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APPROXIMATE FIRST INTEGRALS FOR A SYSTEM OF TWO COUPLED VAN DER POL OSCILLATORS WITH LINEAR DIFFUSIVE COUPLING

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Abstract- The approximate partial Noether operators for a system of two coupled van der Pol oscillators with linear diffusive coupling are presented via a partial Lagrangian approach. The underlying system of two equations, in general, do not admit a standard Lagrangian. However, the approximate first integrals are constructed by utilization of the partial Noether's theorem with the help of approximate partial Noether operators associated with a partial Lagrangian. These approximate partial Noether operators are not approximate symmetries of the system under study and they do not form an approximate Lie algebra. Moreover, we show how approximate first integrals can be constructed for perturbed ordinary differential equations (ODEs) without making use of a standard Lagrangian.

Keywords- Partial Noether operators, partial Lagrangians, first integrals, van der Pol.

1. INTRODUCTION

The system of two coupled van der Pol oscillators with linear diffusive coupling was first proposed by Rand and Holmes [1]

$$y'' + \epsilon(y^2 - 1)y' + y = A(z - y) + B(z' - y'),$$
$$z'' + \epsilon(z^2 - 1)z' + z = A(y - z) + B(y' - z'), \tag{1}$$

where A and B are coupling parameters that give the strength of the interaction and y and z are dependent variables which model the state of the oscillators and prime denotes the differentiation with respect to x. The study of coupled van der Pol oscillators has been of great interest for mathematicians and engineers due to its extensive range of applications. The system of two equations under study frequently arise in nonlinear oscillations, nonlinear dynamics and mathematical physics etc.

The main goal here to study the system (1) for approximate partial Noether operators and the corresponding approximate first integrals. These are important the physics and for reduction of the system. Secondly we show how one can obtain approximate first integrals for perturbed ODEs without a variational principle in a simple manner.

The approximate first integrals for perturbed equations which have Lagrangians can be constructed by using the classical Noether's theorem [2, 3, 4, 5]. Now, for various equations it is difficult to find the standard Lagrangian and many equations that arise in applications do not have Lagrangians. For the system of two weekly coupled nonlinear oscillators

$$y'' = -\omega_1^2 y + \epsilon\alpha_1^2 z,$$

$$z'' = -\omega_2^2 z + 2\epsilon\alpha_1 yz,$$

where ω_1, ω_2, ϵ and α_1 are positive constant and prime denotes the differentiation with respect to x, no variational problem exists. Likewise, for the simple system of two equations $y'' + \epsilon y' + \epsilon^2 z = 0$, $z'' + \epsilon^2 z = 0$, the curve family is also non-extremal. For an account of this theory to the inverse problem, we refer to the interesting paper [6] in which the complete classification for a system of two second-order ODEs that has Lagrangian was presented.

The theory of the approximate partial Noether operators and approximate conservation laws for differential equations with a small parameter has been recently introduced by Johnpillai et al in [7] (see also [8] for ODEs). We give an easy way to construct approximate partial Noether operators and approximate first integrals for perturbed equations without making use of a Lagrangian. We first compute the approximate partial Noether operators and then approximate first integrals are constructed by utilization of the Noether-like theorem with the help of approximate partial Noether operators associated with a partial Lagrangian. Such a system is known as an approximate partial Euler-Lagrange system. This approach will give further impetus for studies in constructing approximate first integrals of perturbed systems of ODEs without making use of standard Lagrangians.

2. PARTIAL NOETHER OPERATORS

In this section we derive the approximate partial Noether operators for zeroth and first order approximations of ϵ for the system of two coupled van der Pol oscillators (1) via a partial Lagrangian.

A partial Lagrangian of system (1) is

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{(A+1)}{2}(y^2+z^2) \tag{2}$$

and the corresponding partial Euler-Lagrange equations are

$$\begin{aligned} \delta L/\delta y &= \epsilon(y^2-1)y' - B(z'-y') - Az, \\ \delta L/\delta z &= \epsilon(z^2-1)z' - B(y'-z') - Ay. \end{aligned} \tag{3}$$

Now, the approximate operator

$$\chi = X_0 + \epsilon X_1 \tag{4}$$

where

$$X_0 = \xi_0\frac{\partial}{\partial x} + \eta_0^1\frac{\partial}{\partial y} + \eta_0^2\frac{\partial}{\partial z} + \zeta_{0x}^1\frac{\partial}{\partial y'} + \zeta_{0x}^2\frac{\partial}{\partial z'} + \cdots, \tag{5}$$

$$X_1 = \xi_1\frac{\partial}{\partial x} + \eta_1^1\frac{\partial}{\partial y} + \eta_1^2\frac{\partial}{\partial z} + \zeta_{1x}^1\frac{\partial}{\partial y'} + \zeta_{1x}^2\frac{\partial}{\partial z'} + \cdots, \tag{6}$$

is said to be an approximate partial Noether operator corresponding to a partial Lagrangian L in (2) if it satisfies the zeroth and first approximation in ϵ determining equations, respectively (cf. [8])

$$\xi_{0y} = 0,\ \xi_{0z} = 0, \tag{7}$$

$$\eta_{0y}^1 - \frac{1}{2}\xi_{0x} = -B\xi_0,\ \eta_{0z}^2 - \frac{1}{2}\xi_{0x} = -B\xi_0,\ \eta_{0z}^1 + \eta_{0y}^2 = 2B\xi_0, \tag{8}$$

$$\eta_{0x}^1 = B\eta_0^1 - B\eta_0^2 + Az\xi_0 + f_{0y},\ \eta_{0x}^2 = -B\eta_0^1 + B\eta_0^2 + Ay\xi_0 + f_{0z}, \tag{9}$$

$$\begin{aligned} \eta_0^1(A+1)y + \eta_0^2(A+1)z + \frac{(A+1)}{2}(y^2+z^2)\xi_{0x} \\ -Az\eta_0^1 - Ay\eta_0^2 - f_{0x} = 0. \end{aligned} \tag{10}$$

$$\xi_{1y} = 0,\ \xi_{1z} = 0, \tag{11}$$

$$\begin{aligned} &\eta_{1y}^1 - \frac{1}{2}\xi_{1x} = -(y^2-1)\xi_0 - B\xi_1, \\ &\eta_{1z}^2 - \frac{1}{2}\xi_{1x} = -(z^2-1)\xi_0 - B\xi_1, \\ &\eta_{1z}^1 + \eta_{1y}^2 = 2B\xi_1, \end{aligned} \tag{12}$$

$$\begin{aligned} &\eta_{1x}^1 = (y^2-1)\eta_0^1 + B\eta_1^1 - B\eta_1^2 + Az\xi_1 + f_{1y}, \\ &\eta_{1x}^2 = (z^2-1)\eta_0^2 - B\eta_1^1 + B\eta_1^2 + Ay\xi_0 + f_{1z}, \end{aligned} \tag{13}$$

$$\eta_1^1(A+1)y + \eta_1^2(A+1)z + \frac{(A+1)}{2}(y^2+z^2)\xi_{1x}$$
$$-Az\eta_1^1 - Ay\eta_1^2 - f_{1x} = 0. \tag{14}$$

Solving equations (7)-(9), we obtain

$$\xi_0 = \alpha(x), \tag{15}$$

$$\eta_0^1 = (\frac{\alpha'}{2} - B\alpha)y + zE(x) + F(x), \tag{16}$$

$$\eta_0^2 = (\frac{\alpha'}{2} - B\alpha)z - yE(x) + 2By\alpha + H(x), \tag{17}$$

$$f_0 = (\frac{\alpha''}{2} - B\alpha')\frac{y^2}{2} + yzE' + yF' - B\frac{y^2}{2}(\frac{\alpha'}{2} - B\alpha)$$
$$-ByzE - ByF - Ayz\alpha + Byz(\frac{\alpha'}{2} - B\alpha) \tag{18}$$
$$-B\frac{y^2}{2}E + B^2\alpha y^2 + BHy + K(x,z),$$

where E, F, H and K are arbitrary functions of x and

$$K(x,z) = \frac{z^2}{4}\alpha'' - \frac{3}{4}Bz^2\alpha' + \frac{B}{2}z^2E + H'z$$
$$+BFz + \frac{B^2}{2}\alpha z^2 - BHz + M(x). \tag{19}$$

The replacement of (15)-(19) in (10) and then comparison of the coefficients of powers of y and z lead to

$$B\alpha' - B^2\alpha - E' + BE = 0, \tag{20}$$

$$\frac{\alpha'''}{4} - \frac{3}{4}B\alpha'' + (A+1)\alpha' + \frac{3}{2}B^2\alpha' - B\alpha - 3AB\alpha - \frac{B}{2}E' + AE = 0, \tag{21}$$

$$\frac{\alpha'''}{4} - \frac{3}{4}B\alpha'' + (A+1)\alpha' + \frac{B^2\alpha'}{2} - AB\alpha - B\alpha + \frac{B}{2}E' - AE = 0, \tag{22}$$

$$\frac{B}{2}\alpha'' - 2A\alpha' - B^2\alpha' + 4AB\alpha + 2B\alpha + E'' - BE' = 0, \tag{23}$$

$$F'' - BF' + BH' + (A+1)F - AH = 0, \tag{24}$$

$$H'' + BF' - BH' + (A+1)H - AF = 0. \tag{25}$$

In order to proceed with equations (20)-(25), two cases arise, viz. $B = 0$ and $B \neq 0$.

Case 1: $B = 0$.

The simple calculations reveal that the following subcases arise when we solve equations (20)-(25) for $B=0$.
Case 1.1: $A=0,\ B=0$.
Case 1.2: $A\neq 0,\ B=0$.

The subcases of Case 1.2 are

Case 1.2.1: $A+1\neq 0,\quad B=0$.
Case 1.2.2: $A+1=0,\quad B=0$.

For Case 1.2.1, there are two subcases

Case 1.2.1.1: $A\neq -1/2,\quad B=0$.
Case 1.2.1.2: $A=-1/2,\quad B=0$.

Case 2: $B\neq 0$.
For the solution of equations (20)-(25) when $B\neq 0$, two subcases arise:

Case 2.1: If $A\neq B^2/2,\quad B\neq 0$.
Case 2.2: $A=B^2/2,\quad B\neq 0$.

One can easily see that two subcases of Case 2.1 arise when we solve for the first order approximation.
Case 2.1.1: If $A\neq -1/2,\quad B\neq 0$.
Case 2.1.2: If $A=-1/2,\quad B\neq 0$.

We provide detailed calculations for the Case 1.1. However, all the possibilities are listed in the table.
Case 1.1: If $A=0$ and $B=0$, then equations (20)-(25) yield

$$\alpha(x)=d_2+d_3\cos 2x+d_4\sin 2x, \tag{26}$$

$$E(x)=d_1,\quad F(x)=d_5\cos x+d_6\sin x, \tag{27}$$

$$H(x)=d_7\cos x+d_8\sin x, \tag{28}$$

where $d_1,\cdots,d_8$ are constants.
Equations (15)-(19) together with (26)-(28) give rise to

$$\xi_0=d_2+d_3\cos 2x+d_4\sin 2x, \tag{29}$$

$$\eta_0^1 = y(-d_3 \sin 2x + d_4 \cos 2x) + d_1 z + d_5 \cos x + d_6 \sin x, \tag{30}$$

$$\eta_0^2 = z(-d_3 \sin 2x + d_4 \cos 2x) - d_1 y + d_7 \cos x + d_8 \sin x, \tag{31}$$

$$f_0 = (-d_3 \cos 2x - d_4 \sin 2x)(y^2 + z^2) + y(-d_5 \sin x + d_6 \cos x) \\ +z(-d_7 \sin x + d_8 \cos x). \tag{32}$$

In order to construct the partial Noether operators for the first order approximation of ϵ for Case 1.1 of system (1), we solve equations (11)-(14).

One can easily construct the solution of determining equations (11)-(14) for the case when $A = 0$, $B = 0$ as

$$d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad d_5 = 0, \quad d_6 = 0, \quad d_7 = 0, \quad d_8 = 0, \tag{33}$$

or

$$\xi_0 = 0, \quad \eta_0^1 = 0, \quad \eta_0^2 = 0, \quad f_0 = 0 \tag{34}$$

and

$$\xi_1 = g_2 + g_3 \cos 2x + g_4 \sin 2x, \tag{35}$$

$$\eta_1^1 = (-g_3 \sin 2x + g_4 \cos 2x)y + g_1 z + g_5 \cos x + g_6 \sin x, \tag{36}$$

$$\eta_1^2 = (-g_3 \sin 2x + g_4 \cos 2x)z - g_1 y + g_7 \cos x + g_8 \sin x, \tag{37}$$

$$f_1 = (y^2 + z^2)(-g_3 \cos 2x - g_4 \sin 2x) + y(-g_5 \sin x + g_6 \cos x) \\ +z(-g_7 \sin x + g_8 \cos x). \tag{38}$$

The approximate partial Noether operators are constructed by setting each constant one by one equal to unity and the remaining constants to zero. In this case the approximate partial Noether operators from equation (4) are

$$\begin{aligned}
&\chi_1 = \epsilon(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}), \quad f = 0, \\
&\chi_2 = \epsilon\frac{\partial}{\partial x}, \quad f = 0, \\
&\chi_3 = \epsilon(\cos 2x\frac{\partial}{\partial x} - y \sin 2x\frac{\partial}{\partial y} - z \sin 2x\frac{\partial}{\partial z}), \quad f = -\epsilon(y^2 + z^2)\cos 2x, \\
&\chi_4 = \epsilon(\sin 2x\frac{\partial}{\partial x} + y \cos 2x\frac{\partial}{\partial y} + z \cos 2x\frac{\partial}{\partial z}), \quad f = -\epsilon(y^2 + z^2)\sin 2x, \\
&\chi_5 = \epsilon \cos x\frac{\partial}{\partial y}, \quad f = -\epsilon y \sin x, \quad \chi_6 = \epsilon \sin x\frac{\partial}{\partial y}, \quad f = -\epsilon y \cos x, \\
&\chi_7 = \epsilon \cos x\frac{\partial}{\partial z}, \quad f = -\epsilon z \sin x, \quad \chi_8 = \epsilon \sin x\frac{\partial}{\partial z}, \quad f = \epsilon z \cos x.
\end{aligned} \tag{39}$$

3. FIRST INTEGRALS

The first integrals for a system of coupled van der Pol oscillators (1) with the help of approximate partial Noether operators corresponding to the partial Lagrangian (2) are determined from the formula (see [8])

$$I = f - [\xi L + (\eta^\alpha - \xi u_x^\alpha)\frac{\partial L}{\partial u_x^\alpha}] + O(\epsilon^2), \quad \alpha = 1, 2, \tag{40}$$

where

$$\xi = \xi_0 + \epsilon\xi_1, \eta^1 = \eta_0^1 + \epsilon\eta_1^1, \, \eta^2 = \eta_0^2 + \epsilon\eta_1^2 \text{ and } f = f_0 + \epsilon f_1.$$

The first integrals, partial Noether operators and guage terms for each case are presented in the following table.

Partial Noether Operators χ_i	Guage-like Terms f	First Integrals I_i
Case 1.1 $(A = 0, \ B = 0)$		
$\chi_1 = \epsilon(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})$,	$f = 0$,	$I_1 = \epsilon(-y'z + yz')$,
$\chi_2 = \epsilon\frac{\partial}{\partial x}$,	$f = 0$,	$I_2 = \frac{\epsilon}{2}(y^2 + z^2 + y'^2 + z'^2)$,
$\chi_3 = \epsilon(\cos 2x\frac{\partial}{\partial x}$ $-y\sin 2x\frac{\partial}{\partial y} - z\sin 2x\frac{\partial}{\partial z})$,	$f = -\epsilon(y^2 + z^2)\cos 2x$,	$I_3 = \epsilon\Bigg(\frac{1}{2}\cos 2x(y'^2 + z'^2 - y^2$ $-z^2) + (yy' + zz')\sin 2x\Bigg)$,
$\chi_4 = \epsilon(\sin 2x\frac{\partial}{\partial x}$ $+y\cos 2x\frac{\partial}{\partial y} + z\cos 2x\frac{\partial}{\partial z})$,	$f = -\epsilon(y^2 + z^2)\sin 2x$,	$I_4 = \epsilon\Bigg(\frac{1}{2}\sin 2x(y'^2 + z'^2 - y^2$ $-z^2) - (yy' + zz')\cos 2x\Bigg)$,
$\chi_5 = \epsilon\cos x\frac{\partial}{\partial y}$,	$f = -\epsilon y\sin x$,	$I_5 = \epsilon(-y\sin x - y'\cos x)$,
$\chi_6 = \epsilon\sin x\frac{\partial}{\partial y}$,	$f = -\epsilon y\cos x$,	$I_6 = \epsilon(y\cos x - y'\sin x)$,
$\chi_7 = \epsilon\cos x\frac{\partial}{\partial z}$,	$f = -\epsilon z\sin x$,	$I_7 = \epsilon(-z\sin x - z'\cos x)$,
$\chi_8 = \epsilon\sin x\frac{\partial}{\partial z}$.	$f = \epsilon z\cos x$.	$I_8 = \epsilon(z\cos x - z'\sin x)$.

Case 1.2.1.1: $(A \neq 0, -1, -1/2, \quad B = 0)$		
$\chi_1 = \epsilon \frac{\partial}{\partial x}$,	$f = -\epsilon A y z$,	$I_1 = \epsilon\Big(- Ayz + \frac{1}{2}(y'^2 + z'^2) + \frac{A+1}{2}(y^2 + z^2)\Big)$,
$\chi_2 = \frac{\epsilon}{2} \cos x (\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = -\frac{\epsilon}{2}(y+z)\sin x$,	$I_2 = \frac{-\epsilon}{2}\Big((y+z)\sin x + (y'+z')\cos x\Big)$,
$\chi_3 = \frac{\epsilon}{2} \sin x (\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = \frac{\epsilon}{2}(y+z)\cos x$,	$I_3 = \frac{\epsilon}{2}\Big((y+z)\cos x - (y'+z')\sin x\Big)$,
$\chi_4 = \epsilon\Big[\cos[(\sqrt{2A+1})x] \times (-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\Big]$,	$f = \epsilon\Big[\sqrt{2A+1} \times \sin[(\sqrt{2A+1})x](y-z)\Big]$,	$I_4 = \epsilon\Big(\sqrt{2A+1}(y-z) \times \sin(\sqrt{2A+1})x + (y'-z')\cos(\sqrt{2A+1})x\Big)$,
$\chi_5 = \epsilon\Big[\sin[(\sqrt{2A+1})x] \times (-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\Big]$.	$f = -\epsilon\Big[\sqrt{2A+1} \times \cos[(\sqrt{2A+1})x](y-z)\Big)$.	$I_5 = \epsilon\Big(- \sqrt{2A+1}(y-z) \times \cos(\sqrt{2A+1})x + (y'-z')\sin(\sqrt{2A+1})x\Big)$.

Case 1.2.1.2: $(A=-1/2,\ \ B=0)$		
$\chi_1=\epsilon\frac{\partial}{\partial x}$,	$f=\frac{\epsilon}{2}yz$,	$I_1=\frac{\epsilon}{2}\Big[yz+\frac{1}{2}(y^2+z^2)+y'^2+z'^2\Big]$,
$\chi_2=\frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y}+\frac{\partial}{\partial z})$,	$f=-\frac{\epsilon}{2}(y+z)\sin x$,	$I_2=\frac{-\epsilon}{2}\Big[(y+z)\sin x+(y'+z')\cos x\Big]$,
$\chi_3=\frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y}+\frac{\partial}{\partial z})$,	$f=\frac{\epsilon}{2}(y+z)\cos x$,	$I_3=\frac{\epsilon}{2}\Big[(y+z)\cos x-(y'+z')\sin x\Big]$,
$\chi_4=\epsilon(-\frac{\partial}{\partial y}+\frac{\partial}{\partial z})$,	$f=0$,	$I_4=\epsilon\Big(y'-z'\Big)$,
$\chi_5=-\epsilon x(\frac{\partial}{\partial y}-\frac{\partial}{\partial z})$,	$f=-\frac{\epsilon}{2}(y-z)$,	$I_5=\epsilon\Big(z-y+x(y'-z')\Big)$,
$\chi_6=-\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$.	$f=\epsilon(\frac{1}{3}(y^3-z^3)-y+z)$.	$I_6=y'-z'+\epsilon\Big[\frac{1}{3}(y^3-z^3)-y+z\Big]$.
Case 1.2.2: $(A+1=0,\ B=0)$		
$\chi_1=\epsilon\frac{\partial}{\partial x}$,	$f=\epsilon yz$,	$I_1=\epsilon\Big(yz+\frac{1}{2}(y'^2+z'^2)\Big)$,
$\chi_2=\frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y}+\frac{\partial}{\partial z})$,	$f=-\frac{\epsilon}{2}(y+z)\sin x$,	$I_2=\frac{-\epsilon}{2}\Big[(y+z)\sin x+(y'+z')\cos x\Big]$,
$\chi_3=\frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y}+\frac{\partial}{\partial z})$,	$f=\frac{\epsilon}{2}(y+z)\cos x$,	$I_3=\frac{\epsilon}{2}\Big[(y+z)\cos x-(y'+z')\sin x\Big]$,
$\chi_4=\epsilon(-\frac{\partial}{\partial y}+\frac{\partial}{\partial z})\exp(x)$,	$f=-\epsilon(y+z)\exp(x)$,	$I_4=\epsilon\Big[-y+z+y'-z'\Big]\times\exp(x)$,
$\chi_5=\epsilon(-\frac{\partial}{\partial y}+\frac{\partial}{\partial z})\exp(-x)$.	$f=-\epsilon(y+z)\exp(-x)$.	$I_5=\epsilon\Big[y-z+y'-z'\Big]\times\exp(-x)$.

Case 2.1.1: $(A \neq B^2/2, -1/2, \quad B \neq 0)$		
$\chi_1 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = -\frac{\epsilon}{2}(y+z)\sin x$,	$I_1 = \frac{-\epsilon}{2}\Big[(y+z)\sin x + (y'+z')\cos x\Big]$,
$\chi_2 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = \frac{\epsilon}{2}(y+z)\cos x$,	$I_2 = \frac{\epsilon}{2}\Big[(y+z)\cos x - (y'+z')\sin x\Big]$,
$\chi_3 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\times \exp(B - \sqrt{-1-2A+B^2})$,	$f = \epsilon\Big[(-y+z)\times (B - \sqrt{-1-2A+B^2}) + 2B(y-z)\Big]\times \exp(B - \sqrt{-1-2A+B^2})$,	$I_3 = \epsilon\Big[-(y-z)\times (B - \sqrt{-1-2A+B^2}) + 2B(y-z) + y' - z'\Big]\times \exp(B - \sqrt{-1-2A+B^2})$,
$\chi_4 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\times \exp(B + \sqrt{-1-2A+B^2})$.	$f = \epsilon\Big[(-y+z)\times (B + \sqrt{-1-2A+B^2}) + 2B(y-z)\Big]\times \exp(B + \sqrt{-1-2A+B^2})$.	$I_4 = \epsilon\Big[-(y-z)\times (B + \sqrt{-1-2A+B^2}) + 2B(y-z) + y' - z'\Big]\times \exp(B + \sqrt{-1-2A+B^2})$.
Case 2.1.2: $(A \neq B^2/2, \quad A = -1/2, \quad B \neq 0)$		
$\chi_1 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = -\frac{\epsilon}{2}(y+z)\sin x$,	$I_1 = \frac{-\epsilon}{2}\Big[(y+z)\sin x + (y'+z')\cos x\Big]$,
$\chi_2 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = \frac{\epsilon}{2}(y+z)\cos x$,	$I_2 = \frac{\epsilon}{2}\Big[(y+z)\cos x - (y'+z')\sin x\Big]$,
$\chi_3 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = 2\epsilon B(y-z)$,	$I_3 = \epsilon\Big(2B(y-z) + y' - z'\Big)$,
$\chi_4 = \frac{\epsilon}{2B}\exp(2Bx) \times(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = 0$,	$I_4 = \epsilon\Big(\frac{1}{2B}\exp(2Bx) (y'-z')\Big)$,
$\chi_5 = -\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.	$f = 2B(y-z) + \epsilon(\frac{y^3}{3} - \frac{z^3}{3} - y + z)$.	$I_5 = 2B(y-z) + y' - z' + \epsilon\Big[\frac{1}{3}(y^3 - z^3) - y + z\Big]$.

Case 2.2: $(A = B^2/2,\ B \neq 0)$		
$\chi_1 = \epsilon(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}) \times \exp(Bx)$,	$f = -\frac{\epsilon B}{2}(y^2 - z^2) \times \exp(Bx)$,	$I_1 = \epsilon\Big[yz' - y'z + \frac{B}{2}(z^2 - y^2)\Big] \times \exp(Bx)$,
$\chi_2 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \times \sin x \exp(Bx)$,	$f = \epsilon\Big[(-y+z)(\cos x + B\sin x) + 2B(y-z)\sin x\Big] \times \exp(Bx)$,	$I_2 = \epsilon\Big[(-y+z)(\cos x + B\sin x) + 2B(y-z)\sin x + (y'-z')\sin x\Big] \times \exp(Bx)$,
$\chi_3 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \times \cos x \exp(Bx)$,	$f = \epsilon\Big[(y-z)(\sin x - B\cos x) + 2B(y-z)\cos x\Big] \times \exp(Bx)$,	$I_3 = \epsilon\Big[(y-z)(\sin x - B\cos x) + 2B(y-z)\cos x + (y'-z')\sin x\Big] \times \exp(Bx)$,
$\chi_4 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$,	$f = \frac{-\epsilon}{2}(y+z)\sin x$,	$I_4 = \frac{-\epsilon}{2} \times \Big((y+z)\sin x + (y'+z')\cos x\Big)$,
$\chi_5 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$.	$f = \frac{\epsilon}{2}(y+z)\cos x$.	$I_5 = \frac{\epsilon}{2} \times \Big((y+z)\cos x - (y'+z')\sin x\Big)$.

4. CONCLUDING REMARKS

We have shown how one can construct approximate first integrals of perturbed ordinary differential equations without a variational principle. The approximate partial Noether operators corresponding to a partial Lagrangian for a system of two coupled van der Pol oscillators were constructed by using the partial Lagrangian approach. Then the approximate first integrals were computed by invoking the partial Noether's theorem with the help of partial Noether operators via a partial Lagrangian for the system under consideration. All the special cases with respect to coupling parameters A and B are listed in the table. Out of all the first integrals obtained here, two were stable and the rest unstable. These approximate partial Noether operators are not approximate symmetry generators of the given system of equations therefore they do not admit an approximate Lie algebra and under what condition the approximate algebra is generated is given elsewhere [8].

5. REFERENCES

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