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VARIATIONAL ITERATION METHOD FOR SOLVING n-th ORDER FUZZY DIFFERENTIAL EQUATIONS

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Abstract- In this paper, the variational iteration method (VIM) is employed to solve a system of fuzzy differential equations of first order. Since every ordinary fuzzy differential equations of higher order can be converted into a fuzzy system of the first order, this method can be used for solving n-th order fuzzy differential equations. Also the convergency of VIM for this system is proved. Finally to more illustrate several examples are solved.
Keywords- Variational iteration method, n-th order fuzzy differential equations.

1. INTRODUCTION

Many authors have been worked about variational iteration method (VIM) that for more information sees [6]-[9]. In this paper, the VIM is extent to solve n-th order fuzzy differential equations and obtain approximate fuzzy solution.
The VIM is proposed by He [9,10] as a modification of a general Lagrange multiplier method [11]. To illustrate its basic idea of the technique, we consider following general nonlinear system:

$$L(u(t)) + N(u(t)) = g(t),$$

where L is a linear operator, N is a nonlinear operator, and g(t) is a given construct a correction functional for the system, which reads

$$u^{[k+1]}(t) = u^{[k]}(t) + \int_a^x \lambda\,[Lu^{[k]}(s) - N\tilde{u}^{[k]}(s) - g(s)]ds$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory [9,10,11], the subscript k denotes the n-th order approximation and $\tilde{u}^{[k]}$ denotes a restricted variation, i.e $\delta\tilde{u}^{[k]} = 0$

The structure of this paper is organized as follows. In section 2, some basic definitions and notations which will be used are brought. In section 3, the numerical method to solve n-th order fuzzy differential equations is proposed. In section 4, convergency of VIM for this system is proved. In section 5, the application of mentioned method VIM is brought by solving some numerical examples and finally the results are compared with exact solutions. Conclusion is drawn in section 6.

2. PRELIMINAR

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ for all $r \in [0,1]$, which satisfy the following requirements [2]

$\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,

$\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,
$\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

2.1. Remark [1]

Let $\tilde{u}(r) = (\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$ be a fuzzy number, we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2}, \; u^d(r) = \frac{\underline{u}(r) - \overline{u}(r)}{2}.$$

It is clear that $u^d(r) \ge 0$ and $\underline{u}(r) = u^c(r) + u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$ also a fuzzy number $\tilde{u} \in E$ is said symmetric if $\tilde{u}(r)$ is independent of r for all $0 \le r \le 1$.

2.2. Remark [1]

Let $\tilde{u}(r) = (\underline{u}(r), \overline{u}(r))$, $\tilde{v}(r) = (\underline{v}(r), \overline{v}(r))$ and also k, s are arbitrary real numbers.

If $\tilde{w} = k\tilde{u} + s\tilde{v}$ then

$$w^c(r) = ku^c(r) + sv^c(r),$$
$$w^d(r) = |k|\, u^d(r) + |s|\, v^d(r).$$

2.3. Definition [2]

Let $\tilde{u} = (\underline{u}(r), \overline{u}(r)), \tilde{v} = (\underline{v}(r), \overline{v}(r))$, be fuzzy numbers then the Hausdorff distance between $\tilde{u}, \tilde{v}$ is

$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}.$$

2.4. Remark [1]

Clearly from remark (2.2) we have

$$|\overline{u}(r) - \overline{v}(r)| \le |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|,$$
$$|\underline{u}(r) - \underline{v}(r)| \le |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|.$$

Hence for all $r \in [0,1]$

$$\max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\} \le |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|,$$

and then

$$D(\tilde{u}, \tilde{v}) \le \sup_{0 \le r \le 1}\{|u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|\}.$$

Therefore if $|u^c(r) - v^c(r)|$ *and* $|u^d(r) - v^d(r)|$ tend to zero then $D(\tilde{u}, \tilde{v})$ tend to zero.
Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded r-level intervals. This means that if $\tilde{v} \in E$ then the r-level set

$$[\tilde{v}]_r = \{s \mid \tilde{v}(s) \ge r\},$$

is a closed bounded interval which is denoted by
$[\tilde{v}]_r = [\underline{v}(r), \overline{v}(r)]$ *for* $r \in (0,1]$,
and $[\tilde{v}]_0 = \overline{\bigcup_{r \in (0,1]} [\tilde{v}]_r}$.
Two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are called equal, $\tilde{u} = \tilde{v}$, if $\tilde{u}(s) = \tilde{v}(s)$ for all $s \in R$ or $[\tilde{u}]_r = [\tilde{v}]_r$ for all $r \in [0,1]$.

2.5. Lemma [13]

If $\tilde{u},\tilde{v}\in E$, then for $r\in(0,1]$,

$$[\tilde{u}+\tilde{v}]_r=[\underline{u}(r)+\underline{v}(r),\overline{u}(r)+\overline{v}(r)],$$
$$[\tilde{u}.\tilde{v}]_r=[\min k,\max k],$$

where

$$k=\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r),\overline{u}(r)\overline{v}(r)\}.$$

2.6. Lemma [13]

Let $[\underline{v}(r),\overline{v}(r)],\ r\in(0,1]$ be a given family of non-empty intervals. If

$$(i)\quad [\underline{v}(r),\overline{v}(r)]\supset[\underline{v}(s),\overline{v}(s)]\ \mathit{for}\ 0<r\le s,$$

and

$$(ii)\quad [\lim_{k\to\infty}\underline{v}(r_k),\lim_{k\to\infty}\overline{v}(r_k),]=[\underline{v}(r),\overline{v}(r)],$$

whenever (r_k) is a non-decreasing sequence converging to $r\in(0,1]$, then the family $[\underline{v}(r),\overline{v}(r)],0<r\le 1,$ represent the r-level sets of a fuzzy number v in E. Conversely if $[\underline{v}(r),\overline{v}(r)],0<r\le 1$, are the r-level sets of a fuzzy number $\tilde{v}\in E,$ then the conditions (i) and (ii) hold true.

2.7. Definition [14]

Let I be a real interval. A mapping $\tilde{v}:I\to E$ is called a fuzzy process and we denote the r-level set by $[\tilde{v}(t)]_r=[\underline{v}(t,r),\overline{v}(t,r)]$. The Seikkala derivative $\tilde{v}'(t)$ of $\tilde{v}$ is defined by

$$[\tilde{v}'(t)]_r=[\underline{v}'(t,r),\overline{v}'(t,r)],$$

provided that is a equation defines a fuzzy number $\tilde{v}'(t)\in E.$

2.8. Definition [14]

The fuzzy integral of fuzzy process $\tilde{v}$, $\int_b^a\tilde{v}(t)dt$ for $a,b\in I$, is defined by

$$[\int_a^x\tilde{v}(t)dt]_r=[\int_a^x\underline{v}(t,r)d(t),\int_a^x\overline{v}(t,r)d(t)]$$

provided that the Lebesgue integrals on the right exist.

3. N-TH ORDER FUZZY DIFFERENTIAL EQUATIONS

In this section, we are going to investigate solution of n-th order fuzzy differential equations. Let

$$\begin{cases}\tilde{y}^{(n)}(x)=a_1(x)\tilde{y}+a_2(x)\tilde{y}'+\ldots+a_n(x)\tilde{y}^{(n-1)},\\ \tilde{y}(a)=\tilde{\alpha}_1,\tilde{y}'(a)=\tilde{\alpha}_2,\ldots,\tilde{y}^{(n-1)}(a)=\tilde{\alpha}_n,\quad a\le x\le b\end{cases}\tag{3.1}$$

where $\tilde{\alpha}_i, i=0,1,\ldots,n-1$ are fuzzy constant numbers and $a_i(x),$ are continuous on $[a,b]$. $\tilde{y}(x)$ is the solution to be determined.

Using the following assumptions

$$\tilde{y}=\tilde{y}_1,\tilde{y}'=\tilde{y}_2,\tilde{y}''=\tilde{y}_3,\ldots,\tilde{y}^{(n-1)}=\tilde{y}_n\ .$$

Then equation (3.1) is transformed to the following system of fuzzy differential equations of first order.

$$\begin{cases} \tilde{y}'_1 = \tilde{y}_2 \\ \tilde{y}'_2 = \tilde{y}_3 \\ \tilde{y}'_3 = \tilde{y}_4 \\ \quad\vdots \\ \tilde{y}'_n(x) = a_1(x)\tilde{y}_1 + a_2(x)\tilde{y}_2 + \ldots + a_n(x)\tilde{y}_n, \end{cases} \qquad a \le x \le b \tag{3.2}$$

with fuzzy initial conditions

$\tilde{y}_1(a) = \tilde{\alpha}_0, \tilde{y}_2(a) = \tilde{\alpha}_1, \ldots, \tilde{y}_n(a) = \tilde{\alpha}_{n-1}$.

Let

$(\underline{y}_1(x;r), \overline{y}_1(x;r)),\ (\underline{y}_2(x;r), \overline{y}_2(x;r)), \ldots, (\underline{y}_n(x;r), \overline{y}_n(x;r)),\ 0 \le r \le 1$ *and* $a \le x \le b$ are parametric form of $\tilde{y}_1(x), \tilde{y}_2(x), \ldots, \tilde{y}_n(x)$ respectively.

Then, parametric form of (3.2) is

$$\begin{cases} \underline{y}'_j(x;r) = \underline{y}_{j+1}(x;r), \quad (1 \le j \le n-1) \\ \underline{y}'_n(x;r) = \sum_{a_i(x)>0} a_i(x)\underline{y}_i(x;r) + \sum_{a_i(x)<0} a_i(x)\overline{y}_i(x;r), \\ \underline{y}'_j(x;r) = \overline{y}_{j+1}(x;r), \\ \underline{y}'_n(x;r) = \sum_{a_i(x)>0} a_i(x)\overline{y}_i(x;r) + \sum_{a_i(x)<0} a_i(x)\underline{y}_i(x;r). \end{cases} \tag{3.3}$$

To solve this system by VIM the following formulas are obtained

$$\underline{y}_j^{[k+1]}(x;r) = \underline{y}_j^{[k]}(x;r) + \int_a^x \underline{\lambda}_j(x,t)[\underline{y}'^{[k]}_j(t;r) - \underline{\tilde{y}}_j^{[k]}(t;r)]dt,$$

$$\underline{y}_n^{[k+1]}(x;r) = \underline{y}_n^{[k]}(x;r) + \int_a^x \underline{\lambda}_n(x,t)[\underline{y}'^{[k]}_j(t;r) - \Sigma_{a_i(t)>0} a_i(t)\underline{\tilde{y}}_i^{[k]}(t;r) - \Sigma_{a_i(t)<0} a_i(t)\tilde{\overline{y}}_i^{[k]}(t;r)]dt$$

$$\overline{y}_j^{[k+1]}(x;r) = \overline{y}_j^{[k]}(x;r) + \int_a^x \overline{\lambda}_j(x,t)[\overline{y}'^{[k]}_j(t;r) - \tilde{\overline{y}}_{j+1}^{[k]}(t;r)]dt,$$

$$\overline{y}_n^{[k+1]}(x;r) = \overline{y}_n^{[k]}(x;r) + \int_a^x \overline{\lambda}_n(x,t)[\overline{y}'^{[k]}_j(t;r) - \Sigma_{a_i(t)>0} a_i(t)\tilde{\overline{y}}_i^{[k]}(t;r) - \Sigma_{a_i(t)<0} a_i(t)\underline{\tilde{y}}_i^{[k]}(t;r)]dt$$

where $\lambda(x,t)$ is a general Lagrangian multiplier which can be identified optimally via variational theory, $\underline{\tilde{y}}^{[k]}, \tilde{\overline{y}}^{[k]}$ denote a restricted variation, i.e. $\delta\underline{\tilde{y}}^{[k]} = 0, \delta\tilde{\overline{y}}^{[k]} = 0$, k is the number of iteration step and note that $\delta\underline{\tilde{y}}^{[k]} = 0, \delta\tilde{\overline{y}}^{[k]} = 0.$

The variation is calculated with respect to $\underline{y}_j^{[k]}, \overline{y}_j^{[k]}$ (j=1, 2, …, n), respectively, then we have

$$\delta \underline{y}_j^{[k+1]}(x;r) = \delta \underline{y}_j^{[k]}(x;r) + \delta\int_a^x \underline{\lambda}_j(x,t)[\underline{y}_j'^{[k]}(t;r) - \widetilde{\underline{y}}_{j+1}^{[k]}(t;r)]dt$$
$$= \delta \underline{y}_j^{[k]}(x;r) + \underline{\lambda}_j(x,t)\delta \underline{y}_j^{[k]}(t;r)|_{t=x} - \int_a^x \frac{\partial \underline{\lambda}_j(x,t)}{dt}\delta \underline{y}_j^{[k]}(t;r)]dt$$
$$= (1+\underline{\lambda}_j(x,x))\delta \underline{y}_j^{[k]}(x;r) + \int_a^x (-\frac{\partial \underline{\lambda}_j(x,t)}{dt})\delta \underline{y}_j^{[k]}(t;r)]dt = 0, \quad j = 1,2,..., n-1.$$
$$\delta \underline{y}_n^{[k+1]}(x;r) = \delta \underline{y}_n^{[k]}(x;r) + \delta\int_a^x \underline{\lambda}_n(x,t)[\underline{y}_j'^{[k]}(t;r) - \sum_{a_i(t)>0} a_i(t)\widetilde{\underline{y}}_i^{[k]}(t;r) - \sum_{a_i(t)<0} a_i(t)\widetilde{\overline{y}}_i^{[k]}(t;r)]dt$$
$$= \delta \underline{y}_n^{[k]}(x;r) + \underline{\lambda}_n(x,t)\delta \underline{y}_n^{[k]}(t;r)|_{t=x} - \int_a^x \frac{\partial \underline{\lambda}_j(x,t)}{dt}\delta \underline{y}_n^{[k]}(t;r)]dt$$
$$= (1+\underline{\lambda}_n(x,x))\delta \underline{y}_n^{[k]}(x;r) + \int_a^x (-\frac{\partial \underline{\lambda}_n(x,t)}{dt})\delta \underline{y}_n^{[k]}(t;r)]dt = 0, \quad j = 1,2,..., n-1.$$

For arbitrary $\delta \underline{y}_j^{[k]}, j = 1,2,...,n,$ the following stationary conditions are obtained

$$-\frac{\partial \underline{\lambda}_i(x,t)}{dt} = -\frac{\partial \underline{\lambda}_2(x,t)}{dt} = ... = -\frac{\partial \underline{\lambda}_n(x,t)}{dt} = 0,$$

and the natural boundary condition

$$1 + \underline{\lambda}_j(x,x) = 0, \qquad j = 1,2,..,n.$$

The Lagrange multipliers, can be identified as

$$\underline{\lambda}_j(x,t) = -1, \qquad j = 1,2,..,n.$$

Similar to above we have

$$\overline{\lambda}_j(x,t) = -1, \qquad j = 1,2,..,n,$$

and the following iteration formula can be obtained as

$$\begin{cases} \underline{y}_j^{[k+1]}\ (x;r) = \underline{y}_j^{[k]}(x;r) - \int_a^x [\underline{y}_j'^{\ [k]}(t;r) - \underline{y}_{j+1}^{[k]}(t;r)]dt, \qquad j = 1,2,...,n-1. \\ \underline{y}_n^{[k+1]}\ (x;r) = \underline{y}_n^{[k]}(x;r) - \int_a^x [\underline{y}_j'^{\ [k]}(t;r) - \sum_{a_i(t)>0} a_i(t)\underline{y}_i^{[k]}(t;r) - \sum_{a_i(t)<0} a_i(t)\overline{y}_i^{[k]}(t;r)]dt, \\ \overline{y}_j^{[k+1]}\ (x;r) = \overline{y}_j^{[k]}(x;r) - \int_a^x [\overline{y}_j^{\ [k]}(t;r) - \overline{y}_{j+1}^{[k]}(t;r)]dt, \\ \overline{y}_n^{[k+1]}\ (x;r) = \overline{y}_n^{[k]}(x;r) - \int_a^x [\overline{y}_j'^{[k]}(t;r) - \sum_{a_i(t)>0} a_i(t)\overline{y}_i^{[k]}(t;r) - \sum_{a_i(t)<0} a_i(t)\underline{y}_i^{[k]}(t;r)]dt. \end{cases} \tag{3.4}$$

Beginning with $\underline{y}_j^{[0]}(a;r) = \underline{\alpha}_j(r), \overline{y}_j^{[0]}(a;r) = \overline{\alpha}_j(r),$ by the iteration formula (3.3), we can obtain the numerical solution of Eq.(3.1).

4. CONVERGENCE

In this section we analyze the convergency of VIM for (3.1). Similar to Remark (2.1), let

$$y^c(r) = \frac{\underline{y}(r) + \overline{y}(r)}{2}, \; y^d(r) = \frac{\underline{y}(r) - \overline{y}(r)}{2}.$$

Then the fuzzy version of (3.1) can be written as

$$\begin{cases} y_j'^{c}\ (x;r) = y_{j+1}^{c}(x;r), \quad (1 \le j \le n-1) \\ y_n'^{c}\ (x;r) = \sum_{a_i(x)>0} a_i(x) y_i^c(x;r) + \sum_{a_i(x)<0} a_i(x) y_i^c(x;r), \\ y_j'^{d}\ (x;r) = y_{j+1}^{d}(x;r), \\ y_n'^{d}\ (x;r) = \sum_{a_i(x)>0} a_i(x) y_i^d(x;r) + \sum_{a_i(x)<0} a_i(x) y_i^d(x;r), \end{cases} \tag{4.5}$$

and

$$\begin{cases} y_j^c(a;r) = \dfrac{\underline{y}_j(a;r) + \overline{y}_j(a;r)}{2} = \dfrac{\underline{\alpha}_j(r) + \overline{\alpha}_j(r)}{2}, \quad (1 \le j \le n) \\ y_j^d(a;r) = \dfrac{\underline{y}_j(a;r) + \overline{y}_j(a;r)}{2} = \dfrac{\underline{\alpha}_j(r) + \overline{\alpha}_j(r)}{2}. \end{cases}$$

Similarly from (3.4) we can obtain the following formula

$$\begin{cases} y_j^{[k+1]c}\ (x;r) = y^{[k]c}(x;r) - \int_a^x [y_j'^{[k]c}(t;r) - y_{j+1}^{[k]c}(t;r)]dt, \quad j = 1,2,\ldots,n-1. \\ y_n^{[k+1]c}\ (x;r) = y_n^{[k]c}(x;r) - \int_a^x [y_j'^{[k]c}(t;r) - \sum_{a_i(t)>0} a_i(t) y_i^{[k]c}(t;r) - \sum_{a_i(t)<0} a_i(t) y_i^{[k]c}(t;r)]dt, \\ y_j^{[k+1]d}\ (x;r) = y_j^{[k]d}(x;r) - \int_a^x [y_j'^{[k]d}(t,r) - \widetilde{y}_{j+1}^{[k]d}(t;r)]dt, \\ y_n^{[k+1]d}\ (x;r) = y_n^{[k]d}(x;r) - \int_a^x [y_j'^{[k]d}(t;r) - \sum_{a_i(t)>0} a_i(t) y_i^{[k]d}(t;r) - \sum_{a_i(t)<0} a_i(t) y_i^{[k]d}(t;r)]dt. \end{cases} \tag{4.6}$$

Let

$$e_j^{[k]c}(x;r) = y_j^{[k]c}(x;r) - y_j^c(x;r),$$

obviously

$$\begin{cases} y_j^c(x;r) = y_j^c(x;r) - \int_a^x [y_j'^c(t;r) - y_{j+1}^c(t;r)]dt, \quad j = 1,2,\ldots,n-1 \\ y_n^c(x;r) = y_n^c(x;r) - \int_a^x [y_j'^c(t;r) - \sum_{a_i(t)>0} a_i(t) y_i^c(t;r) - \sum_{a_i(t)<0} a_i(t) y_i^c(t;r)]dt. \end{cases}$$

Then

$$\begin{cases} e_j^{[k+1]c}(x;r) = e_j^{[k]c}(x;r) - \int_a^x [e_j'^{[k]c}(t;r) - e_{j+1}^{[k]c}(t;r)]dt, \quad j = 1,2,\ldots,n-1 \\ e_n^{[k+1]c}(x;r) = y_n^{[k]c}(x;r) - \int_a^x [e_j'^{[k]c}(t;r) - \sum_{a_i(t)>0} a_i(t) e_i^{[k]c}(t;r) - \sum_{a_i(t)<0} a_i(t) e_i^{[k]c}(t;r)]dt. \end{cases} \tag{4.7}$$

Relation of (4.7) can be written as follow

$$\begin{cases} e_j^{[k+1]c}(x;r) = \int_a^x [e_{j+1}^{[k]c}(t;r)dt, \quad j = 1,2,\ldots,n-1 \\ e_n^{[k+1]c}(x;r) = \int_a^x \sum_{j=1}^n a_j(t) e_j^{[k]c}(t;r)dt. \end{cases}$$

Suppose

$$|e_j^{[k]c}| = \max_{a \le t \le b} |e_j^{[k]c}(t;r)|, |e^{[k]c}| = \max_j |e_j^{[k]c}|, \quad j = 1,2,\ldots,n, \ k = 0,1,\ldots.$$

and

$$A_j = \max_{a \le t \le b} |a_j(t)|, A = \max_j \{A_j\}.$$

Then

$$\begin{cases} | e_j^{[1]c}(x;r) | \le \int_a^x | e_{j+1}^{[0]c}(t;r) dt \le x | e^{[0]c} |, \qquad j = 1, 2, ..., n-1 \\ | e_n^{[1]c}(x;r) | \le \int_a^x \sum_{j=1}^n | a_j(t) || e_j^{[0]c}(t;r) | dt \le nAx | e^{[0]c} |, \end{cases}$$

also

$$\begin{cases} | e_j^{[2]c} | \le \int_a^x | e_{j+1}^{[1]c}(t,r) dt \le \dfrac{x^2}{2!} | e^{[0]c} |, \qquad j = 1,2,...,n-1 \\ | e_n^{[2]c}(x,r) | \le \int_a^x \sum_{j=1}^n | a_j(t) || e_j^{[1]c}(t,r) | dt \le (nA)^2 \dfrac{x^2}{2!} | e^{[0]c} |, \end{cases}$$

and similarly we can obtain

$$\begin{cases} | e_j^{[k]c}(x;r) | \le \dfrac{x^k}{k!} | e^{[0]c} |, \qquad j = 1,2,...,n-1 \\ | e_n^{[k]c}(x;r) | \le (nA)^k \dfrac{x^k}{k!} | e^{[0]c} | . \end{cases}$$

Thus

$$\begin{cases} e_j^{[k]c}(x;r) \to 0 \quad as\ k \to \infty, \qquad j = 1,2,...,n-1 \\ e_n^{[k]c}(x;r) \to 0 \quad as\ k \to \infty. \end{cases} \tag{4.8}$$

In similar way, it can be proven that

$$\begin{cases} e_j^{[k]d}(x;r) \to 0 \quad as\ k \to \infty, \qquad j = 1, 2, ..., n-1 \\ e_n^{[k]d}(x;r) \to 0 \quad as\ k \to \infty. \end{cases} \tag{4.9}$$

and (4.8), (4.9) imply the convergency of method.

5. APPLICATION

In this section, three numerical examples are solved by MATLAB for illustration and the obtained solutions are compared with the exact solutions.

Example 1. Consider the two-order Fuzzy differential equation

$$\begin{cases} \tilde{y}''(x) = (1/2)\tilde{y}'(x) + (1/2)\tilde{y}(x), \\ \tilde{y}(0) = (2+r, 4-r), \ \tilde{y}'(0) = (2+r, 4-r), \end{cases}$$

the exact solution for this problem is $\tilde{y}(x) = (2+r, 4-r)e^x$.

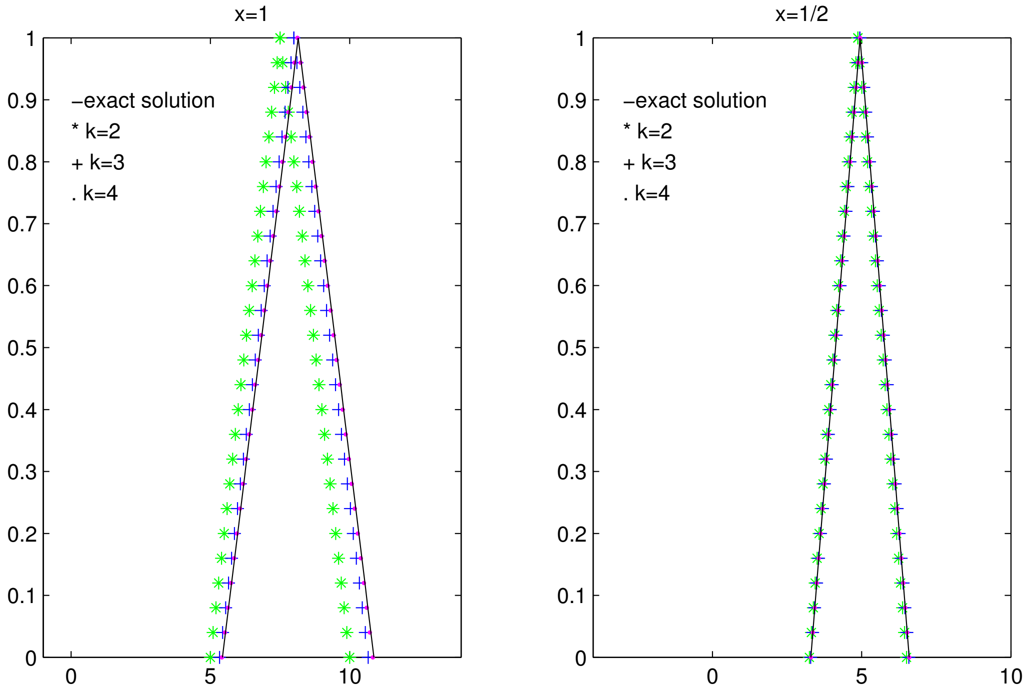

Table1.

$d_H(\tilde{y}^{[k]}, \tilde{y}_{exact})$			
x	k=2	k=5	k=10
0.2	0.0056	3.6597e-007	2.6645e-015
0.4	0.0473	2.4124e-005	4.3476e-012
0.6	0.1685	2.8320e-004	3.8261e-010
0.8	0.4222	0.0016	9.2191e-009
1	0.8731	0.0065	1.0925e-007

Example2. consider the four-order Fuzzy differential equation numerical result for example

$$\begin{cases} \tilde{y}^{(4)}(x) = \tilde{y}(x), \qquad 0 \le x \le 1 \\ \tilde{y}(0) = (r-1, 1-r),\ \tilde{y}'(0) = (r-1\,,1-r),\ \tilde{y}''(0) = (r-1\,,1-r), \\ \tilde{y}^{(3)}(0) = (r-1\,,1-r). \end{cases}$$

The exact solution for this problem is $\tilde{y}(x) = (r-1\,,1-r)e^{x}$.

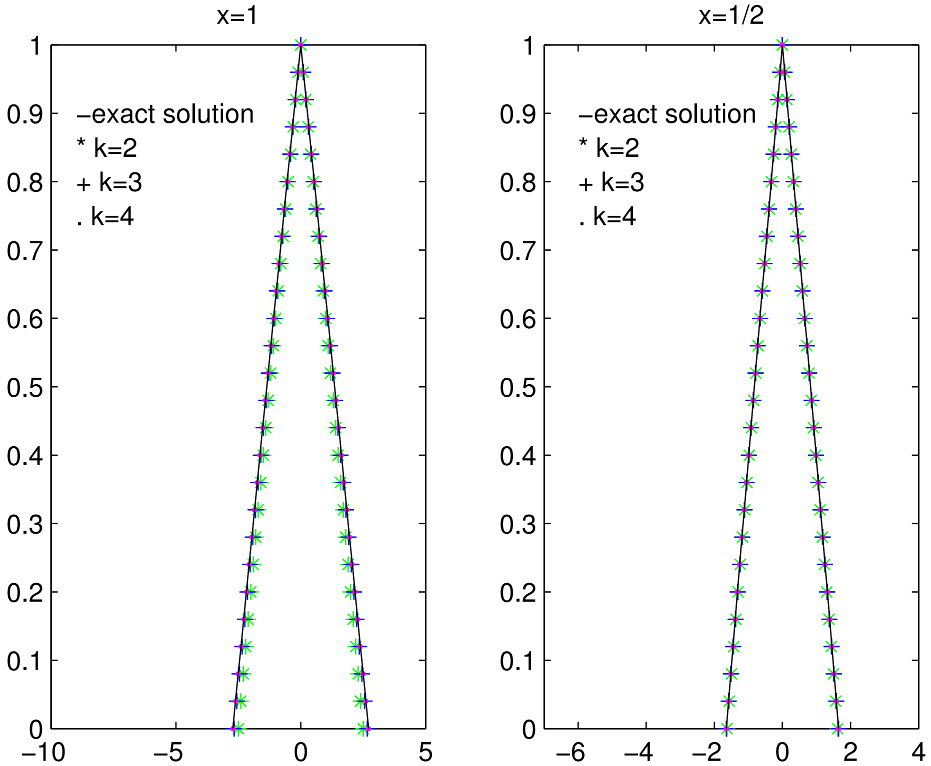

Table2.

$d_H(\tilde{y}^{[k]}, \tilde{y}_{exact})$			
x	k=2	k=5	k=10
0.2	1.4028e-003	9.1494e-008	6.6613e-016
0.4	1.825e-002	6.0310e-006	1.0869e-012
0.6	4.2119e-002	7.0800e-005	9.5652e-011
0.8	1.0554e-001	4.1026e-004	2.3048e-009
1	2.1828e-001	1.6152e-003	2.7313e-008

Example3. Consider the six-order Fuzzy differential equation numerical result for example

$$\begin{cases} \tilde{y}^{(6)}(x) = (x+1)\tilde{y}^{(5)}(x) + (1/24)x\,\tilde{y}^{(3)}(x) + (1/12)x^2\tilde{y}^{(2)} - (2/x^4)\tilde{y}(x), \qquad 1 \le x \le 2 \\ \tilde{y}(1) = (r\,, 2-r),\ \tilde{y}'(1) = (4r, 8-4r),\ \tilde{y}''(1) = (12r\,, 24-12r),\ \tilde{y}^{(3)}(1) = (24r\,, 48-24r), \\ \tilde{y}^{(4)}(1) = (24r\,, 48-24r),\ \tilde{y}^{(5)}(1) = (0,0). \end{cases}$$

The exact solution for this problem is $\tilde{y}(x) = (r, 2-r)x^4$.

Starting with initial approximations $\underline{y}_1^{[0]}(1;r) = r, \overline{y}_1^{[0]}(1;r) = 2-r,\ \underline{y}_2^{[0]}(1;r) = 4r,\ \overline{y}_2^{[0]}(1;r) =$
$8-4r,\ \underline{y}_3^{[0]}(1;r) = 12r,\ \overline{y}_3^{[0]}(1;r) = 24-12r,\ \underline{y}_4^{[0]}(1;r) = 24r, \overline{y}_4^{[0]}(1;r) = 48-24r,\ \underline{y}_5^{[0]}(1;r) =$
$12r, \overline{y}_5^{[0]}(1;r) = 24-12r, \underline{y}_6^{[0]}(1;r) = 0, \overline{y}_6^{[0]}(1;r) = 0.$

We can obtain following results:

$$\underline{y}_1^{[1]}(x;r) = r(4x-3),$$
$$\overline{y}_1^{[1]}(x;r) = (2-r)(4x-3),$$
$$\underline{y}_1^{[2]}(x;r) = r(6x^2-8x+3),$$
$$\overline{y}_1^{[2]}(x;r) = (2-r)(6x^2-8x+3),$$
$$\underline{y}_1^{[3]}(x;r) = r(4x^3-6x^2+4x-1),$$
$$\overline{y}_1^{[3]}(x;r) = (2-r)(4x^3-6x^2+4x-1),$$
$$\underline{y}_1^{[4]}(x;r) = rx^4,$$
$$\overline{y}_1^{[4]}(x;r) = (2-r)x^4.$$

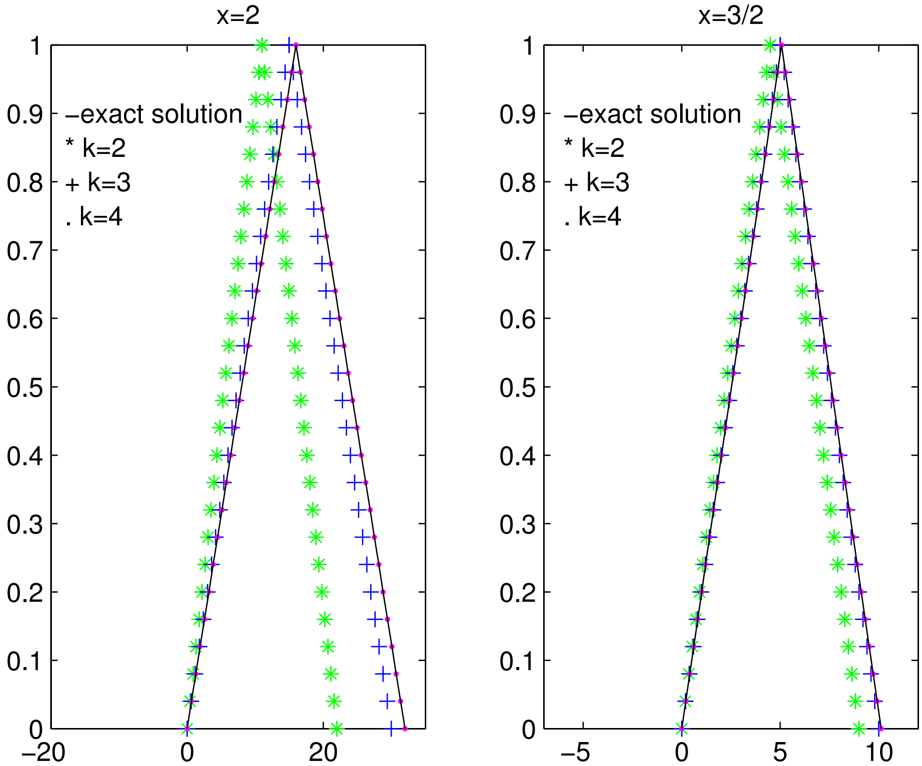

Table3.

$d_H(\widetilde{y}^{[k]}, \widetilde{y}_{exact})$			
x	k=2	k=5	k=10
1.2	0.0672	0.0032	0
1.4	0.0.5632	0.0512	0
1.6	1.9872	0.2591	0
1.8	4.9152	0.8192	0
2	10	2	0

6. CONCLUSION

In this paper, we used He's variational iteration method (VIM) to obtain fuzzy solution of the n-th order fuzzy differential equations. Convergency of VIM for this system is proved. The effectiveness of the method was shown by different examples.

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