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THE EXTENDED G'/G-EXPANSION METHOD AND TRAVELLING WAVE SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS

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Abstract-In this Letter, the G'/G-expansion method [M.L. Wang, X.Z. Li, J.L. Zhang, Phys. Lett. A 372 (2008) 417] is improved and an extended G'/G -expansion method is proposed to seek the travelling wave solutions of nonlinear evolution equations. We choose the mKdV equation to illustrate the validity and advantages of the proposed method. Many new and more general solutions are obtained. Our solutions naturally include those in open literature.

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1. INTRODUCTION

During the past four decades or so searching for explicit solutions of nonlinear evolution equations by using various different methods is the main goal for many researchers, and many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed such as the inverse scattering transform [1], the Backlund/ Darboux transform [2], the tanh-function expansion and its various extension [3], the exp-function expansion method [4-9] and so on, but there is no unified method that can be used to deal with all types of nonlinear evolution equations.

Recently, Wang et al. [10] introduced an expansion technique called the G'/G -expansion method and they demonstrated that it was a powerful technique for seeking analytic solutions of nonlinear partial differential equations. Bekir [11] and Zedan[12] applied this method to obtain traveling wave solutions of various equations. A generalization of the method was given by Zhang et al. [13]. Also, Zhang et al. [14] made a further extension of the method for the evolution equations with variable coefficients.

In this paper, we shall improve the G'/G -expansion method [10] and propose an extended G'/G -expansion method to seek the travelling wave solutions of nonlinear evolution equations.

The rest of the Letter is organized as follows. In Section 2, we describe the extended G'/G -expansion method to seek travelling wave solutions of nonlinear evolution equations, and give the main steps of the method here. In Section 3, we illustrate the method in detail with the celebrated KdV equation. In Section 4, some conclusions are given.

2. DESCRIPTION OF THE EXTENDED G'/G –EXPANSION METHOD

Suppose we have a nonlinear partial differential equation for $u(x,t)$ in the form

$$P(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots)=0, \tag{1}$$

where P is a polynomial in its arguments.

Step 1. By taking $u(x,t)=U(\xi),\xi=x-Vt$, we look for traveling wave solutions of Eq. (1), and transform it to the ordinary differential equation

$$Q(U,U',U'',\cdots)=0, \tag{2}$$

where prime denotes the derivative with respect to ξ.

Step 2. Integrating Eq. (2), if possible, term by term one or more times yields constant(s) of integration. The integration constant(s) can be set to zero for simplicity.

Step 3. Suppose the solution $U(\xi)$ of Eq. (2) can be expressed as a finite series in an extended symmetric form

$$U(\xi)=\sum_{i=-N}^{N}a_i(\frac{G'(\xi)}{G})^i, \tag{3}$$

where a_i are real constants to be determined, N is a positive integer to be determined, and the function $G(\xi)$ is the general solution of the auxiliary linear ordinary differential equation

$$G''(\xi)+\lambda G'(\xi)+\mu=0, \tag{4}$$

where λ,μ are real constants to be determined.

Step 4. By balancing the highest order nonlinear term(s) with the linear term(s) of highest order in Eq. (2), determine N.

Step 5. Get an algebraic equation involving powers of G'/G by substituting (3) together with (4) into Eq. (2). Next, equating the coefficients of each power of G'/G to zero, obtain a system of algebraic equations for a_i,λ,μ and V. Then, to determine these constants, solve the system with the aid of a computer algebra system. Since the solutions of Eq. (4) have been well known for us depending on the sign of the discriminant $\Delta=\lambda^2-4\mu$, the exact solutions of the given Eq. (1) can be obtained.

3. APPLICATIONS

In this section, we will demonstrate the extended G'/G -expansion method on the mKdV equation.

We start with the celebrated mKdV equation in the form

$$u_t-u^2u_x+\delta u_{xxx}=0,\delta>0, \tag{5}$$

The travelling wave variable below

$$u(x,t)=U(\xi),\xi=x-Vt\ , \tag{6}$$

Where the speed V of the travelling wave is to be determined later.

Permits us converting Eq. (5) into an ODE for $u=U(\xi)$

$$-VU'-U^2U'+\delta U'''=0, \tag{7}$$

Integrating it with respect to ξ once yields

$$C - VU - \frac{1}{3}U^3 + \delta U'' = 0, \tag{8}$$

where C is an integration constant that is to be determined later.

Now, we make an ansatz (3) for the solution of Eq.(8). Balancing the terns U'' and U^3 in Eq. (8) yields the leading order $N = 1$. therefore, we can write the solution of Eq.(8) in an extended symmetric form

$$U(\xi) = a_{-1}\left(\frac{G'(\xi)}{G}\right)^{-1} + a_0 + a_1\left(\frac{G'(\xi)}{G}\right), \tag{9}$$

By substituting Eq.(9) and Eq.(4) into Eq. (8) and collecting all terms with the same power of G'/G together, the left-hand side of Eq. (8) is converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $a_{-1}, a_0, a_1, \lambda, \mu, C$ and V as follows:

$$C - \frac{1}{3}a_0^3 + \delta a_1\lambda\mu - Va_0 + \delta a_{-1}\lambda - 2a_1a_0a_{-1} = 0,$$
$$\delta a_{-1}\lambda^2 - a_{-1}a_0^2 - Va_{-1} + 2\delta a_{-1}\mu - a_1a_{-1}^2 = 0,$$
$$3\delta a_{-1}\lambda - a_{-1}^2a_0 = 0,$$
$$-\frac{1}{3}a_{-1}^3 + 2\delta a_{-1}\mu^2 = 0,$$
$$-Va_1 - a_1^2a_{-1} + \delta a_1\lambda^2 - a_1a_0^2 + 2\delta a_1\mu = 0,$$
$$-a_1^2a_0 + 3\delta a_1\lambda = 0,$$
$$-\frac{1}{3}a_1^3 + 2\delta a_1 = 0,$$

Solving the above system with the aid of Maple, we have the following three sets of solutions:

Case 1:
$$a_1 = \pm\sqrt{6\delta}, a_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta},$$
$$V = -\frac{1}{2}\delta\lambda^2 + 2\delta\mu, a_{-1} = 0, C = 0, \tag{10}$$

where λ and μ are arbitrary constants.

Case 2: $$a_{-1} = \pm\mu\sqrt{6\delta}, a_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, V = -\frac{1}{2}\delta\lambda^2 + 2\delta\mu, a_1 = 0, C = 0, \tag{11}$$

where λ and μ are arbitrary constants.

Case 3: $$a_1 = \pm\sqrt{6\delta}, a_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, V = -\frac{1}{2}\delta\lambda^2 - 4\delta\mu, a_{-1} = \pm\mu\sqrt{6\delta}, C = 0, \tag{13}$$

where λ and μ are arbitrary constants.

Substituting the general solutions of Eq.(4) into (9), we have three types of traveling wave solutions of the KdV equation (5) as follow:

Case 1: when $\lambda^2 - 4\mu > 0$,

$$u_{11}=-3\delta(\lambda^2-4\mu)[\frac{C_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}]^2+3\delta\lambda^2+a_0,$$

where $\xi=x-(a_0+8\delta\mu+\delta\lambda^2)t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if

$C_1>0, C_1^2>C_2^2$, then u_1 can be written as

$$u_{11}^{'}=-3\delta(\lambda^2-4\mu)[\sec h^2(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+\xi_0)+12\delta\mu+a_0$$

which is the well-known solitary wave solution of the KdV equation (5) (see Ref. [15]), where $\xi_0=\tanh^{-1}\frac{C_2}{C_1}$, $\xi=x-(a_0+8\delta\mu+\delta\lambda^2)t$.

when $\lambda^2-4\mu<0$,

$$u_{12}=-3\delta(4\mu-\lambda^2)[\frac{-C_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{C_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}]^2+3\delta\lambda^2+a_0,$$

where $\xi=x-(a_0+8\delta\mu+\delta\lambda^2)t$, C_1 and C_2 are arbitrary constants.

when $\lambda^2-4\mu=0$,

$$u_{13}=-12\delta\frac{C_2^2}{(C_1+C_2\xi)^2}+3\delta\lambda^2+a_0,$$

where $\xi=x-(a_0+3\delta\lambda^2)t$, C_1 and C_2 are arbitrary constants.

Case 2: when $\lambda^2-4\mu>0$,

$$u_{21}=\frac{-12\delta\mu^2}{(-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}(\frac{C_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{(C_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}))^2}$$
$$-\frac{12\delta\lambda\mu}{-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}(\frac{C_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{(C_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)})}+a_0,$$

where $\xi=x-(a_0+8\delta\mu+\delta\lambda^2)t$, C_1 and C_2 are arbitrary constants.

when $\lambda^2-4\mu<0$,

$$u_{22}=\frac{-12\delta\mu^2}{(-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}(\frac{-C_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{(C_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}))^2}$$

$$-\frac{12\delta\lambda\mu}{-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}(\frac{-C_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{(C_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)})}+a_0,$$

where $\xi=x-(a_0+8\delta\mu+\delta\lambda^2)t$, C_1 and C_2 are arbitrary constants.

when $\lambda^2-4\mu=0$,

$$u_{23}=\frac{-12\delta\mu^2}{(-\frac{\lambda}{2}+\frac{C_1}{C_1+C_2\xi})^2}-\frac{12\delta\lambda\mu}{-\frac{\lambda}{2}+\frac{C_1}{C_1+C_2\xi}}+a_0,$$

where $\xi=x-(a_0+3\delta\lambda^2)t$, C_1 and C_2 are arbitrary constants.

Here, u_{11}, u_{12} and u_{13} had been given in Ref.[10], u_{21}, u_{22} and u_{23} had not been given in Ref.[10]. So, our solutions naturally include those in open literature.

6. CONCLUSION

The extended $\left(\frac{G'}{G}\right)$-expansion method was successfully used to establish travelling wave solutions of nonlinear evolution equations. The main difference between this method and Wang’s $\left(\frac{G'}{G}\right)$-expansion method is that we assume a new symmetric form $U(\xi)=\sum_{i=-N}^{N}a_i(\frac{G'(\xi)}{G})^i$ for the solutions, instead of $U(\xi)=\sum_{i=0}^{N}a_i(\frac{G'(\xi)}{G})^i$ in his method. So, our solutions naturally include those in open literature by Wang’s $\left(\frac{G'}{G}\right)$-expansion method. The performance of this method is reliable and effective and gives more solutions. This method has more advantages: it is direct and concise. The general solutions of the second order LODE have been well-known for the researchers, and our method is elementary and effective, can be further used in many other nonlinear evolution equations.

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