ and equation (6) reduces to

$$u_{n+1} = \varepsilon(c_n - \int u_n^2 dt) \qquad n = 0,1,2... \tag{9}$$

In applying the iteration formula, an initial guess suitable to the boundary condition should be selected and at each step c_n coefficients have to be determined from the boundary condition. Selecting

$$u_0=1 \tag{10}$$

and using the formula, the approximate solutions at each step are

$$u_1 = 1 - \varepsilon t \tag{11}$$

$$u_2 = 1 - \varepsilon t + \varepsilon^2 t^2 - \frac{1}{3}\varepsilon^3 t^3 \tag{12}$$

$$u_3 = 1 - \varepsilon t + \varepsilon^2 t^2 - \varepsilon^3 t^3 + \frac{2}{3}\varepsilon^4 t^4 - \frac{1}{3}\varepsilon^5 t^5 + \frac{1}{9}\varepsilon^6 t^6 - \frac{1}{63}\varepsilon^7 t^7 \tag{13}$$

Although, iterations up to u_7 are calculated, they are not given here for brevity. In fact, using Mathematica, iterations can be calculated up to any arbitrary order. In Figure 1, exact solution and Taylor series solution with 30 terms is given for ε=2. It is also shown how the successive iterations converge to the exact solution. An excellent match is observed for 7th iteration. Note that using regular perturbation analysis, the Taylor series solution is retrieved. This means that with 7 terms in the perturbation-iteration, much better solution can be obtained than the 30 term regular perturbation expansion.

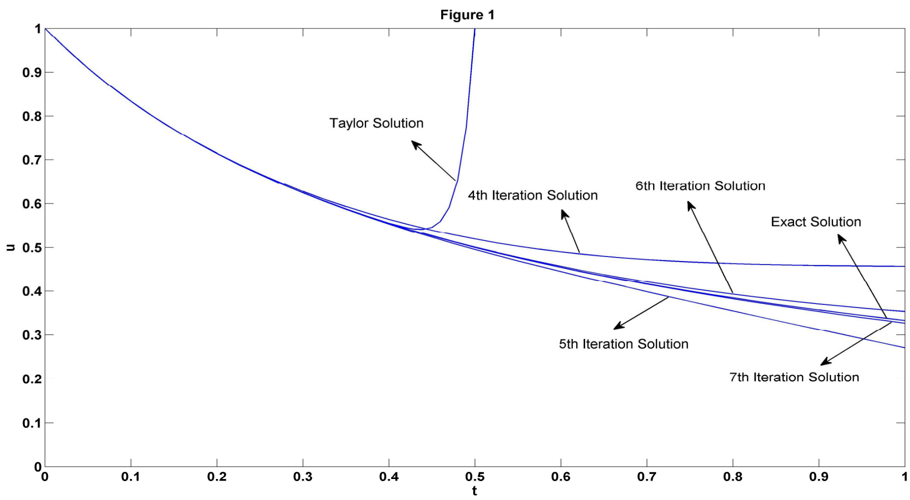

Figure 1- Comparison of perturbation-iteration PIA (1,1) solutions and Taylor series solution (30 terms) with exact solution (ε=2, Example Problem 1)

3. PERTURBATION-ITERATION ALGORITHM PIA(1,2)

In this section, a perturbation-iteration algorithm is obtained by taking one correction term in the perturbation expansion and correction terms up to second derivatives in the Taylor Series expansion, i.e. n=1, m=2. As in the previous case, again only one correction term in the perturbation expansion is taken

$$u_1 = u_0 + \varepsilon u_c + \ldots \tag{14}$$

which upon substitution into (1) and expanding in a Taylor series up to second order derivatives yields after arrangement

$$(F_{\dot{u}} + \varepsilon F_{\dot{u}\varepsilon})\dot{u}_c + (F_u + \varepsilon F_{u\varepsilon})u_c + \frac{1}{2}\varepsilon F_{\dot{u}\dot{u}}\dot{u}_c^2 + \varepsilon F_{u\dot{u}}u_c\dot{u}_c + \frac{1}{2}\varepsilon F_{uu}u_c^2 = -\frac{F}{\varepsilon} - F_\varepsilon - \frac{1}{2}\varepsilon F_{\varepsilon\varepsilon} \tag{15}$$

After solving u_c from above, the iteration scheme is constructed as below

$$u_{n+1} = u_n + \varepsilon (u_c)_n \quad n=0,1,2\ldots \tag{16}$$

Note that, as mentioned before all functions and derivatives are evaluated at ε=0. Since, the equation is nonlinear in u_c, a general solution can not be given as in the previous case.

3.1. Example Problem 1

Consider the same example as in the previous section

$$\dot{u} + \varepsilon u^2 = 0 \qquad u(0)=1 \tag{17}$$

The terms in (15) are evaluated first: $F(u_n, \dot{u}_n, 0) = \dot{u}_n$, $F_u(u_n, \dot{u}_n, 0) = 0$, $F_{\dot{u}}(u_n, \dot{u}_n, 0) = 1$, $F_\varepsilon(u_n, \dot{u}_n, 0) = u_n^2$, $F_{\dot{u}\varepsilon}(u_n, \dot{u}_n, 0) = 0$, $F_{u\varepsilon}(u_n, \dot{u}_n, 0) = 2u_n$, $F_{u\dot{u}}(u_n, \dot{u}_n, 0) = 0$, $F_{\dot{u}\dot{u}}(u_n, \dot{u}_n, 0) = 0$, $F_{uu}(u_n, \dot{u}_n, 0) = 0$, $F_{\varepsilon\varepsilon}(u_n, \dot{u}_n, 0) = 0$. Then (15) takes the simplified form

$$(\dot{u}_c)_n + 2\varepsilon u_n (u_c)_n = -\frac{\dot{u}_n}{\varepsilon} - u_n^2 \tag{18}$$

For the initial assumed function, one may take

$$u_0 = 1 \tag{19}$$

Substituting this function to (18) yields

$$(\dot{u}_c)_0 + 2\varepsilon (u_c)_0 = -1 \tag{20}$$

Solving (20), substituting into (16) and applying the boundary condition yields

$$u_1 = 1 + \frac{1}{2}(e^{-2\varepsilon t} - 1) \tag{21}$$

This solution is substituted into (18)

$$(\dot{u}_c)_1 + 2\varepsilon\left(1 + \frac{1}{2}(e^{-2\varepsilon t} - 1)\right)(u_c)_1 = e^{-2\varepsilon t} - \left(1 + \frac{1}{2}(e^{-2\varepsilon t} - 1)\right)^2 \tag{22}$$

Since the equation to be solved is a variable coefficient equation which is involved, the function in the parentheses of second term is approximated as 1 for simplicity. Solving (22), substituting into the iteration expansion and applying the boundary condition yields

$$u_2 = 1 + \frac{1}{2}(e^{-2\varepsilon t} - 1) + \frac{1}{2}\varepsilon t e^{-2\varepsilon t} + \frac{1}{8}(e^{-4\varepsilon t} - 1) \tag{23}$$

A further iteration yields

$$u_3 = e^{-2\varepsilon t}\left(\frac{161}{384}+\frac{5}{8}\varepsilon t+\frac{5}{16}\varepsilon^2 t^2\right)+e^{-4\varepsilon t}\left(\frac{15}{64}+\frac{3}{8}\varepsilon t+\frac{1}{8}\varepsilon^2 t^2\right)+e^{-6\varepsilon t}\left(\frac{5}{128}+\frac{1}{32}\varepsilon t\right)+\frac{1}{384}e^{-8\varepsilon t}+\frac{39}{128} \tag{24}$$

Using mathematica, up to 7th order iterations are calculated. Since the solutions are very complicated, they are not presented here. Again the seventh iteration compares well with the exact solution whereas the 30 term Taylor expansion which would be obtained by a regular perturbation expansion of 30 correction terms explode and do not represent the real solution after t≥0.2 (See Figure 2).

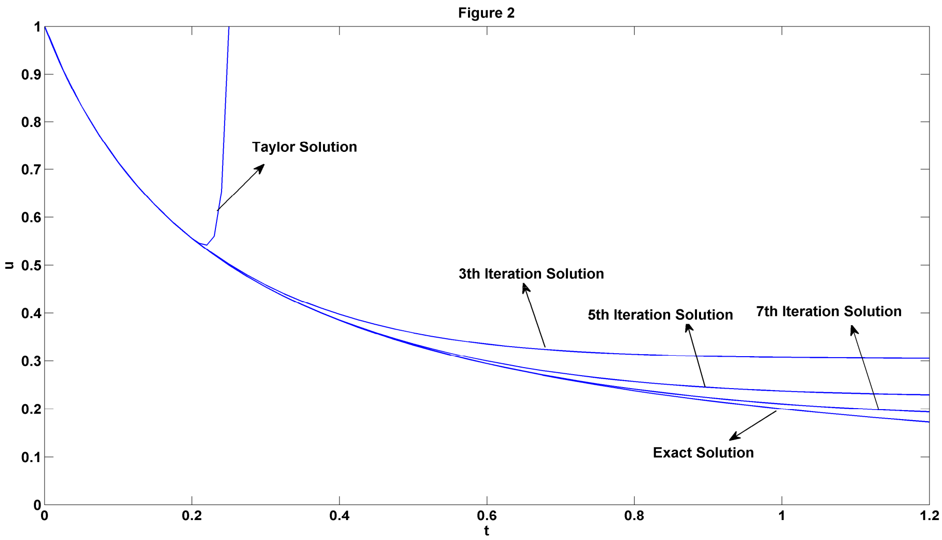

Figure 2- Comparison of perturbation-iteration PIA(1,2) solutions and Taylor series solutions (30 terms) with exact solution (ε=4, Example Problem 1)

3.2. Example Problem 2

Consider the problem of non-Newtonian fluid flow through parallel plates. For constant viscosity and neglecting temperature effects, the equation is [32]

$$\ddot{u}+6\varepsilon(\dot{u})^2\ddot{u}=\Lambda, \qquad u(0)=0, \quad \dot{u}(1/2)=0 \tag{25}$$

for which the first integral is

$$\dot{u}+2\varepsilon(\dot{u})^3=\Lambda\left(y-\frac{1}{2}\right), \quad u(0)=0 \tag{26}$$

where u=u(y) and dot denotes differentiation with respect to variable y. ε is the non-Newtonian coefficient and Λ is the constant pressure gradient. For (26), equation (15) reads

$$(1+6\varepsilon\dot{u}_n^2)(\dot{u}_c)_n=-\varepsilon^{-1}\dot{u}_n-2\dot{u}_n^3+\varepsilon^{-1}\Lambda_3\left(y-\frac{1}{2}\right) \tag{27}$$

For the initial assumption, a simple solution

$$u_0=0 \tag{28}$$

satisfying the remaining boundary condition is selected. Substituting into (27), solving for $(u_c)_0$ and inserting into (16) yields

$$u_1 = \Lambda\left(\frac{y^2}{2} - \frac{y}{2}\right) \tag{29}$$

Using this function, the result for second iteration is

$$u_2 = \frac{\Lambda}{3}(y^2 - y) + \frac{\ln\left[\dfrac{2 + 3\varepsilon\Lambda^2(1-2y)^2}{2+3\varepsilon\Lambda^2}\right]}{36\varepsilon\Lambda} \tag{30}$$

and the result for third iteration is

$$u_3 = \frac{2\Lambda}{9}(y^2 - y) - \frac{\Lambda(104 + 141\,\varepsilon\Lambda^2)}{1800(2+3\varepsilon\Lambda^2)^2} + \frac{(1-2y)^2\Lambda[104 + 141\varepsilon\Lambda^2(1-2y)^2]}{1800[2+3\varepsilon\Lambda^2(1-2y)^2]^2}$$
$$+\frac{729\ln\left[\dfrac{3+2(1-2y)^2\varepsilon\Lambda^2}{3+2\varepsilon\Lambda^2}\right] + 104\ln\left[\dfrac{2+3\varepsilon\Lambda^2}{2+3(1-2y)^2\varepsilon\Lambda^2}\right]}{6000\,\varepsilon\Lambda} \tag{31}$$

Expanding the above expression in a Taylor series and keeping terms up to $O(\varepsilon^3)$

$$u_3^T = \frac{\Lambda}{2}(y^2 - y) + \frac{\varepsilon\Lambda^3}{4}(-2y^4 + 4y^3 - 3y^2 + y) + \frac{\varepsilon^2\Lambda^5}{8}(16y^6 - 48y^5 + 60y^4 - 40y^3$$
$$+15y^2 - 3y) + O(\varepsilon^3) \tag{32}$$

the corresponding result that would be obtained in a regular perturbation problem is retrieved. In Figure 3, it is shown that the third iteration solution is indistinguishable from the numerical solution. Residual errors of each iteration are shown in Figure 4 for ε=10. The errors decrease substantially as the number of iterations increase in the whole domain. In Table 1, regular perturbation (Eq. (32)) and iteration-perturbation (Eq. (31)) solutions are contrasted with the numerical solutions. As ε increases, regular-perturbation solutions diverge much from the numerical ones whereas, perturbation-iteration solutions are consistent with the numerical values even for large ε.

Table 1. Comparison of Perturbation, Perturbation-Iteration and Numerical Results for various ε values

	ε=1			ε=5			ε=10		
y	Numer.	Pert.	Pert.-Iter.	Numer.	Pert.	Pert.-Iter.	Numer.	Pert.	Pert.-Iter.
0.1	0.03578	0.04960	0.035623	0.02644	0.5292	0.02664	0.02241	2.1663	0.02387
0.2	0.06547	0.8259	0.065268	0.04934	0.6888	0.04939	0.04208	2.7872	0.04392
0.3	0.08809	0.10567	0.087879	0.06787	0.7308	0.06778	0.05835	2.9127	0.06100
0.4	0.10245	0.12004	0.102231	0.08064	0.7452	0.08053	0.07001	2.9328	0.07169
0.5	0.10740	0.125	0.107183	0.08543	0.75	0.08532	0.07464	2.9375	0.07632
0.6	0.10245	0.120048	0.102231	0.08064	0.7452	0.08053	0.07001	2.9328	0.07169
0.7	0.08809	0.10567	0.087879	0.06787	0.7308	0.06778	0.05835	2.9127	0.06010
0.8	0.06547	0.08259	0.065268	0.04934	0.6888	0.04939	0.04208	2.7872	0.04392
0.9	0.03578	0.04960	0.035623	0.02644	0.5292	0.02664	0.02241	2.1663	0.02387

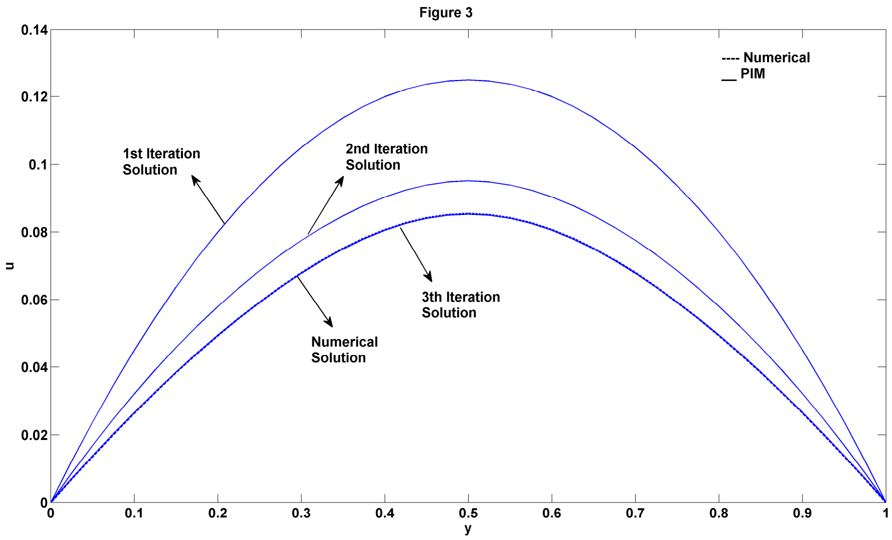

Figure 3. Comparison of perturbation-iteration PIA(1,2) solutions with numerical solutions (ε=5, Λ=-1, Example Problem 2)

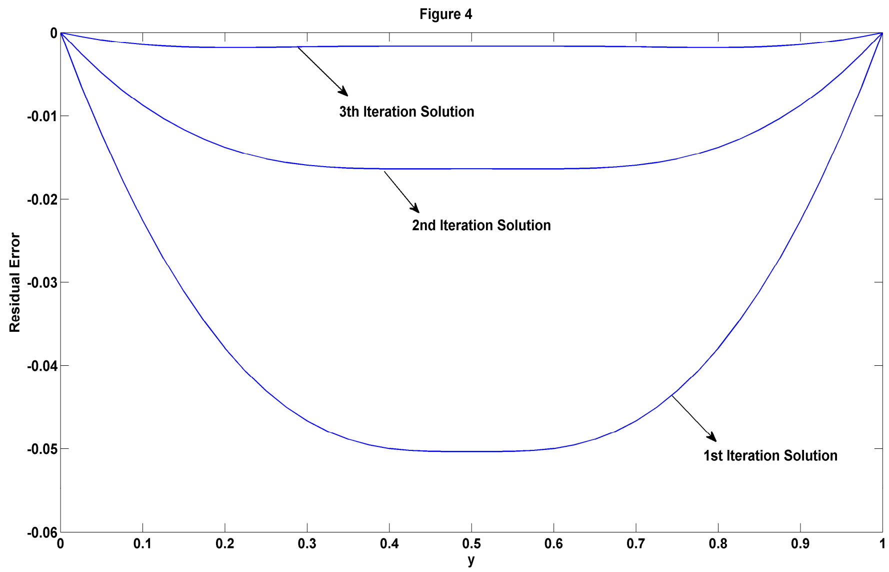

Figure 4. Residual errors for perturbation-iteration PIA(1,2) solutions (ε=10, Λ=-1, Example Problem 2)

3.3. Example Problem 3

The problem given in [33] is reconsidered.

$$(1+\varepsilon u)\dot{u}+u=0 \qquad u(0)=1 \tag{33}$$

where u=u(y). The iteration formula is constructed from (15)

$$(1+\varepsilon u_n)(\dot{u}_c)_n+(1+\varepsilon \dot{u}_n)(u_c)_n=-\frac{\dot{u}_n}{\varepsilon}-\frac{u_n}{\varepsilon}-u_n\dot{u}_n \tag{34}$$

The initial trial function is selected as

$$u_0=1 \tag{35}$$

The first iteration solution is

$$u_1 = e^{-\frac{y}{1+\varepsilon}} \tag{36}$$

Second and third iterations are

$$u_2 = \frac{-e^{-y} + \varepsilon e^{-\frac{2y}{1+\varepsilon}}}{-1+\varepsilon} \tag{37}$$

$$u_3 = \frac{e^{-y}(-6+8\varepsilon+\varepsilon^2-2\varepsilon^3-\varepsilon^4) + e^{-\frac{y(3+\varepsilon)}{1+\varepsilon}}(-9\varepsilon^2+\varepsilon^4) + e^{-2y}(6\varepsilon-2\varepsilon^2) + 4\varepsilon^3 e^{-\frac{4y}{1+\varepsilon}}}{2(-3+\varepsilon)(-1+\varepsilon)^2} \tag{38}$$

In calculating second and third iterations, the terms multiplying ε at the left hand side of the equation are neglected, not to allow variable coefficients in the equations. The Taylor series expansion of (38) corresponding to a regular perturbation solution would be

$$u_3^T = e^{-y} + \varepsilon(e^{-y} - e^{-2y}) + \varepsilon^2\left(\frac{1}{2}e^{-y} - 2e^{-2y} + \frac{3}{2}e^{-3y}\right) \tag{39}$$

Solutions (38) and (39) are compared with the numerical solutions. Regular perturbation solution possesses an initial oscillatory behavior which can not be verified with the numerical solution (See Figure 5). On the contrast, perturbation-iteration solution matches well with the numerical simulations.

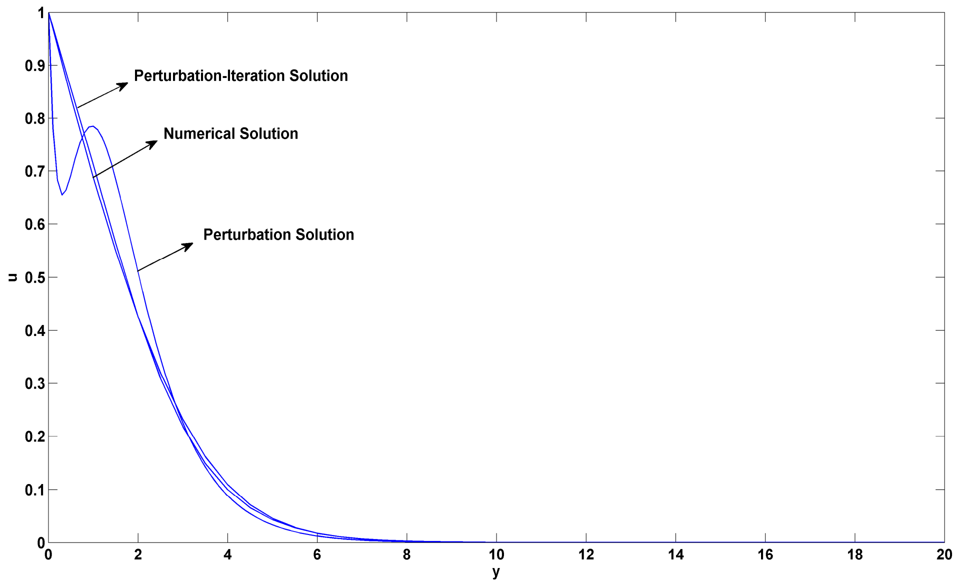

Figure 5. Comparison of perturbation-iteration PIA(1,2) solution and perturbation solution with numerical solution (ε=2, Example Problem 3)

4. CONCLUDING REMARKS

A new approach for constructing perturbation-iteration schemes is presented. Two iteration algorithms are developed. In the first algorithm, one correction term in the perturbation expansion and derivatives up to first order in the Taylor expansion are taken. In the second algorithm, second order derivatives are included in the Taylor series expansion also. Both algorithms are tested using example problems. It is found that iteration results match well with the exact/numerical solutions. Perturbation-iteration algorithms definitely extend the range of validity of perturbation parameter.

Acknowledgment- This work is supported by The Scientific and Technological Research Council of Turkey (TUBITAK) under project number 108M490.

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